# Rate-Improved Permutation Codes for Correcting a Single Burst of Deletions

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***Abstract*— Permutation codes are widely studied due to their promising applications in flash memories. Based on the theory of permutation groups and subgroups, two classes of permutation codes are constructed to correct a single burst deletion of length up to a designated parameter. The proposed codes can achieve larger rates than available codes while maintaining simple interleaving structures. The decoding methods for the proposed codes are provided in proofs and verified by examples.**

***Index Terms*— Flash memories, rank modulation scheme, permutation codes, symbol-invariant deletions, permutation groups.**

## I. Introduction

PERMUTATION codes have recently attracted much attention due to their promising applications in flash storage systems [1]–[4]. To overcome the difficulty of exactly programming each flash memory cell to its designated level, Jiang *et al.* first proposed a rank modulation scheme to represent information in permutations [1]. In such a scheme, the information is stored in the form of ranking of the cells' charges rather than their absolute values.

Error-correcting codes in flash memory have been investigated under rank modulation [2]–[7]. Levenshtein first studied the deletion channel using Varshamov-Tenengolts (VT) codes [8], and constructed a class of asymptotically optimal binary codes against a single deletion [9]. Later, he utilized linear congruence equation to construct a class of binary codes correcting a single burst deletion of length up to two [10]. By using Levenshtein's binary codes, Tenengolts first proposed nonbinary codes against a single deletion [11]. Furthermore, Levenshtein constructed the perfect permutation codes correcting a single deletion [12]. Motivated by hardware implementation of rank modulation in flash memory [13]–[15], Gabrys *et al.* proposed two deletion models: symbol-invariant/permutation-invariant deletion (SID/PID), and constructed permutation codes correcting a single deletion [6].

Burst deletions are a severe type of corruptions occurring in adjacent cells due to capacitative coupling between the cells [16]. Chee *et al.* proposed two classes of codes against a single burst deletion under the PID and SID models, respectively [17]–[19]. Han *et al.* proposed two classes of permutation codes against a burst SID (BSID) [20]. More recently, two classes of multi-permutation codes were proposed against a single burst of unstable deletions [21]. To the best of our knowledge, the optimal permutation codes against a single BSID have not been found in literature. By optimal, it is meant to achieve the highest code rate with code length given.

In this letter, we propose two new constructions of permutation codes against a single BSID of length up to $t$ for $t$ being a code parameter. The first construction is based on the permutation group of length $t$ and can achieve higher code rates than [20, *Construction 1*]. The second construction is based on a stabilizer subgroup and can achieve higher code rates than [20, *Construction 2*]. These four constructions employ an interleaving framework and thus have simpler structures than [19, *Construction 2*].

## II. Preliminaries

### A. Basic Definitions

For integers $m \le n$, define $[m,n] \triangleq \{m, m+1, \dots, n\}$ and $[n] \triangleq \{1,2,\dots,n\}$. A permutation $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$ is defined as a self-bijection on the set $[n]$. Any pair of elements $(\alpha(i), \alpha(j))$ is called an *inversion* in $\alpha$ if $i < j$ and $\alpha(i) > \alpha(j)$. Let $\mathcal{S}_n$ be the full set of permutations over $[n]$. Then, it has cardinality $|\mathcal{S}_n| = n!$. For a subset of positions $\mathcal{P} \subseteq [n]$, define $\alpha(\mathcal{P}) \triangleq \{\alpha(i) : i \in \mathcal{P}\}$, which is an ordered subset of the elements in $\alpha$. For integer $k \in [n]$, define $k(\mathcal{P}) \triangleq k - |\{i \in \mathcal{P} : i < k\}| \in [n]$.

An SID is a stable deletion that does not change the values and relative positions of the surviving symbols. The SID model can be defined as follows [6].

*Definition 1:* For a permutation $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n)) \in \mathcal{S}_n$ and a subset of positions $\mathcal{P} \subseteq [n]$ of size $|\mathcal{P}| = t \in [0, n]$, it is said that the permutation $\alpha$ suffers $t$ SIDs in $\mathcal{P}$, resulting in the vector $\hat{\alpha} = (\hat{\alpha}(1), \hat{\alpha}(2), \dots, \hat{\alpha}(n-t))$, if $\hat{\alpha}(i) = \alpha(k)$ holds for $k \in [n] \backslash \mathcal{P}$ and $i = k(\mathcal{P})$.

*Definition 2:* A code $\mathcal{C} \subseteq \mathcal{S}_n$ is called a $t$-SID permutation code if it can correct up to $t$ SIDs, or a $\le t$-BSID permutation code if it can correct a single burst SID of length up to $t$.

*Definition 3:* A set of $l$ vectors $\rho_i = (\rho_i(1), \rho_i(2), \dots, \rho_i(n_i))$, $i \in [l]$, with lengths $n_1 \ge n_2 \ge \cdots \ge n_l \ge n_1 - 1$, can be interleaved as a vector $\alpha = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_l$ of length $n = \sum_{i=1}^{l} n_i$ by alternately placing the elements of $\rho_1, \rho_2, \dots, \rho_l$ in order. Then, $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$ is called the *interleaved vector* with $\alpha(j) = \rho_i(\lceil j/l \rceil)$ for $j \in [n]$ and $i \equiv j \pmod{l}$, where $\lceil x \rceil$ is the smallest integer equal to or larger than the real $x$ [4].

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### B. Levenshtein's Permutation Codes

Levenshtein's permutation codes were proposed to correct a single SID by signing non-binary vectors as follows [11].

*Definition 4:* The *signature* of a length-$n$ nonbinary vector $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ can be defined as a length-$(n-1)$ binary indicator vector $\mu(\sigma) = (\mu(\sigma_1), \mu(\sigma_2), \dots, \mu(\sigma_{n-1}))$ with

$$\mu(\sigma_i) = \begin{cases} 1, & \sigma_{i+1} \geq \sigma_i \\ 0, & \sigma_{i+1} < \sigma_i \end{cases}, \quad i \in [n-1]. \tag{1}$$

*Proposition 1:* For all $a \in \mathbb{Z}_n \triangleq [0, n-1]$, the class of Levenshtein's permutation codes in $\mathcal{S}_n$ are constructed as

$$C_n^a = \left\{\sigma \in \mathcal{S}_n : \textstyle\sum_{i=1}^{n-1} i \cdot \mu(\sigma_i) \equiv a \pmod{n}\right\}. \tag{2}$$

It was shown that the $n$ Levenshtein's permutation codes in $\mathcal{S}_n$ are disjoint and have the same cardinality of $(n-1)!$, thus partitioning the permutation group $\mathcal{S}_n$ [12, *Theorem 3.1*].

For a permutation code $C$ in $\mathcal{S}_n$, the rate is defined as

$$R(C) = \frac{\log |C|}{\log |\mathcal{S}_n|}. \tag{3}$$

## III. Constructions of $\leq t$-BSID Permutation Codes

In this section, we propose two kinds of permutation codes for correcting a single BSID of length up to $t$. The corresponding decoding methods are included in the proofs.

### A. Construction Based on the Symmetric Groups

*Definition 5:* Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be a set of $n$ distinct integers $b_1 < b_2 < \cdots < b_n$. A permutation over the set $\mathcal{B}$ can be defined from a permutation $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$ over the set $[n]$ as follows

$$f_{\mathcal{B}}(\alpha) = (b_{\alpha(1)}, b_{\alpha(2)}, \dots, b_{\alpha(n)}) \in \mathcal{S}_{\mathcal{B}}, \tag{4}$$

where $\mathcal{S}_{\mathcal{B}}$ denotes the full set of permutations over $\mathcal{B}$. Clearly, $\mathcal{S}_{\mathcal{B}}$ is isomorphic to $\mathcal{S}_n$ and has the same cardinality of $n!$.

*Definition 6:* The *projection* of $\alpha \in \mathcal{S}_n$ onto a subset $\mathcal{A} \subseteq [n]$ is defined as $\alpha_{\downarrow\mathcal{A}} \in \mathcal{S}_{\mathcal{A}}$ that collects from $\alpha$ all the elements in $\mathcal{A}$ while maintaining their relative order in $\alpha$ [6].

*Example 1:* Suppose that $n = 5$ and $\alpha = (4, 1, 3, 5, 2) \in \mathcal{S}_5$. If $\mathcal{B} = \{4, 7, 8, 9, 12\}$, then $f_{\mathcal{B}}(\alpha) = (9, 4, 8, 12, 7) \in \mathcal{S}_{\mathcal{A}}$. If $\mathcal{A} = \{2, 3, 5\} \subseteq [n]$, then $\alpha_{\downarrow\mathcal{A}} = (3, 5, 2) \in \mathcal{S}_{\mathcal{A}}$.

*Construction 1:* Given two positive integers $t$ and $n$, the set $[tn]$ can be partitioned into $t$ order-$n$ congruent classes

$$\mathcal{A}_i = \{j \in [tn] : j \equiv i \pmod{t}\}, \quad i \in [t]. \tag{5}$$

Then, given any integer $a \in \mathbb{Z}_n$ and the symmetric group $\mathcal{S}_t$, a permutation code in $\mathcal{S}_{tn}$ can be constructed from Levenshtein's permutation code $C_n^a$ as

$$\begin{aligned} C_1(t, n, a) = \bigcup_{\pi \in \mathcal{S}_t} \{& f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \circ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \circ \cdots \circ f_{\mathcal{A}_{\pi(t)}}(\alpha_t) : \\ & \alpha_i \in C_n^a, i \in [t]\}. \end{aligned} \tag{6}$$

*Remark 1:* Based on the class of Levenshtein's permutation codes $C_n^a$ in $\mathcal{S}_n$, there exist $n$ distinct codes $C_1(t, n, a)$ in $\mathcal{S}_{tn}$, each having cardinality $|C_1| = |\mathcal{S}_t| \cdot |C_n^a|^t = t![(n-1)!]^t$.

*Theorem 1:* The code $C_1(t, n, a)$ of length $tn$ from *Construction 1* is a $\leq t$-BSID permutation code in $\mathcal{S}_{tn}$.

*Proof:* Suppose that $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(tn))$ is a permutation codeword in $C_1(t, n, a)$ and it suffers a single BSID of length $s \leq t$ over positions $\mathcal{I} = [d, d+s-1]$ for some $d \in [tn - s + 1]$. Let $\alpha' = (\alpha'(1), \alpha'(2), \dots, \alpha'(tn-s))$ be the received permutation of length $tn - s$.

Due to the interleaving codeword structure, a single BSID of length $s \leq t$ in $\alpha$ yields the deletion of at most one symbol in each $\mathcal{A}_i$ for $i \in [t]$. Equivalently, a projection $\alpha'_{\downarrow\mathcal{A}_i}$ of $\alpha'$ onto $\mathcal{A}_i$ has exactly one symbol deleted if and only if it has length $n-1$, otherwise it must have length $n$ without suffering deletion. By *Definitions* 5 and 6, $f_{\mathcal{A}_i}(\alpha_i) = \alpha_{\downarrow\mathcal{A}_i}$. Then, $f_{\mathcal{A}_i}(\alpha_i)$ is recovered from $\alpha'_{\downarrow\mathcal{A}_i}$ of length $n-1$ in two steps:

1) The deleted symbol in $\alpha'_{\downarrow\mathcal{A}_i}$ is identified by comparing $\alpha'_{\downarrow\mathcal{A}_i}$ and $\mathcal{A}_i$;

2) The position of the deleted symbol in $\alpha'_{\downarrow\mathcal{A}_i}$ is determined by using the decoder of Levenshtein's permutation code $C_n^a$.

To determine $\pi \in \mathcal{S}_t$, we check if there exists any $i \in [t]$ such that the deleted symbol in $\alpha'_{\downarrow\mathcal{A}_i}$ is located at the first position of $f_{\mathcal{A}_i}(\alpha_i)$. If so, let $\pi(i) = \alpha'(i) \pmod{t}$ for $i \in [t(n-1)+1, tn]$. Otherwise, let $\pi(i) = \alpha'(i) \pmod{t}$ for all $i \in [t]$.

Finally, interleaving the $t$ permutations $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ in the order specified by $\pi$ yields the unique recovery of $\alpha \in C_1(t, n, a)$. ■

*Example 2:* Consider the $\leq 4$-BSID permutation code $C_1(t, n, a)$ with $t = 4$, $n = 8$, and $a = 0$. By (5), the congruent partition of the set $[tn]$ consists of the $t$ classes

$$\begin{aligned} \mathcal{A}_1 &= \{1, 5, 9, 13, 17, 21, 25, 29\}, \\ \mathcal{A}_2 &= \{2, 6, 10, 14, 18, 22, 26, 30\}, \\ \mathcal{A}_3 &= \{3, 7, 11, 15, 19, 23, 27, 31\}, \\ \mathcal{A}_4 &= \{4, 8, 12, 16, 20, 24, 28, 32\}. \end{aligned} \tag{7}$$

Given $\pi \in \mathcal{S}_4$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C_8^0$, a unique permutation codeword is generated as $\alpha = f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \circ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \circ f_{\mathcal{A}_{\pi(3)}}(\alpha_3) \circ f_{\mathcal{A}_{\pi(4)}}(\alpha_4)$, where $f_{\mathcal{A}_{\pi(i)}}(\alpha_i) = \alpha_{\downarrow\mathcal{A}_{\pi(i)}} \in \mathcal{S}_{\mathcal{A}_{\pi(i)}}$ for $i, \pi(i) \in [1, 4]$.

Suppose that $\pi = (2, 4, 1, 3)$, $\alpha_1 = (1, 7, 6, 5, 4, 3, 2, 8)$, $\alpha_2 = (2, 1, 7, 6, 5, 4, 8, 3)$, $\alpha_3 = (3, 2, 1, 6, 5, 8, 7, 4)$, and $\alpha_4 = (6, 7, 8, 5, 3, 4, 2, 1)$. The four associated permutations $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ can be obtained by (4) and organized in the matrix form

$$\begin{bmatrix} f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \\ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \\ f_{\mathcal{A}_{\pi(3)}}(\alpha_3) \\ f_{\mathcal{A}_{\pi(4)}}(\alpha_4) \end{bmatrix} = \begin{bmatrix} 2 & 26 & 22 & 18 & 14 & 10 & 6 & 30 \\ 8 & 4 & 28 & 24 & 20 & 16 & 32 & 12 \\ 9 & 5 & 1 & 21 & 17 & 29 & 25 & 13 \\ 23 & 27 & 31 & 19 & 11 & 15 & 7 & 3 \end{bmatrix}.$$

Reading the above matrix column-wise yields the permutation codeword $\alpha = (2, 8, 9, 23, 26, 4, 5, 27, 22, 28, 1, 31, 18, 24, 21, 19, 14, 20, 17, 11, 10, 16, 29, 15, 6, 32, 25, 7, 30, 12, 13, 3)$.

Suppose that $\alpha$ suffers a single BSID of length 3 at positions $\mathcal{I} = \{9, 10, 11\}$, i.e., the three adjacent symbols $\{22, 28, 1\}$ in $\alpha$ are deleted, yielding $\alpha' = (2, 8, 9, 23, 26, 4, 5, 27, 31, 18, 24, 21, 19, 14, 20, 17, 11, 10, 16, 29, 15, 6, 32, 25, 7, 30, 12, 13, 3)$.

It is easy to justify the burst length being $s = 3$ by measuring the length of $\alpha'$. To recover $\alpha$ from $\alpha'$, we first obtain the four projections of $\alpha'$, without knowing $\pi$, as follows

$$\begin{aligned} \alpha'_{\downarrow\mathcal{A}_1} &= (9, 5, 21, 17, 29, 25, 13), \\ \alpha'_{\downarrow\mathcal{A}_2} &= (2, 26, 18, 14, 10, 6, 30), \\ \alpha'_{\downarrow\mathcal{A}_3} &= (23, 27, 31, 19, 11, 15, 7, 3), \\ \alpha'_{\downarrow\mathcal{A}_4} &= (8, 4, 24, 20, 16, 32, 12). \end{aligned}$$

By comparing each $\alpha'_{\downarrow\mathcal{A}_i}$ and the corresponding $\mathcal{A}_i$, the deleted symbols are identified to be 1, 22, and 28 in $\alpha'_{\downarrow\mathcal{A}_1}$, $\alpha'_{\downarrow\mathcal{A}_2}$, and $\alpha'_{\downarrow\mathcal{A}_4}$, respectively. By using the decoder of $C_8^0$, the positions of the deleted symbols are determined to be all 3 in the three projections of length 7. Now that $f_{\mathcal{A}_i}(\alpha_i)$ can be recovered from $\alpha'_{\downarrow\mathcal{A}_i}$ for all $i \in [1,4]$.

Since no deletion occurs at the first position of each $f_{\mathcal{A}_i}(\alpha_i)$ for $i \in \{1,2,4\}$, we obtain $\pi = (2,4,1,3)$ by setting $\pi(i) = \alpha'(i) \pmod 4$ for all $i \in [1,4]$. Finally, $\alpha$ is uniquely recovered by interleaving the four recovered permutations $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ in the order specified by $\pi$.

*Corollary 1:* For $t$ fixed, the code $C_1(t,n,a)$ of length $tn$ from *Construction 1* is asymptotically optimal as $n \to \infty$.

*Proof:* By Stirling's formula $n! \sim n^n$, it is derived that

$$R(C_1) = \frac{\ln(t![(n-1)!]^t)}{\ln(tn)!} \sim \frac{tn\ln n + O(n)}{tn\ln n + O(n)} \sim 1. \tag{8}$$

## B. Construction Based on the Stabilizer Subgroups

*Definition 7:* For $i \in [n]$, the *stabilizer subgroup* of $\mathcal{S}_n$ about the $i$th coordinate is defined as [22]

$$\mathcal{S}_n^{(i)} = \{\pi = (\pi(1),\ldots,\pi(i),\ldots,\pi(n)) \in \mathcal{S}_n : \pi(i) = i\}, \tag{9}$$

where $|\mathcal{S}_n^{(i)}| = (n-1)!$.

*Example 3:* When $n = 3$, there are three stabilizer subgroups $\mathcal{S}_n^{(1)} = \{(1,2,3),(1,3,2)\}$, $\mathcal{S}_n^{(2)} = \{(1,2,3),(3,2,1)\}$, and $\mathcal{S}_n^{(3)} = \{(1,2,3),(2,1,3)\}$. Clearly, they have the identity permutation $(1,2,3)$ in common, and thus are not disjoint.

*Definition 8:* An *even permutation* in $\mathcal{S}_n$ is a permutation in $\mathcal{S}_n$ that has an even number of inversions. Denote by $\mathcal{S}_n^{even}$ the set of all the even permutations in $\mathcal{S}_n$, then $|\mathcal{S}_n^{even}| = n!/2$.

*Construction 2:* Given two positive integers $t$ and $n$, the set $[tn]$ can be partitioned into $t$ disjoint order-$n$ classes $\mathcal{A}_i$ as shown in (5). Then, given any Levenshtein's permutation code $C_n^a$ in $\mathcal{S}_n$ and the stabilizer subgroup $\mathcal{S}_t^{(1)}$ of $\mathcal{S}_t$, a permutation code in $\mathcal{S}_{tn}$ can be constructed as

$$\begin{aligned} C_2(t,n,a) = \bigcup_{\pi\in\mathcal{S}_t^{(1)}} \{&f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \circ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \circ \cdots \circ f_{\mathcal{A}_{\pi(t)}}(\alpha_t) : \\ &\alpha_1 \in C_n^a;\ \alpha_k \in \mathcal{S}_n^{even}, k \in [2,t]\}. \end{aligned} \tag{10}$$

*Remark 2:* Similarly, based on the class of Levenshtein's permutation codes $C_n^a$ in $\mathcal{S}_n$, there also exist $n$ distinct codes $C_2(t,n,a)$ in $\mathcal{S}_{tn}$, each of which has cardinality

$$|C_2| = |\mathcal{S}_t^{(1)}| \cdot |C_n^a| \cdot |\mathcal{S}_n^{even}|^{t-1} = (t-1)!(n-1)!\,(n!/2)^{t-1}.$$

*Theorem 2:* The code $C_2(t,n,a)$ of length $tn$ from *Construction 2* is a $\leq t$-BSID permutation code in $\mathcal{S}_{tn}$.

*Proof:* Suppose that the transmitted permutation $\alpha = (\alpha(1),\alpha(2),\ldots,\alpha(tn)) \in C_2(t,n,a)$ suffers a single BSID of length $s \leq t$ over positions $\mathcal{I}=[i,i+s-1]$ for some $i \in [tn-s]$, yielding a received sequence $\alpha'=(\alpha'(1),\alpha'(2),\ldots,\alpha'(tn-s))$.

Suppose that $\alpha$ is obtained by interleaving the $t$ permutations $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ with $\alpha_1 \in C_n^a$ and $\alpha_i \in \mathcal{S}_n^{even}$, $i \in [2,t]$. The interleaving order is specified by $\pi \in \mathcal{S}_t^{(1)}$. To recover $\alpha$ from $\alpha'$, we first determine the burst length $s$ from the length of $\alpha'$, and then consider the following two cases.

**Case 1 ($s = t$):** Firstly, a single BSID of length $t$ in $\alpha$ must have exactly one symbol in each $\mathcal{A}_i$ deleted from $\alpha$. By the structure of the stabilizer subgroup, $\pi$ can be recovered from $\alpha'$ by setting $\pi(i) = \alpha'(i) \pmod t$ for all $i \in [t]$.

Secondly, each projection $\alpha'_{\downarrow\mathcal{A}_i}$ of $\alpha'$ must have exactly one deleted symbol in $\mathcal{A}_i$. Thus, the deleted symbols can be found simply by comparing $\alpha'_{\downarrow\mathcal{A}_{\pi(i)}}$ and $\mathcal{A}_{\pi(i)}$ for all $i \in [t]$.

Thirdly, the position $p$ of the deleted symbol in $\alpha'_{\downarrow\mathcal{A}_{\pi(1)}}$ can be determined by using the decoder of Levenshtein's permutation code $C_n^a$. Thus, $f_{\mathcal{A}_{\pi(1)}}(\alpha_1)$ is recovered from $\alpha'_{\downarrow\mathcal{A}_{\pi(1)}}$.

Next, by the interleaving structure of $\alpha$ and the characteristic of BSID, it is easily deduced that the position of the deleted symbol in $\alpha'_{\downarrow\mathcal{A}_{\pi(2)}}$ must be either $p-1$ or $p$, and only one of them can lead to an even permutation after inserting the deleted symbol into $\alpha'_{\downarrow\mathcal{A}_{\pi(2)}}$. Therefore, $\alpha_{\downarrow\mathcal{A}_{\pi(2)}}$ and hence $f_{\mathcal{A}_{\pi(2)}}(\alpha_2)$ can be uniquely recovered from $\alpha'_{\downarrow\mathcal{A}_{\pi(2)}}$. Similarly, $\alpha_{\downarrow\mathcal{A}_{\pi(i)}}$ and hence $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ can be uniquely recovered from $\alpha'_{\downarrow\mathcal{A}_{\pi(i)}}$ for all $i \in [3,t]$.

Finally, the transmitted permutation can be uniquely determined as $\alpha = f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \circ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \circ \cdots \circ f_{\mathcal{A}_{\pi(t)}}(\alpha_t)$.

**Case 2 ($s < t$):** Firstly, define a sequence $\alpha''$ with $tn-s$ elements $\alpha''(i) = \alpha'(i) \pmod t \in [t]$, $i \in [tn-s]$. By Construction 2, the permutation $\pi \in \mathcal{S}_t^{(1)}$ will appear periodically in $\alpha''$ except for the elements congruent to the deleted symbols from $\alpha$, thus making $\pi$ easily and uniquely determined.

Secondly, since a single BSID does not change the symbols at other positions, the deleted symbols $\{a_1,a_2,\ldots,a_s\}$ from $\alpha$ can easily be identified by comparing $\alpha'$ and $[tn]$.

Thirdly, the starting position $p$ of the single BSID in $\alpha$ can be determined in the following three disjoint cases.

1) If $\alpha'(1) \not\equiv 1 \pmod{(t)}$, then $p = 1$, due to the fact that $\pi(1) = 1$ such that $\varphi_{\mathcal{A}_{\pi(1)}}(\alpha_1) = \varphi_{\mathcal{A}_1}(\alpha_1)$;

2) If there exists $i \in [2,tn-s]$ such that $\alpha'(i) - \alpha'(i-1) \not\equiv \pi(i \pmod t) - \pi(i-1 \pmod t) \pmod t$, then $p = i$;

3) Otherwise, $p = tn-s+1$, i.e., the single BSID occurs at the last $s$ positions in $\alpha$.

Upon knowing the positions of the single BSID, the deleted symbols $\{a_1,a_2,\ldots,a_s\}$ can be arranged as $a_{k_1},a_{k_2},\ldots,a_{k_s}$ with $a_{k_i} \equiv \pi(p+i-1 \pmod t) \pmod t$ for all $i \in [s]$. Finally, the transmitted permutation $\alpha$ is uniquely determined. ■

*Example 4:* Consider the code $C_2(t,n,a)$ with $t = 4$, $n = 8$, and $a = 0$. The congruent partition $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4$ of the set $[tn]$ is shown in (7). Given $\pi \in \mathcal{S}_4^{(1)}$, $\alpha_1 \in C_8^0$, and $\alpha_2,\alpha_3,\alpha_4 \in \mathcal{S}_8^{even}$, a unique permutation codeword can be generated as $\alpha = f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \circ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \circ f_{\mathcal{A}_{\pi(3)}}(\alpha_3) \circ f_{\mathcal{A}_{\pi(4)}}(\alpha_4)$, where $f_{\mathcal{A}_{\pi(i)}}(\alpha_i) = \alpha_{\downarrow\mathcal{A}_{\pi(i)}} \in \mathcal{S}_{\mathcal{A}_{\pi(i)}}$ for $i,\pi(i) \in [1,4]$.

Suppose that $\pi = (1,4,3,2)$, $\alpha_1 = (3,2,1,5,4,8,7,6)$, $\alpha_2 = (2,1,4,3,6,5,8,7)$, $\alpha_3 = (1,3,2,5,4,8,6,7)$, and $\alpha_4 = (1,2,5,4,3,8,6,7)$. The four associated permutations $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ can be obtained by (4) and organized in the matrix form

$$\begin{bmatrix} f_{\mathcal{A}_{\pi(1)}}(\alpha_1) \\ f_{\mathcal{A}_{\pi(2)}}(\alpha_2) \\ f_{\mathcal{A}_{\pi(3)}}(\alpha_3) \\ f_{\mathcal{A}_{\pi(4)}}(\alpha_4) \end{bmatrix} = \begin{bmatrix} 9 & 5 & 1 & 17 & 13 & 29 & 25 & 21 \\ 8 & 4 & 16 & 12 & 24 & 20 & 32 & 28 \\ 3 & 11 & 7 & 19 & 15 & 31 & 23 & 27 \\ 2 & 6 & 18 & 14 & 10 & 30 & 22 & 26 \end{bmatrix}.$$

Reading the above matrix column-wise yields the permutation codeword $\alpha = (9,8,3,2,5,4,11,6,1,16,7,18,17,12,19,14,13,24,15,10,29,20,31,30,25,32,23,22,21,28,27,26)$.

Suppose that $\alpha$ suffers a single BSID of length $s \leq t$ at positions $\mathcal{I} = [i, i+s-1]$ for some $i \in [tn-s]$, yielding a received sequence $\alpha'$ of length $tn-s$. To show how to recover $\alpha$ from $\alpha'$, we consider the following two cases.

*1) $s = t = 4$:* Suppose that $\mathcal{I} = \{9, 10, 11, 12\}$, i.e., the four adjacent symbols $\{1, 16, 7, 18\}$ are deleted from $\alpha$, yielding $\alpha' = (9, 8, 3, 2, 5, 4, 11, 6, 17, 12, 19, 14, 13, 24, 15, 10, 29, 20, 31, 30, 25, 32, 23, 22, 21, 28, 27, 26)$.

Letting $\pi(i) = \alpha'(i) \pmod 4$ for all $i \in [1, 4]$ yields $\pi = (1, 4, 3, 2) \in \mathcal{S}_4^{(1)}$. It follows that

$$\begin{aligned} \alpha'_{\downarrow \mathcal{A}_{\pi(1)}} &= (9, 5, 17, 13, 29, 25, 21), \\ \alpha'_{\downarrow \mathcal{A}_{\pi(2)}} &= (8, 4, 12, 24, 20, 32, 28), \\ \alpha'_{\downarrow \mathcal{A}_{\pi(3)}} &= (3, 11, 19, 15, 31, 23, 27), \\ \alpha'_{\downarrow \mathcal{A}_{\pi(4)}} &= (2, 6, 14, 10, 30, 22, 26). \end{aligned}$$

By comparing $\alpha'_{\downarrow \mathcal{A}_{\pi(i)}}$ and $\mathcal{A}_{\pi(i)}$ for all $i \in [4]$, the deleted symbols are identified to be 1, 16, 7, and 18, respectively in the four projections above. By using the decoder of Levenshtein's permutation code $C_8^0$, the position of the deleted symbol in $\alpha'_{\downarrow \mathcal{A}_{\pi(1)}}$ is uniquely determined as $p = 3$. The position of the deleted symbol in $\alpha'_{\downarrow \mathcal{A}_{\pi(2)}}$ is also determined as $p = 3$ since only the insertion of 16 into that position of $\alpha'_{\downarrow \mathcal{A}_{\pi(2)}}$ gives rise to an even permutation. Similarly, the positions of the deleted symbols 7 in $\alpha'_{\downarrow \mathcal{A}_{\pi(3)}}$ and 18 in $\alpha'_{\downarrow \mathcal{A}_{\pi(4)}}$ are all determined as 3.

Now that $f_{\mathcal{A}_{\pi(i)}}(\alpha_i)$ is uniquely recovered for all $i \in [4]$, and their interleaving in order leads to the unique recovery of $\alpha$.

*2) $s = 3 < t$:* Suppose that $\mathcal{I} = \{9, 10, 11\}$, yielding $\alpha' = (9, 8, 3, 2, 5, 4, 11, 6, 18, 17, 12, 19, 14, 13, 24, 15, 10, 29, 20, 31, 30, 25, 32, 23, 22, 21, 28, 27, 26)$ of length 29. By comparing $\alpha'$ and $[1, 32]$, the three adjacent symbols deleted from $\alpha$ can be identified as $\{1, 16, 7\}$ without knowing their order.

Let $\alpha''(i) = \alpha'(i) \pmod 4$ such that $\alpha''(i) \in [1, 4]$ for $i \in [1, 29]$. Then, $\alpha'' = (1, 4, 3, 2, 1, 4, 3, 2, 2, 1, 4, 3, 2, 1, 4, 3, 2, 1, 4, 3, 2, 1, 4, 3, 2, 1, 4, 3, 2)$, in which it is easily detected that the permutation $\pi = (1, 4, 3, 2) \in \mathcal{S}_4^{(1)}$ appears periodically except for the ninth position.

Let $p$ be the starting position of the single BSID in $\alpha$. Since $\alpha'(1) \equiv \pi(1) \pmod{(t)}$, then $p \neq 1$. It is searched sequentially in $[2, 29]$ that only $p = 9$ satisfies the inequality

$$\alpha'(p) - \alpha'(p-1) \not\equiv \pi(p \pmod 4) - \pi(p-1 \pmod 4) \pmod 4,$$

since $\alpha'(9) - \alpha'(8) \equiv 4 \pmod 4$, whereas $\pi(9 \pmod 4) = \pi(1)$, $\pi(8 \pmod 4) = \pi(4)$, and $\pi(1) - \pi(4) \equiv 3 \pmod 4$. Therefore, the three positions are determined to be 9, 10, and 11.

Finally, since $\pi(9 \pmod 4) \equiv 1$, $\pi(10 \pmod 4) \equiv 16$, and $\pi(11 \pmod 4) \equiv 7$, all in modulo-4, we obtain the unique arrangement that $\{1, 16, 7\}$, whose insertion into the position $p = 9$ of $\alpha'$ completes the recovery of $\alpha$.

*Corollary 2:* For $t$ fixed, the code $C_2(t, n, a)$ of length $tn$ from *Construction 2* is asymptotically optimal as $n \to \infty$.

*Proof:* Similarly by $n! \sim n^n$, it is derived that

$$R(C_2) = \frac{\ln[(t-1)!(n-1)!(\frac{n!}{2})^{t-1}]}{\ln(tn)!} \sim \frac{tn \ln n + O(n)}{tn \ln n + O(n)} \sim 1. \tag{11}$$

TABLE I

RATE COMPARISON BETWEEN DIFFERENT $\leq t$-BSID PERMUTATION CODES IN $\mathcal{S}_n$

| **Codes** | **Constructions** | **Constraints** | **Rates** $R(C_i)$ |
|---|---|---|---|
| $C_1$ | Construction 1 | $t \mid n$ | $\ln\{t![(\frac{n}{t}-1)!]^t\}/\ln n!$ |
| $C_2$ | Construction 2 | $t \mid n$ | $\ln\{(t-1)!(\frac{n}{t}-1)![\frac{1}{2}(\frac{n}{t})!]^{t-1}\}/\ln n!$ |
| $C_3$ | [19, Construction 2] | $2t! \mid n$ | $\ln \frac{t!n!}{[(4t)!]^2 n^{2t-1}} / \ln n!$ |
| $C_4$ | [20, Construction 1] | $t \mid n$ | $\ln[(\frac{n}{t}-1)!]^t/\ln n!$ |
| $C_5$ | [20, Construction 2] | $(t+1) \mid n$ | $\ln[(\frac{n}{t+1})!]^{t+1}/\ln n!$ |

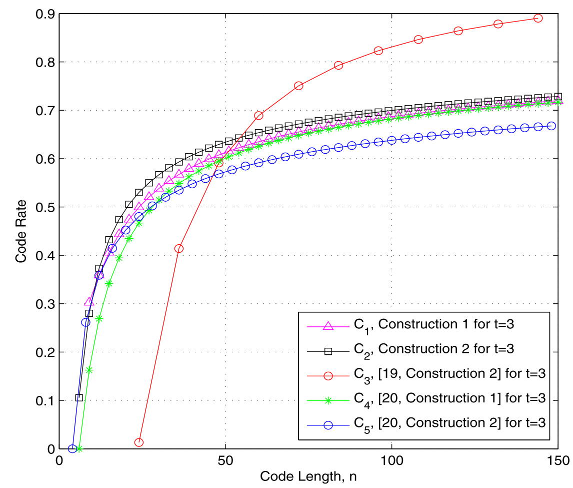


Fig. 1. Rate comparison among the constructions proposed here and in [19] and [20] for the code length $n \in [3, 150]$ and the burst length $t = 3$.

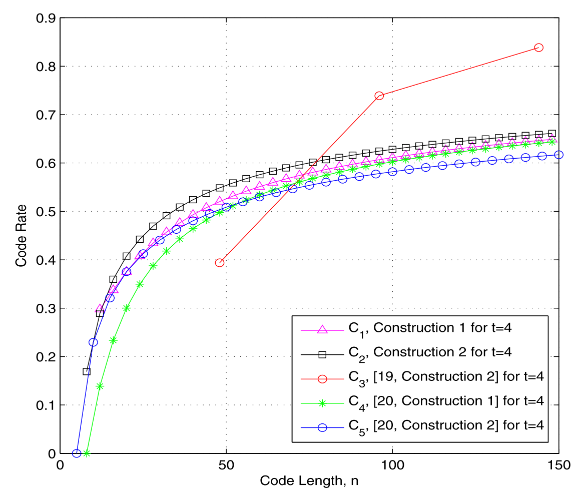


Fig. 2. Rate comparison among the constructions proposed here and in [19] and [20] for the code length $n \in [3, 150]$ and the burst length $t = 4$.

## C. Rate Comparison for Different Codes

For fair comparison, consider a unified pair of $n$ and $t$ for comparing different $\leq t$-BSID permutation codes in $\mathcal{S}_n$. Table I presents the rates and constraints for five codes, where the rate for $C_3$ is a lower bound [19]. Given $t$, it is shown that $R(C_2) - R(C_1) = \frac{(t-1)\ln\frac{n}{2} - t\ln t}{\ln n!} > 0$ for $n > 2t^{\frac{t}{t-1}}$, $R(C_1) - R(C_4) = \frac{\ln t!}{\ln n!} > 0$ for $n > t$, and $R(C_2) - R(C_3) = \frac{\ln\{2[\frac{n^2}{2}(\frac{n}{t})!]^t[(4t)!]^2\} - \ln[n^2 n!]}{\ln n!}$, the third comparison result being not straightforward.

Intuitively, Figs. 1 and 2 show the rate distributions versus $n \in [3, 150]$ for $t = 3$ and $t = 4$, respectively. It is seen that

$$R(C_2) > R(C_1) \text{ for } (n > 10, t = 3) \text{ and } (n > 12, t = 4);$$
$$R(C_2) > R(C_3) \text{ for } (n < 60, t = 3) \text{ and } (n < 96, t = 4);$$
$$R(C_1) > R(C_3) \text{ for } (n < 60, t = 3) \text{ and } (n < 96, t = 4);$$
$$R(C_2) > R(C_5) \text{ for } n > t; \text{ and } R(C_1) > R(C_4) \text{ for } n > t.$$

However, for $(n \geq 60, t = 3)$ and $(n \geq 96, t = 4)$, both the proposed codes yield lower rates than $C_3$. In these cases, the proposed codes may still be favored due to their advantage in flexibly matching the code parameters $t$ and $n$, in addition to the fact that they achieve much higher rates than $C_3$ over a wider range of $n$ as $t$ grows.

## IV. Conclusion

Two classes of permutation codes have been constructed under the interleaving framework for correcting a single BSID of length up to $t$. The use of a symmetric group or a stabilizer subgroup enlarges the code cardinality as much as possible, while maintaining the advantage in flexibly choosing the code parameters for practical applications in flash memories.

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