Chapter 5 Linear System: Random Process

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5 Linear Systems: Random Processes

5.1 Introduction

In many scientific disciplines the description of input-output relationships for various types of systems is paramount to understanding the system. The inputs could be many different things. However, they are many times considered some form of excitation to the system, with the output representing the response of the system. System description can take many forms. When the input to a system can be thought of as a realization from a particular random process, the output signal can be determined by using the system's definition. Each realization of the process in turn generates an output signal. In this way there exists a mapping through the system from a set of outcomes governing the input random process and thus defining an output random process.

In this chapter we will explore the relationships that exist between input process characterization and output process characterizations for a special class of systems called linear systems, while in the next chapter, we will explore the same problem for nonlinear systems. For example, the following question is the main theme: Knowing the mean, autocorrelation function, first-order density, and stationarity of the input process, what are the mean, autocorrelation function, first-order density, stationarity of the output process? It will be seen that a particular partial characterization of type may not be sufficient for determining corresponding partial characterization of one the same type for the output process. The presentation begins with definitions of signals and systems pertinent to this development and is followed by deeper exploration of the partial characterizations of the output process for linear time-varying and time-invariant systems of both discrete time and continuous time representations.

5.2 Classification of Systems

A systems can be thought of as a mapping from a closed set, S_I, called the input signal set to a closed set, S_O, called the output signal set. If the input and output signals are continuous time signals, then the systems represented by T[·] is called a **continuous time system**. Similarly, if the inputs and output signals are discrete time signals, the system is called a **discrete time system**. It is conventional to indicate continuous time input and output signal as错误！未定义书签。错误！未定义书签。 $x(t)$错误！未定义书签。 and $y(t)$, respectively, whereas discrete time input and output signals are indicated by $x[n]$ and $y[n]$ respectively. The input and output functional relationships for both continuous time and discrete time systems will be indicated by $T[\cdot]$ as shown in Figure 5.1.Also shown are two other types of mixed continuous time and continuous-discrete time systems. Examples are digital to analog and analog to digital convertors.

Let us consider some special characteristics that these systems possess, for example linearity ,causality, and time invariance. Letting $T[\cdot]$ represent the mapping of a system and $x(t)$错误！未定义书签。 and $y(t)$,the input and output, respectively, a continuous time system can be represented as follows:

$$y(t) = T[x(t)] \tag{5.1}$$

Although the following presentation users continuous time system formality, the concepts and definitions that follow will apply to any of the four types of systems presented in Figure 5.1.

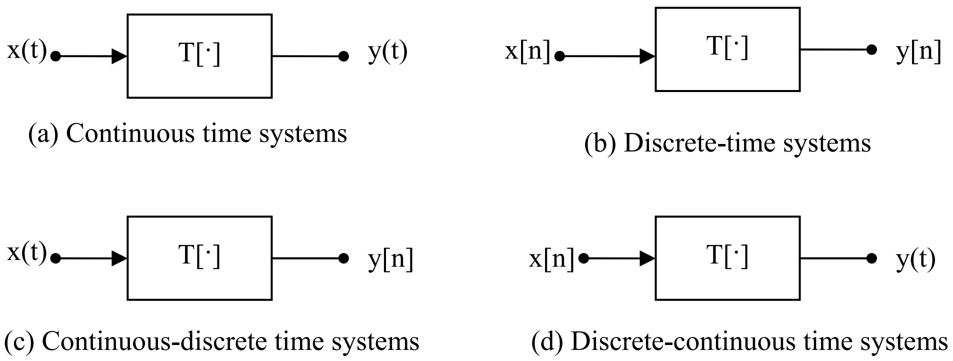

Figure 5.1 Systems

A system is called a **linear** if for every $x_1(t)$ and $x_2(t) \in S_I$, the input set, and all constants a_1 and a_2,

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)] \tag{5.2}$$

If a system is no linear, it is called a **nonlinear** system.

Let $x(t)$ and $y(t)$ represent the input and output of the system. A system is described as being **time invariant** if the response to a time-translated version of that input signal results in a time-translated version of the output signal for all choices of input signal. For all $x(t) \in S_1$,

$$\text{If } y(t) = T[x(t)], \text{ then } T[x(t-\alpha)] = y(t-\alpha) \tag{5.3}$$

A system that is not time invariant will be called **time varying**.

A system is called **causal** if its output at time t_0 , $y(t_0)$,resulting from input $x(t)$ dose not depend on $x(t)$ for any time $t \ge t_0$, for all t_0 and all inputs $x(t)$. This means that the output of a causal filter cannot precede or anticipate the input signal to produce the output.

A system is called a **random system** if the output depends not only on the input but on the outcome of an underlying random experiment. In many cases its response can be seen to be a function of one or more random variables. If a system is not random, it is called a **deterministic system**.

5.2.1 Linear Time-Invariant Systems

A linear continuous time-invariant filter with input $x(t)$ and output $y(t)$ can be characterized by the following rule of correspondence (the convolution integral):

$$y(t) = T[x(t)] = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \tag{5.4}$$

By making a change of variables the convolution integral can also be written as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{5.5}$$

If the input $x(t)$ is an impulse function $\delta(t)$, then the output $y(t)$, after using the sampling property for $\delta(t)$, equals $h(t)$ is referred to as the impulse response of filter.

The linear time-invariant system specified by (5.4) above will be causal if $h(t) = 0$ for all $t < 0$. For a causal filter the input output equations are easily seen to effect the limits for the integrals above and can be written as

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau \tag{5.6}$$

5.2.2 Linear Time-varying Systems

A certain class of linear time-varying filters can be characterized by an $h(t,\tau)$ which represents the response to an impulse at time τ. The response of a system to an impulse an time τ_0 is not a translated version of the response of the system to an impulse at 0. An example of an $h(t,\tau)$ is shown in Figure 5.2. It is seen that the response to an impulse at time τ_2 is longer and lower than the response of the system to an impulse at time τ_1, which could represent a system that becomes sluggish as time increases. The $h(t,\tau_1)$ and $h(t,\tau_2)$ are just two of the profiles that make up $h(t,\tau)$, The input-output relationship for such filters is given by

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)x(\tau)d\tau \tag{5.7}$$

The linear time varying filter will be defined to be **causal** if $h(t,\tau) = 0$ for $t < \tau$.

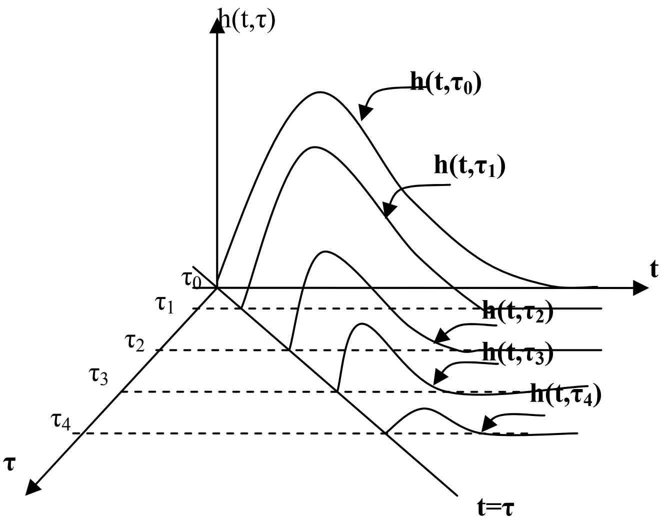

Figure 5.2 An example of the impulse response function h(t,τ) for a linear time varying system

5.3 Continuous Linear Time-Invariant Systems (Random Inputs)

Consider a random process $X(t,e)$ to be the input to a nonrandom linear filter with impulsive response $h(t)$. For every e_i an input waveform $X(t,e_i)$ is specified, and an output waveform $Y(t,e_i)$ can be given by the convolution integral of the realization with the impulse response as

$$Y(t,e_i)=\int_{-\infty}^{\infty}X(t-\tau,e_i)h(\tau)d\tau \tag{5.8}$$

Thus $Y(t,e)$ is a mapping from S, the sample space of the underlying experiment for $X(t)$, to a set of time functions and is, therefore, a random process $Y(t,e)$. For notation purposes the expression above can be written as a function of the input random process $X(t)$ in the following the way:

$$Y(t)=\int_{-\infty}^{\infty}X(t-\tau)h(\tau)d\tau \tag{5.9}$$

The problem of prime importance is the statistical description of the output process $Y(t)$ in terms of the given characterizations of the input process $X(t)$.For example, we might ask what is the mean $\eta_y(t)$ of the output process, and is the knowing of the mean $\eta_x(t)$ of the input process sufficient to determine it? Similar questions may be asked about autocorrelation functions, first-order densities, and nth-order densities. These questions are answered in the following sections.

5.3.1 Mean in-mean out (Linear Time-Invariant Filters)

Taking the expected value of both sides of Eq.(5.9) give the mean of the output random process as

$$\eta_Y(t)=E[Y(t)]=E\left[\int_{-\infty}^{\infty}X(t-\tau)h(\tau)d\tau\right] \tag{5.10}$$

If $E[X(t-\tau)h(\tau)]$ is finitely integrable, it is possible to interchange the expected value operator, which itself is an integral operator, with the integral for the convolution. Interchanging the expected value and integration gives

$$\eta_Y(t)=E[Y(t)]=\int_{-\infty}^{\infty}E[X(t-\tau)h(\tau)d\tau] \tag{5.11}$$

Assuming that the linear filter is not random we can take $h(\tau)$ outside the expected value to obtain

$$\eta_Y(t)=\int_{-\infty}^{\infty}E[X(t-\tau)h(\tau)d\tau] \tag{5.12}$$

Since $E[X(t-\tau)]=\eta_x(t-\tau)$, the following very important result is obtained:

$$\eta_Y(t)=\int_{-\infty}^{\infty}\eta_X(t-\tau)h(\tau)d\tau=\eta_X(t)*h(t) \tag{5.13}$$

When the mean of the input process is a constant, the mean of the output process will be constant as well:

$$\eta_Y(t) = \eta_X \int_{-\infty}^{\infty} h(\tau) d\tau \triangleq \eta_Y \tag{5.14}$$

If the system transfer function is defined as the Fourier transform of the impulse response, the mean of the output process can written as

$$\eta_Y = \eta_X H(j0)$$

$$\text{Where} \quad H(jw) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \tag{5.15}$$

Thus it is seen that the mean of the output process $Y(t)$ can be obtained by multiplying the input mean η_X by the dc gain $H(j0)$.

5.3.2 Autocorrelation in-Autocorrelation out (Linear Time-invariant Filters)

If the input process has an autocorrelation function $R_{XX}(t,u)$, a procedure paralleling the development given in the previous section is used to obtain the output correlation function $R_{YY}(t,u)$ and thus the relationship between the input autocorrelation and the output autocorrelation. Using the definition of the autocorrelation of the output process and (5.9) representing the linear filter, we have

$$\begin{aligned} R_{YY}(t,u) &= E[Y(t)X(u)] \\ &= E\left[\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha \int_{-\infty}^{\infty} X(u-\beta)h(\beta)d\beta\right] \end{aligned} \tag{5.16}$$

Interchanging integrals and expected values, and assuming a deterministic filter, we rewrite (5.16) as

$$\begin{aligned} R_{YY}(t,u) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\alpha)X(u-\beta)]h(\alpha)h(\beta)d\alpha d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t-\alpha,u-\beta)h(\alpha)h(\beta)d\alpha d\beta \end{aligned} \tag{5.17}$$

If the input process is wide sense stationary, its autocorrelation function $R_{XX}(\tau)$ is a function of the time difference only. By letting $t-u=\tau$ and using the fact that $X(t)$ is a wide sense stationary process, we can treat $R_{YY}(t,u)$ as a function of the time difference $t-u$ alone. This we denote by $R_{YY}(\tau)$:

$$\begin{aligned} R_{YY}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\beta-\alpha)h(\alpha)h(\beta)d\alpha d\beta \\ &= h(\tau) * R_{XX}(\tau) * h(-\tau) \\ or \quad R_{YY}(\tau) &= F_\beta^{-1}[H(p)\Phi_{XX}(p)H(-p)] \end{aligned} \tag{5.18}$$

Equation (5.18) expresses the output autocorrelation function $R_{YY}(\tau)$ in terms of the convolution of $h(\tau), R_{XX}(\tau)$ and the time reversed impulse function $h(-\tau)$.By taking the Fourier transform of both sides of (5.18), the relationship between in the input and output power spectral densities can be obtained as

$$S_{YY}(\omega) = H(j\omega)S_{XY}(\omega)H^*(j\omega) \\ = |H(j\omega)|^2 S_{XX}(\omega) \tag{5.19}$$

Example 5.1

Let the random process $X(t)$ be the input to a linear system represented by its impulse response $h(t)$ resulting in the output process $Y(t)$.The input was applied at minus infinite. We have $h(t)$ given by $h(t) = 2e^{-t}\mu(t)$, and that $X(t)$ is a random telegraph process with autocorrelation function $R_{xx}(\tau) = e^{-2|\tau|}$.Find the power spectral density $S_{YY}(\omega)$ for the output process.

Solution

The output power spectral density depends upon the input power spectral density and the magnitude squared of the transfer function .The power spectral density of the input process is determined as

$$S_{XX}(\omega) = F[R_{XX}(\tau)] = \int_{-\infty}^{+\infty} e^{-2|\tau|} e^{-j\omega\tau} d\tau = \frac{4}{\omega^2 + 4}$$

The transfer function $H(jw)$ is obtained from the impulse response as

$$H(j\omega) = F[h(t)] = \int_{-\infty}^{+\infty} e^{-\tau}\mu(t)e^{-j\omega\tau} dt = \frac{1}{j\omega + 1}$$

Therefore from (5.19) the power spectral density of the output can be written as

$$S_{YY}(\omega) = |H(j\omega)|^2 S_{XX}(\omega) \\ = \left|\frac{1}{j\omega+1}\right|^2 \frac{4}{\omega^2+4} = \frac{4}{(\omega^2+1)(\omega^2+4)}$$

The power spectral density could have also been obtained in terms of the Laplace transform as follows:

$$S_{XX}(\omega) = [H(p)\Phi_{XX}(P)H(-p)]|_{p=j\omega}$$

5.3.3 Cross Correlation of the Input and Output

If $X(t)$ represents the input process to a linear time-invariant filter characterized by an impulse response $h(t)$, the output $Y(t)$ can be expressed by the convolution integral given in (5.9). The cross-correlation function for $X(t)$ and $Y(t)$ is computed by taking the expected value of the product of $X(t)$ and $Y(u)$, where $Y(u)$ is expressed by the convolution integral. This results in

$$R_{XY}(t,u) = E[X(t)Y(u)] = E\left[X(t)\int_{-\infty}^{\infty} h(\beta)X(u-\beta)d\beta\right] = \int_{-\infty}^{\infty} h(\beta)E[X(t)X(u-\beta)]d\beta \tag{5.20}$$

Recognizing the expected value to be the autocorrelation function for the input process $X(t)$, the cross-correlation function between the input and output becomes

$$R_{XY}(t,u) = \int_{-\infty}^{\infty} h(\beta)R_{xx}(t,u-\beta)d\beta \tag{5.21}$$

Thus the cross-correlation function between the input and output process can be determined by knowing the input process autocorrelation and the impulse response defining the system.

If the input process is wide sense stationary, the autocorrelation function can be written in terms of the time difference only as

$$R_{XY}(t,u) = \int_{-\infty}^{\infty} h(\beta)R_{xx}(t-u+\beta)d\beta \tag{5.22}$$

Thus the cross-correlation function between input and output processes for a wide sense stationary input process is a function of time difference $\tau = t-u$ only, and it can be written as

$$R_{XY} = \int_{-\infty}^{\infty} h(\beta)R_{XX}(\tau+\beta)d\beta = \int_{-\infty}^{\infty} R_{XX}(\tau-\beta)h(-\beta)d\beta \tag{5.23}$$

The $R_{XY}(\tau)$ is recognized as a convolution integral of the autocorrelation function of the input process and a time-reversed version of the impulse response of the system:

$$R_{XY} = R_{XX}(\tau) * h(-\tau) \quad or \quad R_{XY}(\tau) = \Psi_{\beta}^{-1}[\Phi_{XX}(p)H(-p)] \tag{5.24}$$

Following a similar development for the cross-correlation between $Y(t)$ and $X(t)$, it can be shown that $R_{YX}(\tau)$ is

$$R_{YX}(\tau) = h(\tau) * R_{XX}(\tau) \quad or \quad R_{YX}(\tau) = \Psi_{\beta}^{-1}[H(p)\Phi_{XX}(p)] \tag{5.25}$$

This result can also be obtained by replacing τ by $-\tau$ in (5.24) and using the fact that $R_{XX}(\tau)$ is an even function of τ.

Since the cross-correlation function has been shown to be a function of time difference only, the input autocorrelation function is a function of time difference only , and the output autocorrelation function is a function of time difference only ,the input and output process $X(t)$ and $Y(t)$ are jointly wide sense stationary.

As a nemonic notice that if the right variable is changed from X to Y passing through a system with impulsive response $h(\tau)$ that $R_{XX}(\tau)$ is convolved with a time-reversed impulsive response

$h(-\tau)$ or if a left variable is changed from X to Y passing through a system with impulse response $h(t)$,that $h(\tau)$ is convolved with $R_{XX}(\tau)$.This relationship is shown in Figure 5.3.

Notice that (5.18) can be easily confirmed by using this nemonic as both right and left variables will be changed .This approach can be used for ease in calculating cross-correlation and autocorrelation functions for outputs of linear systems with multiple inputs and multiple outputs.

Example 5.2

A given linear time-invariant causal filter has impulse response $h(t)=e^{-t}\mu(t)$, and its input $X(t)$ is a zero mean wide sense stationary process characterized by its autocorrelation function $R_{XX}(\tau)=1/2e^{-2|\tau|}$,Suppose that the input was applied at time $t=-\infty$,resulting in the output process $Y(t)$. Find the following:

(a) The mean $\eta_Y(t)$ of the output process $Y(t)$.

(b) The cross-correlation function $R_{YX}(\tau)$.

(c) The cross-correlation function $R_{XY}(\tau)$.

(d) The output power spectral density $S_{YY}(w)$.

(e) The output autocorrelation function $R_{YY}(\tau)$

Solution

(a) The mean of the output process can be found by using (5.13) and is determined as

$$\eta_X(t)=\eta_X(t)*h(t)=0*e^{-\tau}\mu(t)=0$$

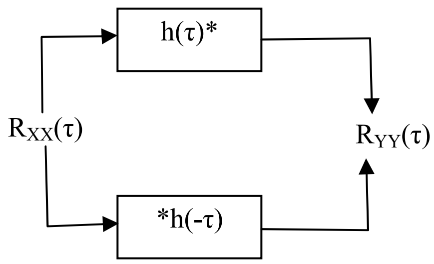

Figure 5.3 Relationships between input ,output ,and cross correlation functions for a linear time invariant system

(b) The cross-correlation function $R_{YX}(\tau)$ between output process $Y(t)$ and input process $X(t)$ can be determined from (5.25) as

$$R_{YX}(\tau)=F_{\beta}^{-1}\left[H(p)\Phi_{XX}(p)\right]=F_{\beta}^{-1}\left\{F_u\left[e^{-t}\mu(t)\right]\cdot F_{\beta}\left(\frac{1}{2}e^{-2|\tau|}\right)\right\}$$
$$=F_{\beta}^{-1}\left[\frac{1}{p+1}\cdot\frac{1}{2}\left(\frac{1}{p+2}+\frac{1}{-p+2}\right)\right]$$
$$=F_{\beta}^{-1}\left[\frac{2}{(p+1)(p+2)(-p+2)}\right]$$

To obtain the inverse transform, a partial fraction expansion for the term in the brackets is performed followed by the inverse bilateral Laplace transform to get

$$R_{YX}(\tau)=F_{\beta}^{-1}\left[\frac{-2/3}{(p+1)}+\frac{-1/2}{(p+2)}+\frac{1/6}{(-p+2)}\right]$$
$$=\left(-\frac{2}{3}e^{-\tau}-\frac{1}{2}e^{-2\tau}\right)\mu(t)+\frac{1}{6}e^{2\tau}\mu(-\tau)$$

(c) To find $R_{XY}(\tau)$,we can use the fact that $R_{XY}(\tau)=R_{YX}(-\tau)$ to obtain

$$R_{XY}(\tau)=R_{YX}(-\tau)=\left(-\frac{2}{3}e^{\tau}-\frac{1}{2}e^{2\tau}\right)\mu(-\tau)+\frac{1}{6}e^{-2\tau}\mu(t)$$

(d) To find $S_{YY}(\omega)$, the power spectral density $S_{XX}(\omega)$ of the input must first be found. From the definition of power spectral density, we have

$$S_{XX}(\omega)=F\left[R_{XX}(\tau)\right]=F\left[\frac{1}{2}e^{-2|\tau|}\right]=\frac{2}{\omega^2+4}$$

From (5.19) the power spectral density $S_{YY}(\omega)$ of the output process $Y(t)$ in terms of the input power spectral density is

$$S_{YY}(\omega)=|H(j\omega)|^2\,S_{XX}(\omega)=\left|\frac{1}{j\omega+1}\right|^2\cdot\frac{2}{(\omega^2+4)}$$
$$=\frac{2}{(\omega^2+1)(\omega^2+4)}$$

(e) The autocorrelation function $R_{XX}(\tau)$ can be found from Eq. (5.18) as the convolution or as the inverse transform. The easiest way is to use the Bilateral inverse as follows:

$$R_{YY}(\tau)=F_{\beta}^{-1}\left[H(p)\Phi_{XX}(p)H(-p)\right]$$
$$=F_{\beta}^{-1}\left\{\frac{1}{(p+1)}\frac{2}{(p+2)(-p+2)}\frac{1}{(-p+1)}\right\}$$

A partial fraction expansion is performed for the function in {}, and then an inverse bilateral Laplace transform to give the output autocorrelation function as

$$
\begin{aligned}
R_{YY}(\tau) &= \Psi_{\beta}^{-1}\left[\frac{1/3}{(p+1)}+\frac{1/3}{(-p+1)}+\frac{-1/3}{(p+2)}+\frac{-1/3}{(-p+2)}\right] \\
&= \frac{1}{3}e^{-\tau}\mu(\tau)+\frac{1}{3}e^{\tau}\mu(-\tau)-\frac{1}{6}e^{-2\tau}\mu(\tau)-\frac{1}{6}e^{2\tau}\mu(-\tau) \\
&= \frac{1}{3}e^{-|\tau|}-\frac{1}{6}e^{-2|\tau|}
\end{aligned}
$$

5.3.4 nth-Order Densities in-nth-Order Densities Out

To arrive at the output densities, it is worthwhile to discuss the first-order density of the output. Rewriting the input-output relationship (5.4) as

$$
Y(t)=\int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau \tag{5.26}
$$

it is easily seen that the output random variable $Y(t)=Y_t$ defined at any time t not only depends on the input $X(t)$ at time t but on $X(t)$ for all $-\infty < t < \infty$. Therefore, in general, the determination of the output first-order density requires a complete characterization of the input process. Although the second-order density is just as complicated, it should be noted that the mean and autocorrelation of the output can be determined,(5.13) and (5.18), without knowing the first-or second-order densities of the input process provided the mean and autocorrelation of the input process are known .This is an important result , and it demonstrates that for linear systems the mean and autocorrelation of the input process are sufficient for the same partial characterizations of the output .Later we will see for the special case where the input process is Gaussian that the first ,second ,and nth order densities of the output process $Y(t)$ can be easily determined.

5.3.5 Stationary of the Output process

For a wide sense stationary process as input to a linear time-invariant system it has been shown, (5.14) and (5.18), that the mean of the output process is a constant and not a function of time and that the autocorrelation function depends only on the time difference, t-u or τ. These results allow the following statement to be made:

If the input to a linear time-invariant filter is a wide sense stationary random process, the output is also a wide sense stationary random process.

We also might ask if the input process is stationary of order one or two, what can we say about the output? From the argument given in the previous section, there is no reason to believe that the first –order density would be independent of time or that the second-order densities would be equal for each pair of times t_i and t_j。Thus an output process is not necessarily stationary of order two just because the input is.

5.4 Continuous Time-Varying Systems with Random Input

If the input to a linear time-varying system, characterized by its $h(t,\tau)$, is a random process $X(t)$ with known partial characterizations, the characterizations of the output process are desired. The partial characteristics of the output process consist of mean, autocorrelation, first-and higher-order densities along with various stationarity properties.

5.4.1 Mean in-Mean out (Linear Time-Varying Filter)

Taking the expected value of both side of (5.7), the input-output equation for a linear time varying filter with impulse response $h(t,\tau)$, the mean of the output random process becomes

$$\eta_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(t,\tau)X(\tau)d\tau\right] \qquad (5.27)$$

Interchanging the expected value and the integration operations give

$$\eta_Y(t) = E[Y(t)] = \int_{-\infty}^{\infty} E[h(t,\tau)X(\tau)]d\tau \qquad (5.28)$$

We assume that the linear filter is not random, so $h(t,\tau)$ can be taken outside the expected value to yield

$$\eta_Y(t) = \int_{-\infty}^{\infty} h(t,\tau)E[X(\tau)]d\tau \qquad (5.29)$$

Now, recognizing that $E[X(\tau)] = \eta_X(\tau)$, we write the mean of the output process as

$$\eta_Y(t) = \int_{-\infty}^{\infty} h(t,\tau)\eta_X(\tau)d\tau \qquad (5.30)$$

When the input process is stationary in mean, $\eta_X(t)$ is constant, the mean of the output process is

$$\eta_Y(t) = \eta_X \int_{-\infty}^{\infty} h(t,\tau)d\tau \qquad (5.31)$$

From this result it is seen that the mean $\eta_Y(t)$ of the output process $Y(t)$ will be a function of time, since the integral of $h(t,\tau)$ with respect to τ is a function of time. Thus the output process is not stationary in the mean even if the input process is stationary in mean.

5.4.2 Autocorrelation in-Autocorrelation out (Linear Time-Varying Filter)

Assume that the input process has an autocorrelation function $R_{XX}(t,u)$, and the output a correlation function $R_{YY}(t,u)$; the relationship between the input autocorrelation and the output autocorrelation is now determined. Using the definition of the autocorrelation of the output process and (5.7), representing the linear filter, we write the output autocorrelation as

$$\begin{aligned} R_{YY}(t,u) &= E[Y(t)Y[u]] \\ &= E\left[\int_{-\infty}^{\infty} h(t,\alpha)X(\alpha)d\alpha \int_{-\infty}^{\infty} h(u,\beta)X(\beta)d\beta\right] \end{aligned} \tag{5.32}$$

After interchanging the integrals and expected values, and assuming a deterministic filter, we rewrite (5.32) as

$$\begin{aligned} R_{YY}(t,u) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,\alpha)h(u,\beta)E[X(\alpha)X(\beta)]d\alpha d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,\alpha)h(u,\beta)R_{XX}(\alpha,\beta)d\alpha d\beta \end{aligned} \tag{5.33}$$

If the input process is wide sense stationary, its autocorrelation function $R_{XX}(\tau)$ is a function of the time difference only. Thus the output process autocorrelation function, $R_{YY}(t,u)$, can be written as

$$R_{YY}(t,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,\alpha)h(u,\beta)R_{XX}(\alpha-\beta)d\alpha d\beta \tag{5.34}$$

The equation above expresses the output autocorrelation function $R_{YY}(t,u)$ in terms of the autocorrelation function of the input process, $R_{XX}(\tau)$, but since $h(t,\alpha)h(u,\beta)$ is not necessarily a function of $t-u$, the double integral will in general not be a function of $t-u$. Thus the output process will not be stationary in autocorrelation even if the input process is stationary in autocorrelation.

5.4.3 Cross Correlation of the Input and Output (Linear Time-Varying Filter)

For an input random process $X(t)$ a linear time-varying filter produces an output process $Y(t)$.These two processes are in some ways statistically related .The cross-correlation function provides a statistical measure of the correlation that exists between the processes .For a time-varying linear filter characterized by its $h(t,\tau)$, the output $Y(t)$ is given by (5.7). Thus the cross correlation can be written as

$$\begin{aligned} R_{XY}(t,u) &= E[X(t)Y(u)] = E\left[X(u)\int_{-\infty}^{\infty} h(u,\beta)X(\beta)d\beta\right] \\ &= \int_{-\infty}^{\infty} h(u,\beta)E[X(t)X(\beta)]d\beta \end{aligned} \tag{5.35}$$

Recognizing the autocorrelation function for the input process $X(t)$, the cross-correlation function between input and output becomes

$$R_{XY}(t,u) = \int_{-\infty}^{\infty} h(u,\beta)R_{XX}(t,\beta)d\beta \tag{5.36}$$

Thus the cross-correlation function between input and output can be determined by knowing only the input autocorrelation function $R_{XX}(t,u)$ and the system' s impulse response $h(t,\tau)$.

If $X(t)$, the input process, is wide sense stationary, the $R_{XX}(t,\beta)$ can be written in terms of the time difference. The cross-correlation function of (5.36) becomes

$$R_{XY}(t,u)=\int_{-\infty}^{\infty} h(u,\beta)R_{XX}(t-\beta)d\beta \tag{5.37}$$

The cross-correlation function can be written in a slightly different form by making a change of variables $t-\beta=\alpha$ in (5.37), resulting in

$$R_{XY}(t,u)=\int_{-\infty}^{\infty} h(u,t-\alpha)R_{XX}(\beta)d\alpha \tag{5.38}$$

The cross correlation between $Y(t)$ and $X(u)$ can be similarly found to be

$$\begin{aligned} R_{YX}(t,u) &= E[Y(t)X(t)] \\ &= \int_{-\infty}^{\infty} h(t,\beta)R_{XX}(u,\beta)d\beta \end{aligned} \tag{5.39}$$

Clearly, even if the input process is wide sense stationary, the input and output processes are not jointly wide sense stationary for a linear time-varying system.

5.4.4 nth-Order Densities in-nth-Order Densities out (Linear Time-Varying Filter)

By a similar argument to that given in section 5.3.4 for a linear time-invariant system, the first- and higher-order densities of the output of a linear time-varying filter cannot be determined knowing only the same order characterizations of the input. In general, a total characterization of the input process would be required to even get the first-order densities.

5.4.5 Stationarity of the Output Process (Linear Time-varying Filter)

From (5.31), it is seen that the mean of the output process will be a function of time even if the input process is stationary in its mean, and from (5.34), that the auto-correlation function does not depend only on the time difference $t-u$ even when the input process is stationary in autocorrelation. From these results the following statement can be made.

If the input to a linear time-varying filter is a wide sense stationary random process, the output random process will not necessarily be stationary in any sense.

5.5 Discrete Time-Invariant Linear Systems With Random Inputs

In the study of linear continuous time systems with random inputs, relationships were obtained for the output mean and the autocorrelation function in terms of the input autocorrelation and mean. Similar formulas will now be developed for discrete time linear shift-invariant systems characterized by the following input-output relationships (discrete convolution sum).

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]or \\ = \sum_{k=-\infty}^{\infty} h[n-k]x[k] \tag{5.40}$$

The derivations assume that the input process was applied at time $-\infty$, and results are interpreted as being in steady state. In (5.40) the $h[n]$ is the unit-sample response. If $h[n]$ is zero for all $n<0$, the discrete system is causal, and Eq.(5.40) can be written as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \tag{5.41}$$

5.5.1 Mean in-Mean Out

The mean of the output random sequence is seen from (5.40) for a nonrandom $h[n]$ to be

$$E[Y[n]] = \sum_{k=-\infty}^{\infty} h[k]E[x[k-n]] \\ = h[n] * \eta_X[n] \tag{5.42}$$

In other words, the output mean is the convolution of the input mean with the unit-sample response $h[n]$, similar to the result for continuous time systems.

If the input process is wide sense stationary with mean $E[X(n)] = \eta_X$, for all n, then $E[Y[n]]$ can be written as

$$E[Y[n]] = \eta_X \sum_{k=-\infty}^{\infty} h[k] \\ = \eta_X Z(h[n])|_{Z=1} \equiv \eta_X H(1) \tag{5.43}$$

where $H(1) = H(Z)|_{Z=1}$ and $H(Z)$ is the Z-transform of $h[n]$.

This result compares with that for the continuous case, the difference being $z=1$ in $H(Z)$ instead of $p=0$ in the $H(p)$.

5.5.2 Autocorrelation in-Autocorrelation Function Out

In a straightforward manner, the output autocorrelation function $R_{YY}[k_1,k_2]$ defined by

$$R_{XX}[k_1,k_2] = E[Y[k_1]Y[k_2]] \tag{5.44}$$

can be found using (5.40). Assuming that the linear filter is not random, we can take $h[n]$ and $h[m]$ outside the expected value to yield

$$R_{YY}[k_1,k_2]=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}h[n]h[m]R_{XX}[k_1-n,k_2-m] \tag{5.45}$$

For a wide sense stationary input sequence $X[n]$, the output autocorrelation function $R_{YY}[k_1,k_2]$ can be written as a function of $k_1-k_2=k$:

$$R_{YY}[k]=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}h[n]h[m]R_{XX}[k-n+m] \tag{5.46}$$

Equation (5.46) is the discrete form of a double convolution sum and can be written more compactly as

$$\begin{aligned}R_{YY}[k]&=h[k]*R_{XX}[k]*h[-k]\\&=Z^{-1}\left[H(z)\Psi_{XX}(z)H\left(z^{-1}\right)\right]\end{aligned} \tag{5.47}$$

and the Z-transform of (5.47) can be expressed as

$$\Psi_{YY}(z)=H(z)\Psi_{XX}(z)H\left(z^{-1}\right) \tag{5.48}$$

where $\Psi_{XX}(z)$, $\Psi_{YY}(z)$, and $H(z)$ represent the Z-transforms of $R_{XX}[k]$, $R_{YY}[k]$, and $h[k]$, respectively.

The **power spectral density** for a random sequence $X[n]$ is defined as

$$\psi_{XX}(\omega)\triangleq\Psi_{XX}(z)\Big|_{z=e^{j\omega}} \tag{5.49}$$

Therefore from (5.48) and (5.49) it is possible to write the power spectral density of the output process in terms of the power spectral density of the input process as

$$\begin{aligned}\psi_{YY}(\omega)&=H(z)\Psi_{XX}(z)H\left(z^{-1}\right)\Big|_{z=e^{j\omega}}\\&=\left|H\left(e^{j\omega}\right)\right|^2\psi_{XX}(\omega)\end{aligned} \tag{5.50}$$

The results above can be compared to those determined for a continuous random input process $X(t)$ and a continuous system characterized by $h(t)$ with output process $Y(t)$. The formulas are analogous ,with the discrete time convolution replacing the continuous time convolution, the Z-transform replacing the Laplace transform, and the $e^{j\omega}$ replacing the $j\omega$.

5.5.3 Cross-correlation Functions

As in the continuous time case the corresponding cross-correlation functions of input and output random processes for discrete time can be determined as convolutions:

$$\begin{aligned}&R_{YX}[k]=h[k]*R_{XX}[k] \quad or \quad R_{YX}[k]=Z^{-1}\left[H(z)\Psi_{XX}(z)\right]\\&R_{XY}[k]=R_{XX}[k]*h[-k] \quad or \quad R_{YX}[k]=Z^{-1}\left[\Psi_{XX}(z)H\left(z^{-1}\right)\right]\end{aligned} \tag{5.51}$$

Their Z-transform equivalents are determined as

$$\begin{aligned}\Psi_{YX}(z) &= Z(R_{YX}[n]) = H(z)\Psi_{XX}(z) \\ \Psi_{XY}(z) &= Z(R_{XY}[k]) = \Psi_{XX}(z)H(z^{-1})\end{aligned} \tag{5.52}$$

5.5.4 nth-Order Densities

By arguments similar to those for the continuous case the first-,second- , and $nth-$ order densities can be examined .Since the output $Y[n]$ at any given n is a function of the input process $X[k]$ random variables for all k form $-\infty$ to ∞, the first-order density cannot be determined without knowing the nth-order joint densities for all input random variables $X[k]$. Thus a first-order characterization of the input process is insufficient information to obtain the first-order densities for the output process. Similarly the second- and *n*th-order densities can not be determined unless the input process is totally characterized. However, if the input process is a Gaussian random process and characterized by its mean and autocorrelation function, it is totally characterized. Thus the corresponding first-, second-, and all higher-order densities can be determined, totally characterizing the output.

5.5.5 Stationarity

If the input process is stationary in mean, stationary in autocorrelation, or wide sense stationary the output process is correspondingly stationary in mean, stationary in autocorrelation, and wide sense stationary. This is verified by the functional relationships for the output mean in terms of the input mean given in (5.43) and the output autocorrelation in terms of the input autocorrelation function given in (5.47) .For a discrete time-invariant linear system specified by its impulse response $h[n]$, the mean and autocorrelation of the input are sufficient to determine the mean and output autocorrelation function ,so only a partial characterization of the input process is required. However, input stationarity of order 1 does not imply output stationrity of order 1, nor does input stationarity of order *n* imply output stationarity of order *n*.

5.5.6 MA, AR, and ARMA Random Processes

Give a discrete time linear system with input $x[n]$ and output $y[n]$ characterized by a difference equation as

$$y[n] = -\sum_{k=1}^{p} a_k y[n-k] + \sum_{k=0}^{q} b_k x[n-k] \qquad \text{for all } n \tag{5.53}$$

The system function or transfer function can be obtained by first taking an inverse Z-transform of both sides of Eq. (5.53), ignoring initial conditions to obtain the following:

$$Y(z) = -\sum_{k=1}^{p} a_k z^{-k} Y(z) + \sum_{k=0}^{q} b_k z^{-k} X(z) \tag{5.54}$$

We then rearrange (5.54) to obtain $H(z)$ as $Y(z)$ over $X(z)$:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{q}b_k z^{-k}}{1+\sum_{k=1}^{q}a_k z^{-k}} \tag{5.55}$$

Thus the transfer function $H(z)$ is a rational expression in z^{-1}, where q is assumed less than p. The impulse response of such a system is obtained by taking the inverse Z-transform of $H(z)$ which is written as $h[n]=Z^{-1}(H(z))$.If the input process is a white random sequence, then the output process is an ARMA (p,q) process.

If all the $b_k=0$ for $k=1,2,\cdots,q$, then the transfer function can be written in terms of an **all pole model** as

$$H(z)=\frac{b_0}{1+\sum_{k=1}^{p}a_k z^{-k}} \tag{5.56}$$

If the input to a system represented by the $H(z)$ above is a white random sequence then the output process is an autoregressive process of order p, $AR(p)$.

If all the $a_k=0$ for $k=1,2,\cdots,p$, then the transfer function can be written in terms of the **all zero model** as

$$H(z)=\sum_{k=0}^{q}b_k z^{-k} \tag{5.57}$$

and if the input process is a white random sequence, the output process is said to be a moving average process, MA(q).

Example 5.3

Find the steady state mean, autocorrelation and power spectral density for the $MA(1)$, processes where the input process $X[n]$ has a zero mean for all n and an autocorrelation function given by $E\{X[m+k]x[m]\}=\sigma^2\delta[k]$ for all m.

Solution

The mean of the output process is obtained from $Eq.(5.42)$ as the convolution of the impulse response of the system with the mean of the input process. Since the mean of the input process is zero for all n, the convolution is zero for all n, all thus $E\{Y[n]\}=0$ for all n.

From $Eq.(5.47)$ the output autocorrelation function of the output process $Y[n]$ is a convolution of the impulse response, the input autocorrelation function, and the time-reversed impulse response. It can also be determined as the inverse Z-transform of the product of their transforms.

$$R_{YY}[k] = h[k] * R_{XX}[k] * h[-k] = Z^{-1}\left[H(z)\Psi_{XX}(z)H\left(z^{-1}\right)\right] \tag{5.58}$$

Where $H(z)$ from (5.57) is $H(z) = b_0 + b_1 z^{-1}$ and $\Psi_{XX}[z] = Z\left(\sigma^2\delta[m]\right) = \sigma^2$ .The autocorrelation function from (5.58) is seen to be

$$\begin{aligned} R_{YY}[k] &= Z^{-1}\left\{\left(b_0 + b_1 z^{-1}\right)\sigma^2\left(b_0 + b_1 z\right)\right\} \\ &= \sigma^2 Z^{-1}\left\{b_0 b_1 z + \left(b_0^2 + b_1^2\right) + b_1 b_0 z^{-1}\right\} \\ &= \sigma^2 b_0 b_1 \delta[k+1] + \sigma^2\left(b_0^2 + b_1^2\right)\delta[k] + \sigma^2 b_1 b_0 \delta[k-1] \end{aligned} \tag{5.59}$$

The values for $k = 0$ and $k = 1$ check with the results determined in Example 4.6, as do the zero values for $k > 1$ and $k < -1$.

The power spectral density $\phi_{YY}(w)$ is obtained from $\Psi_{YY}(z)$ using (5.50) as follows:

$$\begin{aligned} \psi_{YY}(w) &= \Psi_{YY}(z)|_{z=e^{jw}} = H(z)\Psi_{XX}(z)H\left(z^{-1}\right)|_{z=e^{jw}} \\ &= \left\{\left(b_0 + b_1 z^{-1}\right)\sigma^2\left(b_0 + b_1 z\right)\right\}|_{z=e^{jw}} \\ &= \left(b_0^2 + b_1^2 + 2b_0 b_1 \cos(w)\right)\sigma^2 \end{aligned} \tag{5.60}$$

Similarly the autocorrelation function and power spectral density for the $MA(q)$ process are easily determined using the power transfer functions in (5.57) and (5.50). The following example finds the power spectral density for the $MA(q)$ process:

Example 5.4

Find the power spectral density for $MA(q)$ process.

Solution

The power spectral density for the $MA(q)$ process uses the transfer function given in (5.57) and is determined as follows:

$$\begin{aligned} \psi_{YY}(w) &= \Psi_{YY}(z)|_{z=e^{jw}} = H(z)\Psi_{XX}(z)H\left(z^{-1}\right)|_{z=e^{jw}} \\ &= \left\{\sum_{k=0}^{q} b_k z^{-k}\left(\sigma^2\right)\sum_{k=0}^{q} b_k z^k\right\}|_{z=e^{iw}} \\ &= \left|\sum_{k=0}^{q} b_k e^{-ikw}\right|^2 \sigma^2 \end{aligned} \tag{5.61}$$

Example5.5

Find the steady state mean, autocorrelation, and power spectral density for the $AR(1)$, processes where the input process $X[n]$ has a zero mean for all n and an autocorrelation function given by $E\{X[m+k]x[m]\} = \sigma^2\delta[k]$ for all m.

Solution

The mean of the output process is given from $Eq.(5.42)$ as the convolution of the impulse response of the system with the mean of the input process. Since the mean of the input process is zero for all n, the convolution is zero for all n, and thus $E\{X[n]\}=0$ for all n.

Using $Eq.(5.47)$ where $H(z)$ from (5.56) is $H(z)=b_0/\left(1+a_1z^{-1}\right)$ and $\Psi_{XX}[z]=Z\left(\sigma^2\delta[m]\right)=\sigma^2$, the autocorrelation function $R_{YY}[k]$ is seen to be

$$\begin{aligned} R_{XX}[k] &= Z^{-1}\left\{H(z)\Psi_{XX}(z)H\left(z^{-1}\right)\right\} \\ &= Z^{-1}\left\{\frac{b_0}{\left(1+a_1z^{-1}\right)}\sigma^2\frac{b_0}{\left(1+a_1z^{-1}\right)}\right\} \\ &= \frac{b_0^2\sigma^2}{a_1}Z^{-1}\left\{\frac{z}{(z+a_1)(z+1/a_1)}\right\} \end{aligned} \tag{5.62}$$

Expanding the rational expression in terms of a partial fraction expansion gives

$$\begin{aligned} R_{YY}[k] &= \frac{b_0^2\sigma^2}{a_1}Z^{-1}\left\{z\left(\frac{A_1}{(z+a_1)}-\frac{A_1}{(z+1/a_1)}\right)\right\} \\ where \quad A_1 &= \frac{a_1}{1-a_1^2} \end{aligned} \tag{5.63}$$

After taking the inverse Z-transform and recognizing the first term as a positive time sequence and the second as a negative time sequence, we find the desired autocorrelation function $R_{YY}[k]$ to be

$$R_{YY}[k]=\frac{b_0^2\sigma^2}{1-a_1^2}\left\{(-a_1)^{-k}u[k]+\left(\frac{-1}{a_1}\right)^{-k}u[-k-1]\right\} \tag{5.64}$$

Evaluating $R_{YY}[k]$ for $k\geq 0$ gives

$$R_{YY}[k]=\frac{b_0^2\sigma^2}{1-a_1^2}(-a_1)^k \tag{5.65}$$

Which checks with the result determined in (4.179) by a time domain method.

The power spectral density can be determined directly from this autocorrelation by taking the Z-transform and evaluating at $z=e^{iw}$, or directly from (5.50) which gives $\psi_{YY}(w)$ as

$$\begin{aligned} \psi_{YY}(w) &= \Psi_{YY}(z)|_{z=e^{iw}} = H(z)\Psi_{XX}(z)H\left(z^{-1}\right)|_{z=e^{jw}} \\ &= \left\{\frac{b_0}{\left(1+a_1z^{-1}\right)}\sigma^2\frac{b_0}{\left(1+a_1z^{-1}\right)}\right\}|_{z=e^{jw}} \\ &= b_0^2\sigma^2\left\{\frac{1}{\left(1+a_1e^{-jw}\right)\left(1+a_1e^{jw}\right)}\right\} \end{aligned} \tag{5.66}$$

Multiplying the terms in the denominator and simplifying the power spectral density reduces to

$$\psi_{YY}(w)=\frac{b_0^2\sigma^2}{1+a_1^2+2a_1\cos(w)} \tag{5.67}$$

Example 5.6

Find the steady state power spectral density for the $AR(p)$ process where the input process $X[n]$ has a zero mean for all n and an autocorrelation function given by $E\{X[m+k]X[m]\}=\sigma^2\delta[k]$ for all m.

Solution

The transfer function for the general $AR(p)$ process from $Eq.(5.56)$ is

$$H(z)=\frac{b_0}{1+\sum_{k=1}^{p}a_k z^{-k}} \tag{5.68}$$

Thus the power spectral density can be determined as

$$\begin{aligned}\psi_{YY}(w)&=\Psi_{YY}(z)|_{z=e^{jw}}=H(z)\Psi_{XX}(z)H(z^{-1})|_{z=e^{jw}}\\&=\left\{\frac{b_0}{1+\sum_{k=1}^{p}a_k z^{-k}}\sigma^2\frac{b_0}{1+\sum_{k=1}^{p}a_k z^{k}}\right\}|_{z=e^{jw}}\\&=\frac{b_0^2\sigma^2}{|1+\sum_{k=1}^{p}a_k e^{-jkw}|^2}\end{aligned} \tag{5.69}$$

Example 5.7

Find the steady state mean, autocorrelation, and power spectral density for the $ARMA(1,1)$, processes where the input process $X[n]$ has a zero mean for all n and an autocorrelation function given by $E\{X[m+k]X[m]\}=\sigma^2\delta[k]$ for all m.

Solution

Since the mean of the input process is zero the mean of the output process will also be zero. The transfer function for the $ARMA(1,1)$ process from $Eq.(5.55)$ is

$$H(z)=\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}} \tag{5.70}$$

Thus the autocorrelation function in the steady state can be found from $Eq.(5.64)$ by using the inverse Z-transform as follows:

$$
\begin{aligned}
R_{YY}[k] &= Z^{-1}\left\{H(z)\Psi_{XX}(z)H\left(z^{-1}\right)\right\} \\
&= Z^{-1}\left\{\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}}\sigma^2\frac{b_0+b_1z}{1+a_1z}\right\} \\
&= \frac{\sigma^2}{a_1}\left\{\frac{b_0b_1z^2+\left(b_0^2+b_1^2\right)z+b_0b_1}{(z+a_1)(z+1/a_1)}\right\}
\end{aligned}
\tag{5.71}
$$

Expanding the term in brackets, divided by z, and expanding into partial fractions gives

$$
R_{YY}[k] = \frac{\sigma^2}{a_1}Z^{-1}\left\{z\left[\frac{A}{z}+\frac{B}{z+a_1}+\frac{C}{z+1/a_1}\right]\right\} \tag{5.72}
$$

Where A, B, and C are determined by partial fraction expansion method to yield

$$
\begin{aligned}
A &= b_0b_1 \\
B &= \frac{b_0b_1a_1^2-a_1b_0^2-a_1b_1^2+b_0b_1}{a_1^2-1} \\
C &= -\frac{b_0b_1a_1^2-a_1b_0^2-a_1b_1^2+b_0b_1}{a_1^2-1}
\end{aligned}
\tag{5.73}
$$

Multiplying out the z and taking the inverse Z-transform shown in $Eq.(5.71)$, we have the autocorrelation function $R_{YY}[k]$ for $ARMA(1,1)$ process as

$$
R_{YY}[k] = \frac{\sigma^2}{a_1}\left\{A\delta[k]+B(-a_1)^k\mu[k]-C\left(-a_1^{-1}\right)^{-k-1}\mu[-k-1]\right\} \tag{5.74}
$$

Thus the $\delta(k)$ term gives a contribution at $k=0$ only, while away from the origin the autocorrelation function is a power of $(-a_1)$ for positive k with coefficient $\sigma A/a_1$ as indicated by the second term and the third term is the autocorrelation function for time index less than zero. If we evaluate the autocorrelation at zero, we must add contributions of both of the first terms:

$$
R_{YY}[0] = \frac{\sigma^2(A+B)}{a_1} \tag{5.75}
$$

Substituting the A and B from (5.73) and simplifying gives the $R_{YY}[0]$ as

$$
R_{YY}[0] = \frac{\left(b_0^2+b_1^2-2b_0b_1a_1\right)\sigma^2}{1-a_1^2} \tag{5.76}
$$

The formula above for $R_{YY}[0]$ verifies the result determined in (4.209). After evaluating (5.74) for $k>0$ using the B of (5.73), the $R_{YY}[k]$ can be simplified to give

$$
\begin{aligned}
R_{YY}[k] &= \frac{\sigma^2}{a_1}\left\{B(-a_1)^k\mu[k]\right\} \\
&= \frac{\sigma^2\left(a_1^2b_0b_1-a_1b_0^2-a_1b_1^2+b_0b_1\right)}{1-a_1^2}(-a_1)^{k-1}
\end{aligned}
\tag{5.77}
$$

This result verifies that determined in the steady state development given in Chapter 4, $Eq.(4.209)$ for $k=1$ and (4.211) for $k>1$, It is also easily shown that $R_{YY}[-k]=R_{YY}[k]$ by evaluating (5.74) for negative k.

The power spectral density can be determined directly from this autocorrelation by taking the Z-transform and evaluating at $z=e^{jw}$ or determined directly from (5.50), which gives $\phi_{YY}(w)$ as

$$\begin{aligned}\psi_{YY}(w)&=\Psi_{YY}(z)|_{z=e^{jw}}=H(z)\Psi_{XX}(z)H(z^{-1})|_{z=e^{jw}}\\&=\left\{\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}}\sigma^2\frac{b_0+b_1z^1}{1+a_1z}\right\}|_{z=e^{jw}}\\&=b_0^2\sigma^2\left\{\frac{(b_0+b_1e^{-jw})(b_0+b_1e^{jw})}{(1+a_1e^{-jw})(1+a_1e^{jw})}\right\}\end{aligned}\tag{5.78}$$

Multiplying out the terms in the numerator and denominator and simplifying the power spectral density reduces to

$$\psi_{YY}(w)=\frac{b_0^2+b_1^2+2b_0b_1\cos(w)}{1+a_1^2+2a_1\cos(w)}b_0^2\sigma^2\tag{5.79}$$

Example 5.8

Find the steady state power spectral density for the $ARMA(p,q)$ process where the input process $X[n]$ has a zero mean for all n and an autocorrelation function given by $E\{X[m+k]X[m]\}=\sigma^2\delta[k]$ for all m.

Solution

The power spectral density for the $ARMA(p,q)$ is easily found by substituting the transfer function for the system $H(z)$ given in $Eq.(5.55)$ into $Eq.(5.50)$ to get

$$\begin{aligned}\psi_{YY}(w)&=\Psi_{YY}(z)|_{z=e^{jw}}=H(z)\Psi_{XX}(z)H(z^{-1})|_{z=e^{jw}}\\&=\left\{\frac{\sum_{k=0}^{q}b_kz^{-k}}{1+\sum_{k=0}^{p}a_kz^{-k}}\sigma^2\frac{\sum_{k=0}^{q}b_kz^{k}}{1+\sum_{k=0}^{p}a_kz^{k}}\right\}|_{z=e^{jw}}\\&=\frac{\left|\sum_{k=0}^{q}b_ke^{-jkw}\right|^2}{\left|1+\sum_{k=0}^{p}a_ke^{-jkw}\right|^2}b_0^2\sigma^2\end{aligned}\tag{5.80}$$

5.6 Discrete Time-Varying Linear Systems with Random Inputs

In the study of linear continuous time systems with random inputs relationships were obtained for the output mean and the autocorrelation function in terms of the input autocorrelation and mean .Similar formulas will now be developed for discrete time-varying linear systems that can be characterized by the following input-output relationship:

$$y[n]=\sum_{m=-\infty}^{\infty}h[n,m]x[m] \tag{5.81}$$

where $h[n,m]$ is the response of the unit sample impulse at time *m*. If $h[n,m]$ is zero for all $m>n$, the discrete time-varying systems is causal, and Eq.(5.81) can be rewritten as

$$y[n]=\sum_{m=-\infty}^{n}h[n,m]x[m] \tag{5.82}$$

5.6.1 Mean in-Mean out (Time-Varying Discrete Time System)

The mean of the output random sequence is seen from (5.81) for a nonrandom $h[m,n]$ to be

$$\begin{aligned} E[Y[n]] &= \sum_{m=-\infty}^{\infty}h[n,m]E[X[m]] \\ &= \sum_{m=-\infty}^{\infty}h[n,m]\eta_X[m] \end{aligned} \tag{5.83}$$

In other words, the output mean is the result of taking the input mean through the system similar to the result for continuous time-varying systems.

If the input process is stationary in the mean, $E[X[m]]=\eta_X$, for all *m*, then $E[Y[n]]$, from (5.83) can be written as

$$\begin{aligned} E[Y[n]] &= \eta_X\sum_{m=-\infty}^{\infty}h[n,m] \\ &= \eta_X Z_m(h[n,m])\big|_{z=1} \equiv \eta_X H(n,1) \end{aligned} \tag{5.84}$$

This result can be compared with that determined for the continuous time-varying case.

5.6.2 Autocorrelation in-Autocorrelation Function out (Time-Varying Discrete Time System)

The definition of the output autocorrelation function $R_{YY}[k_1,k_2]$ is given by

$$R_{YY}[k_1,k_2]=E[Y[k_1]Y[k_2]] \tag{5.85}$$

We substitute $Y(k_1)$ and $Y(k_2)$ from (5.81) into (5.85), and assuming that the linear filter is not random, we now have

$$R_{YY}[k_1,k_2]=\sum_{m=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}h[k_1,m]h[k_2,n]R_{XX}[m,n] \tag{5.86}$$

For a wide sense stationary input random sequence $X[n]$, the output autocorrelation function $R_{YY}[m,n]$ can be written as

$$R_{YY}[k_1,k_2]=\sum_{m=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}h[k_1,n]h[k_2,m]R_{XX}[n-m] \tag{5.87}$$

There is no reason to believe that this double sum will be just a function of the difference between k_1 and k_2.Even though the input process is wide sense stationary the output of a linear discrete time-varying system will not necessarily be wide sense stationary.

5.6.3 Cross-correlation Functions (Time-Varying Discrete Time System)

Using $Y(k_1)$ and $Y(k_2)$ from (5.81) in the definitions of the cross-correlation functions between the input and output process, we can show the $R_{YX}[k_1,k_2]$ and $R_{XY}[k_1,k_2]$ to be

$$\begin{aligned} R_{YX}[k_1,k_2]&=E[Y(k_1)X(k_2)]=\sum_{m=-\infty}^{\infty}h[k_1,m]R_{XX}[m,k_2] \\ R_{XY}[k_1,k_2]&=E[X(k_1)Y(k_2)]=\sum_{m=-\infty}^{\infty}h[k_2,m]R_{XX}[k_2,m] \end{aligned} \tag{5.88}$$

5.6.4 nth-Order Densities

Since the output process at any time *n* is a function of the input process random variables at all times, a total characterization is required to determine the first-order densities of the output random process. Thus a first-order characterization of the input process is insufficient information to determine the first-order densities of the output process. Similarly the second and nth-order densities cannot be determined unless the input process is totally characterized If the input process is a Gaussian random process with known mean and autocorrelation function, the output process will be Gaussian and thus totally characterized by its mean and autocorrelation functions given in (5.83) and (5.86).

5.6.5 Stationarity

For a discrete linear time-varying system characterized by its unit-sample response, the output process is not necessarily correspondingly stationary in mean, stationary in autocorrelation, or stationary in wide sense if the input process is stationary in mean, stationary in autocorrelation, or stationary in a wide sense. For a discrete time-varying linear system, specified by an $h(n,m)$, knowing the second-order information of the input process is, however, sufficient information to determine the second-order output information with the relationships given in (5.83) and (5.86).

5.7 Linear System Identification

If the input $X(t)$ to a linear time-invariant filter, specified by its impulse response $h(t)$, is a white process with autocorrelation function

$$R_{XX}(\tau)=\frac{N_0}{2}\delta(\tau) \tag{5.89}$$

the cross-spectral density $S_{YX}(j\omega)$ can be obtained from (5.25) as

$$S_{YX}(\omega)=S_{XX}(\omega)H(j\omega) \tag{5.90}$$

Dividing through by $S_{XX}(\omega)$ which is $N_0/2$ gives the transfer function $H(j\omega)$ as

$$H(j\omega)=\frac{S_{YX}(\omega)}{S_{XX}(\omega)}=\frac{2}{N_0}S_{YX}(\omega) \tag{5.91}$$

That is, by measuring the cross-spectral density between output and input when white noise is the input, the transfer function of an unknown linear system can be determined.

5.8 Derivatives of Random Processes

Let $X(t)$ be a random process with known characterization in terms of mean and autocorrelation function. If realization of this process is the input to a differentiator an output signal is obtained, thus establishing a mapping from the sample space to the output and specifying the output as a random process. The derivative process is denoted as

$$Y(t)=\frac{dX(t)}{dt} \tag{5.92}$$

The logical question is : What are the statistical characterizations of this derivative process? The derivative operation can be viewed as a linear system with transfer function $H(p)=p$, so the formulas given earlier can be used to find the autocorrelation between input and output can be expressed as

$$\begin{aligned} R_{YX}(\tau)&=h(\tau)*R_{XX}(\tau)=\frac{d\delta(\tau)}{d\tau}*R_{XX}(\tau) \\ &=\frac{d}{d(\tau)}R_{XX}(\tau) \end{aligned} \tag{5.93}$$

The autocorrelation function of the output is determined as

$$\begin{aligned} R_{YY}(\tau)&=\Psi^{-1}\left[H(p)\Phi_{XX}(p)H(-p)\right] \\ &=\Psi^{-1}\left[-p^2\Phi_{XX}(p)\right] \end{aligned} \tag{5.94}$$

A property of the inverse Laplace transform is that

$$\Psi^{-1}\left[p^n F(p)\right]=\frac{d^n f(t)}{d^n t} \tag{5.95}$$

Using (5.95) in (5.94), we obtain the autocorrelation function of the output process as

$$R_{YY}(\tau)=-\frac{d^2}{d^2\tau}R_{XX}(\tau) \tag{5.96}$$

In a similar fashion the autocorrelation function of $Z(t)$ can be found, where $Z(t)$ is defined as the nth-order derivative of a random process $X(t)$ and denoted by

$$Z(t)=\frac{d^n X(t)}{d^n t} \tag{5.97}$$

The autocorrelation function for $Z(t)$ can be shown to be

$$R_{ZZ}(t)=(-1)^n\frac{d^{2n}}{d^2\tau}R_{XX}(\tau) \tag{5.98}$$

The derivatives of wide sense stationary random processes are statistically related and it is easily shown, similar to the previous derivations, that their cross-correlation function are given as

$$E\left[\frac{d^m X(t+\tau)}{d^m}\frac{d^m X(t)}{d^n t}\right]=(-1)^n\frac{d^{(m+n)}R_{XX}(\tau)}{d^{(m+n)}\tau} \tag{5.99}$$

A word of caution on using these formulas is necessary as all derivatives of random processes do not necessarily exist, but if they do, the formulas are meaningful.

5.9 Multi-Input, Multi-output Linear Systems

Multiple-input and multiple-output linear systems are becoming increasingly significant as systems are becoming more complicated. An example of two-input two-output system is shown in Figure 5.4. The input signals are $x_1(t)$ and $x_2(t)$, and the output signals are $y_1(t)$ and $y_2(t)$. Each output is the sum of two signals. For example, $y_1(t)$ is the sum of one signal resulting from passing $x_1(t)$ through a linear system with transfer function $H_{11}(p)$ and the other from passing $x_2(t)$ through a linear system with transfer function $H_{21}(p)$. Similarly $y_2(t)$ is the sum of a two responses one due to passing $x_1(t)$ through $H_{12}(p)$ and the other from passing $x_2(t)$ through $H_{22}(p)$. In general, $H_{12}(p)$ and $H_{21}(p)$ are not the same. Thus the two-input, two-output system is specified by a 2 by 2 matrix of transfer functions. The basic problem is knowing the means of the inputs, the autocorrelation functions of the inputs, and the cross-relation between inputs determine the corresponding characterizations of the outputs.

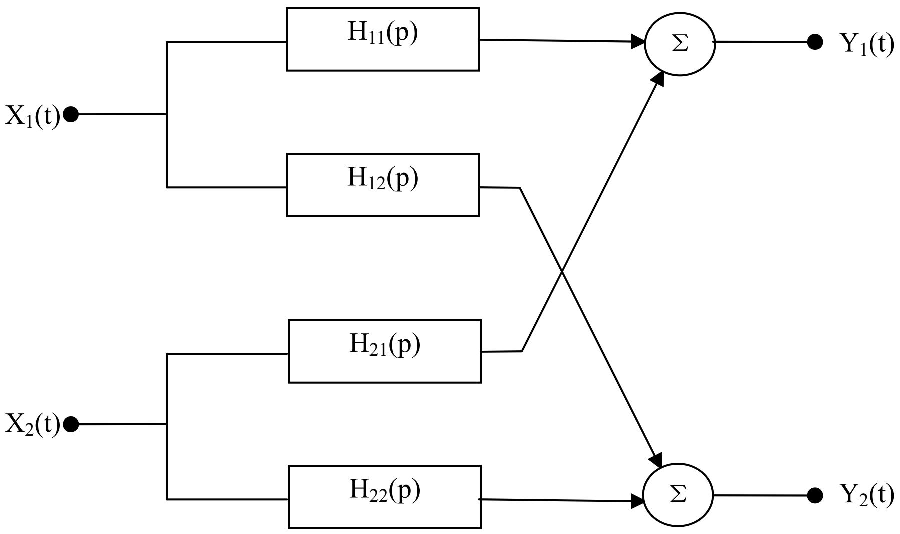

Figure 5.4 A general linear two-input two output system denoted by MIMO(2.2)

5.9.1 Output Means for MIMO(2,2)

If the inputs to the two-input, two-output linear system are random processes $X_1(t)$ and $X_2(t)$, the output processes $Y_1(t)$ and $Y_2(t)$ are easily seen to be the sum of two convolutions:

$$\begin{aligned} Y_1(t) &= \int_{-\infty}^{\infty} X_1(t-\alpha)h_{11}(\alpha)d\alpha + \int_{-\infty}^{\infty} X_2(t-\beta)h_{21}(\beta)d\beta \\ Y_2(t) &= \int_{-\infty}^{\infty} X_1(t-\alpha)h_{12}(\alpha)d\alpha + \int_{-\infty}^{\infty} X_2(t-\beta)h_{22}(\beta)d\beta \end{aligned} \tag{5.100}$$

The means are determined by taking the expected value of the equations in (5.100) and interchanging the expected value operator and the integral sign to get

$$\begin{aligned} E[Y_1(t)] &= \int_{-\infty}^{\infty} E[X_1(t-\alpha)]h_{11}(\alpha)d\alpha + \int_{-\infty}^{\infty} E[X_2(t-\beta)h_{21}(\beta)d\beta] \\ E[Y_2(t)] &= \int_{-\infty}^{\infty} E[X_1(t-\alpha)]h_{12}(\alpha)d\alpha + \int_{-\infty}^{\infty} E[X_2(t-\beta)h_{22}(\beta)d\beta] \end{aligned} \tag{5.101}$$

Recognizing the integrals as convolutions permits us to write the means in terms of convolutions as

$$\begin{aligned} E[Y_1(t)] &= E[X_1(t)] * h_{11}(\alpha) + E[X_2(t)] * h_{21}(\beta) \\ E[Y_2(t)] &= E[X_1(t)] * h_{12}(\alpha) + E[X_2(t)] * h_{22}(\beta) \end{aligned} \tag{5.102}$$

When the input processes are jointly wide sense stationary, the input means are constant, and the output means can be written in terms of the various transfer functions evaluated at zero:

$$\begin{aligned} \eta_{Y_1} &= \eta_{X_1} H_{11}(0) + \eta_{X_2} H_{21}(0) \\ \eta_{Y_2} &= \eta_{X_1} H_{12}(0) + \eta_{X_2} H_{22}(0) \end{aligned} \tag{5.103}$$

5.9.2 Cross-correlation Functions for MIMO(2,2) Linear Systems

There are four cross-correlation functions that need to be examined for the two-input, two-output linear system given by $R_{Y_1X_1}(t,u), R_{Y_1X_2}(t,u), R_{Y_2X_1}(t,u)$ and $R_{Y_2X_2}(t,u)$. Consider the first one of these, which can be written as

$$\begin{aligned} R_{Y_1X_1}(t,u) &= E[Y_1(t)X_1(u)] \\ &= E\left[\left(\int_{-\infty}^{\infty} h_{11}(\alpha)X_1(t-u)d\alpha + \int_{-\infty}^{\infty} h_{21}(\beta)X_2(t-\beta)d\beta\right)X_1(u)\right] \end{aligned} \tag{5.104}$$

After we multiply out and interchange the integral and expected value operations, (5.104) becomes

$$R_{Y_1X_1}(t,u) = \int_{-\infty}^{\infty} h_{11}(\alpha)R_{X_1X_1}(t-\alpha,u)d\alpha + \int_{-\infty}^{\infty} h_{21}(\beta)R_{X_2X_1}(t-\beta,u)d\beta \tag{5.105}$$

Let us assume that $X_1(t)$ and $X_2(t)$ are jointly wide sense stationary. Then this cross-correlation function can be written in terms of $\tau = t-u$ as follows:

$$R_{Y_1X_1}(\tau) = h_{11}(\tau) * R_{X_1X_1}(\tau) + h_{21}(\tau) * R_{X_2X_1}(\tau) \tag{5.106}$$

The other input-output cross-correlation functions for jointly wide sense stationary inputs can be similarly shown to be

$$\begin{aligned} R_{Y_1X_2}(\tau) &= h_{11}(\tau) * R_{X_1X_2}(\tau) + h_{21}(\tau) * R_{X_2X_2}(\tau) \\ R_{Y_2X_1}(\tau) &= h_{12}(\tau) * R_{X_1X_2}(\tau) + h_{22}(\tau) * R_{X_2X_2}(\tau) \\ R_{Y_2X_2}(\tau) &= h_{12}(\tau) * R_{X_1X_2}(\tau) + h_{22}(\tau) * R_{X_2X_2}(\tau) \end{aligned} \tag{5.107}$$

As was done in previous sections, these convolutions may be conveniently calculated by using inverse Laplace transforms of various products of Laplace transforms of the system impulse responses and various autocorrelation and cross-correlation functions of the input processes.

5.9.3 Autocorrelation Functions for Outputs of MIMO(2,2) Linear Systems

The autocorrelation function for the output $Y_1(t)$ can be written as

$$\begin{aligned} R_{Y_1Y_1}(t,u) &= E[Y_1(t)Y_1(u)] \\ &= E\left[\left(\int_{-\infty}^{\infty} h_{11}(\alpha)X_1(t-\alpha)d\alpha + \int_{-\infty}^{\infty} h_{21}(\beta)X_2(t-\beta)d\beta\right)\right. \\ &\qquad \left.\left(\int_{-\infty}^{\infty} h_{11}(\gamma)X_1(u-\gamma)d\gamma + \int_{-\infty}^{\infty} h_{21}(\theta)X_2(t-\theta)d\theta\right)\right] \end{aligned} \tag{5.108}$$

Multiply out the terms in parentheses to get four terms, taking the expected value through the integral, and evaluating the expected value gives the autocorrelation function $R_{Y_1Y_1}(t,u)$ as

$$\begin{aligned} R_{Y_1Y_1}(t,u) &= \int_{-\infty}^{\infty} h_{11}(\alpha)h_{11}(\gamma)R_{X_1X_1}(t-\alpha,u-\gamma)d\alpha d\gamma \\ &+ \int_{-\infty}^{\infty} h_{11}(\alpha)h_{21}(\theta)R_{X_1X_2}(t-\alpha,u-\theta)d\alpha d\theta \\ &+ \int_{-\infty}^{\infty} h_{21}(\beta)h_{11}(\gamma)R_{X_2X_1}(t-\beta,u-\gamma)d\beta d\gamma \\ &+ \int_{-\infty}^{\infty} h_{21}(\beta)h_{21}(\theta)R_{X_2X_2}(t-\beta,u-\theta)d\beta d\theta \end{aligned} \tag{5.109}$$

If the input processes are jointly wide sense stationary, then the equation above can be written as

$$\begin{aligned} R_{Y_1Y_1}(t,u) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{11}(\alpha)h_{11}(\gamma)R_{X_1X_1}(\tau-(\alpha-\gamma))d\alpha d\gamma \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{11}(\alpha)h_{21}(\theta)R_{X_1X_2}(\tau-(\alpha-\theta))d\alpha d\theta \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{21}(\beta)h_{11}(\gamma)R_{X_2X_1}(\tau-(\beta-\gamma))d\beta d\gamma \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{21}(\beta)h_{21}(\theta)R_{X_2X_2}(\tau-(\beta-\theta))d\beta d\theta \end{aligned} \tag{5.110}$$

As before, for the single-input, single-output linear system, the integrals above are recognized as convolutions, and thus the autocorrelation function becomes

$$\begin{aligned} R_{Y_1Y_1}(\tau) &= h_{11}(\alpha)*R_{X_1X_1}(\tau)*h_{11}(-\tau)+h_{11}(\tau)*R_{X_1X_1}(\tau)*h_{21}(-\tau) \\ &+ h_{22}(\tau)*R_{X_2X_1}(\tau)*h_{12}(-\tau)+h_{22}(\tau)R_{X_2X_2}(\tau)*h_{22}(-\tau) \end{aligned} \tag{5.111}$$

In a similar development the autocorrelation function for the other output $Y_2(t)$ can be shown to be

$$\begin{aligned} R_{Y_1Y_1}(\tau) &= h_{12}(\alpha)*R_{X_1X_1}(\tau)*h_{12}(-\tau)+h_{12}(\tau)*R_{X_1X_2}(\tau)*h_{22}(-\tau) \\ &+ h_{22}(\tau)*R_{X_2X_1}(\tau)*h_{12}(-\tau)+h_{22}(\tau)R_{X_2X_2}(\tau)*h_{22}(-\tau) \end{aligned} \tag{5.112}$$

As before, these cross-correlation functions are more conveniently determined in the Laplace domain.

5.9.4 Cross-correlation Functions for Outputs of MIMO(2.2) Linear Systems

The cross-correlation function for the two outputs $Y_1(t)$ and $Y_2(t)$ is determined in a similar fashion to that for the autocorrelation functions:

$$\begin{aligned} R_{Y_1Y_2}(t,u) &= E[Y_1(t)Y_2(u)] \\ &= E\left[\left(\int_{-\infty}^{\infty} h_{11}(\alpha)X_1(t-\alpha)d\alpha + \int_{-\infty}^{\infty} h_{21}(\beta)X_2(t-\beta)d\beta\right)\right. \\ &\quad \left.\left(\int_{-\infty}^{\infty} h_{12}(\gamma)X_1(u-\gamma)d\gamma + \int_{-\infty}^{\infty} h_{22}(\theta)X_2(u-\theta)d\theta\right)\right] \end{aligned} \tag{5.113}$$

After we multiply out the integrals, assuming jointly wide sense stationary inputs, the cross correlation between outputs can be written as

$$R_{Y_1Y_2}(\tau)=h_{11}(\alpha)*R_{X_1X_1}(\tau)*h_{12}(-\tau)+h_{11}(\tau)*R_{X_1X_2}(\tau)*h_{22}(-\tau)$$
$$+h_{21}(\tau)*R_{X_2X_1}(\tau)*h_{12}(-\tau)+h_{21}(\tau)R_{X_2X_2}(\tau)*h_{22}(-\tau) \quad (5.114)$$

As the orders of the $MIMO(n,n)$ system increase, the autocorrelation and cross-correlation functions become very complex, simplifying only when the input processes are all uncorrelated.

5.10 Transients in Linear Systems

Let $x(t)$ be the input and $y(t)$ the output of a linear system that is governed by the following linear constant coefficient differential equation:

$$y(t)=-\sum_{k=1}^{n}a_k\frac{d^k y(t)}{dt^k}+x(t), \qquad t>0 \quad (5.115)$$

If the input is a random process, then the output is also a random process described from the mapping of each realization of the input process to a realization of the output process. Assume that $y(t)=0$ for all t < 0 and that the initial conditions are

$$\frac{d^k y(t)}{dt^k}\Big|_{t=0} \qquad \text{for} \quad k=0 \quad \text{to} \quad n\text{-}1 \quad (5.116)$$

This means that the system is initially at rest.

If $x(t)$ is an input random process $X(t)$, then the output $y(t)$ is $Y(t)$ a random process.

5.10.1 Mean of the Output Process

Substituting $X(t)$ as the input and $Y(t)$ as the output of Eq.(5.115), and taking the expected value, gives the following differential equation for the mean of output process:

$$E[Y(t)]=-\sum_{k=1}^{n}a_k E\left[\frac{d^k Y(t)}{d^k t}\right]+E[X(t)], \qquad t>0 \quad (5.117)$$

The expected value of the derivatives of the process $X(t)$ has been shown to be the derivatives of the expected value of the process as following:

$$E\left[\frac{d^k Y(t)}{d^k t}\right]=\frac{d^k E[Y(t)]}{d^k t} \quad (5.118)$$

Substituting (5.118) into (5.117), we arrive at a differential equation in terms of the mean of the output process:

$$E[Y(t)]+\sum_{k=1}^{n}a_k\left[\frac{d^k E[Y(t)]}{d^k t}\right]=\eta_X(t), \qquad t>0 \quad (5.119)$$

Now, taking the expected value of the initial conditions given in Eq.(5.116), gives zero conditions on the mean:

$$\frac{d^k E[Y(t)]}{dt^k}\Big|_{t=0} = 0 \quad \text{for} \quad k = 0 \quad \text{to } n\text{-}1 \tag{5.120}$$

Equation (5.119) and (5.120) give a differential equation with initial conditions that must be solved to obtain the mean $E[Y(t)]$ of the output process.

5.10.2 Autocorrelation of the Output Process

Similar to the procedure for obtaining the autocorrelation function for the AR process in Chapter 4, the procedure to find the output autocorrelation function $R_{XX}(t_1,t_2)$ consists of two steps: first solving for the cross-correlation function and then from that result finding the autocorrelation function .Evaluation both sides of (5.115) at t_2, multiplying by $X(t_1)$, and taking the expected value gives the cross-correlation function at t_1 and t_2 as

$$\begin{aligned} R_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] \\ &= E\left[X(t_1)\left(-\sum_{k=1}^{n} a_k \frac{d^k Y(t)}{dt^k}\right)_{t=t_2}\right] + E[X(t_1)X(t_2)] \qquad t_1,t_2>0 \end{aligned} \tag{5.121}$$

Moving $X(t_1)$ into the summation, and taking the expected value, the first term of (5.121) becomes

$$E\left[x(t_1)\left(\frac{d^k y(t)}{d^k t}\right)_{t=t_2}\right] = \frac{d^k R_{XY}(t_1,t_2)}{d^k t_2} \tag{5.122}$$

Using (5.122) in Eq.(5.121) and rearranging yields the following differential equation for $R_{XY}(t_1,t_2)$:

$$\sum_{k=1}^{n} \frac{d^k R_{XY}(t_1,t_2)}{d^k t_2} + R_{XY}(t_1,t_2) = R_{XX}(t_1,t_2), \qquad t_2 > 0 \tag{5.123}$$

The initial conditions are obtained by multiplying Eq.(5.120) by $X(t_1)$ and taking the expected value. This gives

$$\frac{d^k R_{XY}(t_1,t_2)}{d^k t_2}\Big|_{t_2=0} \qquad \text{for } k=0 \quad \text{to} \quad n-1 \tag{5.124}$$

Thus the solution of the differential equation (5.123) with initial conditions (5.124) gives the cross-correlation function $R_{XY}(t_1,t_2)$ for $t_1,t_2>0$.

Having obtained $R_{XY}(t_1,t_2)$, we are now in position to solve for $R_{YY}(t_1,t_2)$. Multiplying (5.115) evaluated at t_1 by $Y(t_2)$ and taking the expected value gives

$$R_{YY}(t_1,t_2)=E[Y(t_1)Y(t_2)]$$
$$=E\left[Y(t_1)\left(-\sum_{k=1}^{n}a_k\frac{d^kY(t)}{d^kt}\right)_{t=t_2}\right]+E[X(t_2)Y(t_1)] \qquad t_1,t_2>0 \qquad (5.125)$$

As in the development of cross correlation, (5.132) simplifies to

$$\sum_{k=1}^{n}\frac{d^kR_{YY}(t_1,t_2)}{d^kt}+R_{YY}(t_1,t_2)=R_{XY}(t_1,t_2), \qquad t_1>0 \qquad (5.126)$$

with zero initial condition:

$$\frac{d^kR_{YY}(t_1,t_2)}{d^kt_1}\bigg|_{t_1=0}=0 \qquad \text{for} \quad k=0 \text{ to } n-1 \qquad (5.127)$$

Eq.(5.126) and (5.127) give a differential equation and initial conditions to solve for $R_{YY}(t_1,t_2)$ in terms of the $R_{XY}(t_1,t_2)$ as a driving function.

Example5.9

Given that $y(t)$ is the solution to the following differential equation representing a transfer function $H(p)=1/(p+1)$

$$\frac{dy(t)}{dt}=-y(t)+x(t), \qquad t>0$$

Assume that $y(t)=0$ for all $t<0$ and that we have the initial condition $y(0)=0$.Let the input $x(t)$ be a random process $X(t)$ with mean and autocorrelation function given by

$$\eta_X(t)=E[X(t)]=1, \qquad \mathrm{R}_{XX}(\tau)=1/2\,e^{-2|\tau|}+1$$

Find the mean η_Y of the output process $Y(t)$.

Solution

From $Eqs.(5.119)$ and (5.120),the mean, $E[Y(t)]=\eta_Y(t)$,of the output process satisfies the following differential equation and initial condition:

$$\frac{d\eta_Y(t)}{dt}+\eta_Y(t)=1, \qquad t>0$$
$$\eta_Y(0)=0$$

Upon solving the above for $\eta_Y(t)$,we have

$$\eta_Y(t)=\left[1-e^{-t}\right]\mu(t)$$

Example 5.10

Given that $y(t)$ is the solution to the following differential equation representing a transfer function $H(p)=1/(p+1)$ and the input $x(t)$:

$$\frac{dy(t)}{dt} = -y(t) + x(t), \qquad t > 0$$

Assume that $y(t) = 0$ for all $t < 0$ and that we have the initial condition $y(0) = 0$.Let $x(t)$ be a random process $X(t)$ with mean and autocorrelation function given by

$$\eta_X(t) = E[X(t)] = 0, \qquad R_{XX}(\tau) = \delta(\tau)$$

Find the autocorrelation function $R_{YY}(t_1, t_2)$ of the output process $Y(t)$.

Solution

To obtain the $R_{YY}(t_1, t_2)$, we first find the cross correlation $R_{XY}(t_1, t_2)$ by solving a differential equation. Then from the result we find the $R_{YY}(t_1, t_2)$ by solving another differential equation. We can use the Laplace transform to solve both differential equation. From $Eq.(5.123)$,the $R_{YY}(t_1, t_2)$ is the solution of

$$\frac{dR_{XY}(t_1, t_2)}{dt_2} + R_{XY}(t_1, t_2) = \delta(t_2 - t_1), \qquad t_2 \ge 0$$

With initial conditions from (5.124) as $R_{XY}(t_1, t_2)|_{t_2=0} = 0$ .Taking the Laplace transform with respect to the t_2 variable and using the initial condition results in

$$p\Phi_{XY}(t_1, p) + p\Phi_{XY}(t_1, p) = e^{-pt_2}$$

Solving for $\Phi_{XY}(t_1, p)$ gives

$$\Phi_{XY}(t_1, p) = \frac{e^{-pt_1}}{p+1}$$

Taking the inverse Laplace transform gives us the cross correlation $R_{XY}(t_1, t_2)$ as

$$R_{XY}(t_1, t_2) = e^{-(t_2 - t_1)}\mu(t_2 - t_1)$$

This result is the driving function for the differential equation for $R_{XY}(t_1, t_2)$ given in $Eq.(5.126)$.So we have the following :

$$\frac{dR_{YY}(t_1, t_2)}{dt_1} + R_{YY}(t_1, t_2) = e^{-(t_2 - t_1)}\mu(t_2 - t_1), \qquad t_1 \ge 0$$

Taking the Laplace transform of the equation above with respect to the t_1 variable and using the zero initial condition gives

$$\begin{aligned} p\Phi_{YY}(p, t_2) + \Phi_{YY}(p, t_2) &= e^{-t_2}\int_0^{t_2} e^{+t_1}e^{-pt_1}\,dt_1 \\ &= e^{-t_2}\left[\frac{e^{-(p-1)t_1}}{-(p-1)}\right]_0^{t_2} \\ &= e^{-t_2}\left[\frac{e^{-(p-1)t_2}}{-(p-1)} + \frac{1}{p-1}\right] \end{aligned}$$

Then, solving for $\Phi_{YY}(p,t_2)$, multiplying out and performing a partial fraction expansion of both terms yields

$$\begin{aligned}\Phi_{YY}(p,t_2) &= e^{-t_2}\left[-\frac{e^{-(p-1)t_2}}{p-1}+\frac{1}{p-1}\right]\frac{1}{p+1}\\ &= -\frac{e^{-pt_2}}{(p-1)(p+1)}+\frac{e}{(p-1)(p+1)}\\ &= \frac{\frac{1}{2}e^{-pt_2}}{p-1}-\frac{\frac{1}{2}e^{-pt_2}}{p+1}+\frac{\frac{1}{2}e^{-t_2}}{p-1}-\frac{\frac{1}{2}e^{-t_2}}{p+1}\end{aligned}$$

Finally, taking the inverse bilateral Laplace transform using the t_1 variable gives us the autocorrelation function $R_{XY}(t_1,t_2)$ as

$$R_{YY}(t_1,t_2)=\frac{1}{2}e^{(t_1-t_2)}\mu(-(t_1-t_2))+\frac{1}{2}e^{-(t_1-t_2)}\mu(t_1-t_2)$$

Upon evaluating for $t_1>0$ and $t_2>0$ and regions $t_1>t_2$ and $t_2>t_1$, we find, the $R_{YY}(t_1,t_2)$ from the equation above to be

$$R_{YY}(t_1,t_2)=\begin{cases}\frac{1}{2}e^{-(t_1-t_2)}-\frac{1}{2}e^{-(t_1+t_2)}, & t_1>t_2\\ \frac{1}{2}e^{-(t_2-t_1)}-\frac{1}{2}e^{-(t_1+t_2)}, & t_2>t_1\end{cases}$$

To obtain the steady state results for the autocorrelation function, we let both t_1 and t_2 approach infinity but keep $t_1-t_2=\tau$. The second term goes to zero and the first term remains. Thus, for $t_1-t_2=\tau$, the steady state autocorrelation function $R_{YY}(\tau)$ of the output $Y(t)$ is

$$R_{YY}(\tau)=\frac{1}{2}e^{-|\tau|}$$

This result checks with the result obtained by using the steady state formula given in $Eq.(5.18)$

$$\begin{aligned}R_{YY}(\tau) &= F_\beta^{-1}\left[H(p)*\Phi_{XX}(p)*H(-p)\right]\\ &= F_\beta^{-1}\left[\frac{1}{p+1}1\frac{1}{-p+1}\right]=\frac{1}{2}e^{-|\tau|}\end{aligned}$$

5.11 Summary

The main topic of this chapter is the interaction of random processes with linear systems. In particular, the characterization of the output process is in terms of the input process characterization. Formulas were derived for the output mean and autocorrelation function of linear time-invariant systems to a wide sense stationary input. The mean of the output was determined as a convolution of the input mean with the impulse response of the system, and the output autocorrelation function was shown to be a convolution of the input autocorrelation function, the impulse response, and the time-reversed impulse of the system. Determination of the convolutions in many cases was best done in the Laplace domain for continuous time systems and the Z domain for discrete time systems The output power spectral density was shown to be determined by multiplying the input power spectral density by magnitude squared of the system transfer function $H(jw)$ for the continuous time system and $H(e^{jw})$ for the discrete time system.

Formulas for the mean and autocorrelation function of the output of a linear time varying system in terms of the mean and autocorrelation function of the input process for both continuous and discrete time systems were presented.

The AR, MA, and ARMA processes generated by passing white noise through special discrete-time linear systems were analyzed using the steady state techniques described above. The results obtained for mean and autocorrelation by using the steady state results in the Z-domain were shown to be equal to those steady state results in the time domain.

In using differential equations to model continuous time invariant systems with random processes as inputs it is necessary to obtain the statistical properties of the derivatives of the input processes. Ignoring some issues with existence the auto-and cross-correlation functions for derivatives of a random process were developed using the Laplace transform domain in terms of the autocorrelation of the input process. For mathematically more rigorous presentations on this topic the interested reader could explore Papoulis [1] or Van Trees [7].

Also considered in this chapter was the transient response of a continuous linear time invariant system to an input process applied at equal zero. For the special case where the differential equation does not contain derivatives of the input process the output mean was easily obtained as a convolution; however, the output autocorrelation function required the solution of two differential equations one to get the cross-correlation function between input and output and the second with the cross-correlation function as the input to obtain the output autocorrelation function. The solution was facilitated by using the Laplace transform. The examples illustrated that as t becomes large, the output autocorrelation and mean approach those determined using the steady state methods involving transfer functions.

The chapter concluded with a brief discussion of multiple input, multiple output systems and their output means, autocorrelation functions, and cross-correlation functions to random inputs and results were shown for wide sense stationary input processes.

----------This is the end of Chapter05----------