*4.8.8 Accumulator-Based Code Design (*基于 RA 结构的 LDPC 码*)*

4.8.8.1 Repeat-Accumulate (RA) Codes

RA码是由Divsalar, McEliece等于1998年提出的一种非常简单的串行级联Turbo码，its encoder consists of three trivial encoding blocks, as shown in Fig. 4.8.22.

- The outer code is a repetition code of rate $1/q$, which simply repeats the information bits q times.
- The resulting sequence is then permuted by a large pseudo-random permutation Π.
- The inner code is a rate-1 two-state recursive convolutional code with generator $g(D)=1/(1+D)$. This acts as an accumulator, because the output is the mod-2 sum (accumulation) of all previous input bits.

图4.8.22所示的是非系统RA码的编码器结构，the accumulator output, **p**, is the codeword and the code rate is $1/q$. RA码也可以是系统的。In this case, the information word, **u**, is combined with **p** to yield the codeword **c** = [**u**, **p**] and so that the code rate is $1/(1+q)$.

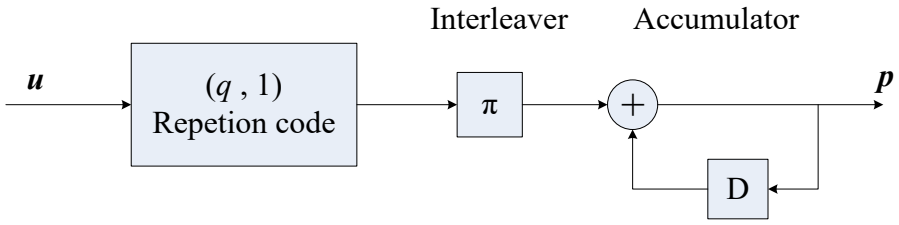

Figure 4.8.22 A repeat-accumulator encoder

RA码同时具有turbo码与LDPC码的特征。我们将在后面看到，从校验矩阵的角度看，RA codes and other accumulator-based codes are LDPC codes. 因此This code could be decoded much like turbo codes using iterated BCJR algorithms,也可以基于因子图（校验矩阵）采用迭代SPA进行并行译码。（that can be decoded as serial turbo codes or as LDPC codes）.

为了保证较好的性能，RA码的码率通常不超过1/2。为了获得高码率，我们可以对RA码的输出做puncture处理，这等价于在ACC之前增加合并器（即进行（q, q-1）SPC编码并只保留校验位，而删余信息位）（for every parity-bit, a of these repeated message bits are summed *mod* 2, a process that we implement using a combiner）。一个通用的系统形式编码器结构如图4.8.23所示。Instead of puncturing, equivalently, we can use a single parity check (SPC(a)) which adds a input bits and outputs one parity, followed by an accumulator without puncturing.

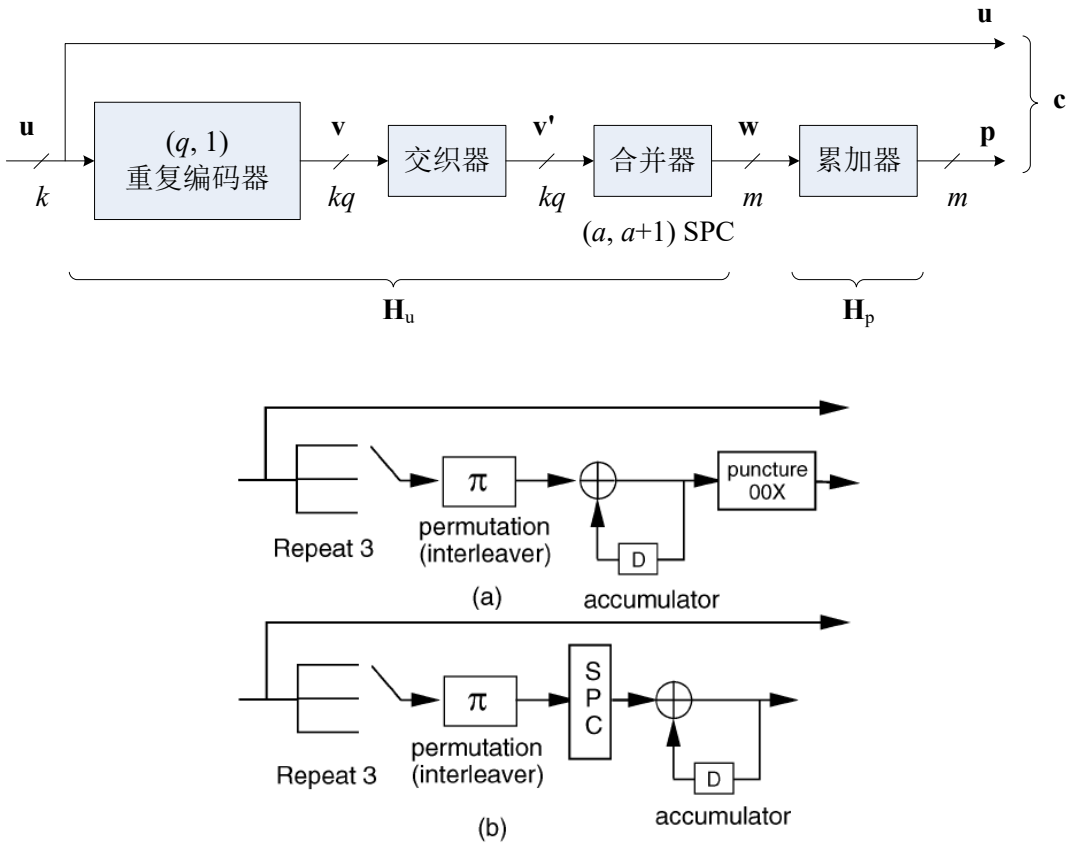

Fig. 4.8.23. Systematic RA code. (a) Encoder using puncturing. (b) Equivalent encoder using SPC.

The encoding process is as follows. Denote the length-k information sequence to be encoded by $\mathbf{u}=(u_1,u_2,...,u_k)$. After repetition encoding, the qk bits at the output of the repetition encoder are q coyies of the k information bits, in the form

$$\mathbf{v}=(v_1,v_2,...,v_{qk})$$

$$=(u_1,...,u_1,\ u_2,...,u_2,\ \cdots,\ u_k,...,u_k)$$

The interleaver, $\pi(i)$, defines a permutation of the input bits, $\mathbf{v}=(v_1,v_2,...,v_n)$, to the output bits

$$\mathbf{v}'=(v_1',v_2',...,v_n')=(v_{\pi(1)},v_{\pi(2)},...,v_{\pi(n)})$$

The bits at the output of the interleaver are combined, modulo-2, in sets of a bits, before being passed to the accumulator. Thus the $m=kq/a$ bits, $\mathbf{w}=(w_1,w_2,...,w_m)$, at the output of the combiner, are given by

$$w_i=v'_{(i-1)a+1}\oplus v'_{(i-1)a+2}\oplus\cdots\oplus v'_{ia},\quad i=1,2,...,m$$

where " " denotes modulo-2 addition.

Finally, the m parity bits, $\mathbf{p}=(p_1,p_2,...,p_m)$, at the output of the accumulator are

produced by

$$p_i = p_{i-1} \oplus w_i, \quad \text{i=1,2,…,m}$$

对于图4.8.23所示的*systematic* RA codes, the final codeword is

$$\mathbf{c} = \left(u_1, u_2, ..., u_k, p_1, p_2, ..., p_m\right)$$

and thus we have a code with length $n = k(1+q/a)$ and rate $R_c = a/(a+q)$.

RA码的校验矩阵与因子图表示：

RA码也可以看作是一种LDPC code with an upper triangular form already built into the parity-check matrix during the code design.

根据图x中的编码器结构，An $m \times n$ RA code parity-check matrix $\mathbf{H}$ has two parts:

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_u & \mathbf{H}_p \end{bmatrix}$$

where $\mathbf{H}_u$是一个由重复编码、合并器、交织器确定的二元矩阵，$\mathbf{H}_p$ is an $m \times m$ "dual-diagnol" matrix,

$$\mathbf{H}_p = \begin{bmatrix} 1 & & & & (1) \\ 1 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & 0 & 1 & 1 & \\ & & & 1 & 1 \end{bmatrix}$$

（the upper-right 1 is included for tailing-biting accumulators.），它对应于图x中的累加器。

对于规则RA码，$\mathbf{H}_u$是一个列重为q，行重为a的regular matrix, ‘1’的位置由交织器决定。(行重由合并器决定，列重由重复编码决定) The parity-check matrix of an RA code is called (q, a)-*regular* if the weight of all the rows of $\mathbf{H}_u$ are the same a, and the weight of all the columns of $\mathbf{H}_u$ are the same q. Note that a regular RA parity-check matrix has columns of weight 2, and one column of weight 1, in $\mathbf{H}_p$ and so is not regular in the sense of (dv, dc)-regular LDPC codes. An irregular RA code will have an irregular column weight distribution in $\mathbf{H}_u$, with $\mathbf{H}_p$ the same as for a regular code.

As for LDPC codes, the factor graph of an RA code is defined by $\mathbf{H}$. The Tanner graph of an RA code consists of $m = kq/a$ check nodes and variable nodes.

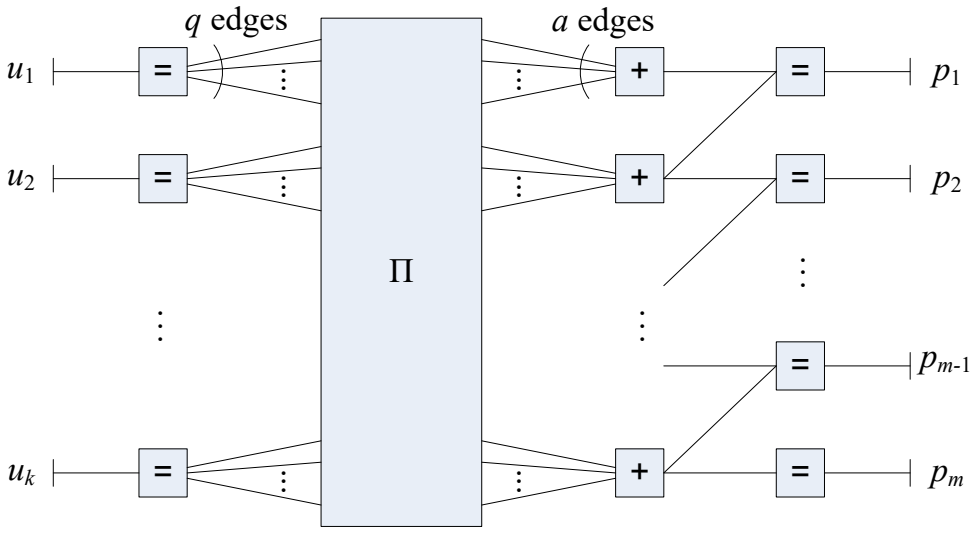

Figure 4.8.24 RA码的Forney-style factor graph

小结：

- Encoding RA codes: 可按串行级联码或基于**H**矩阵编码

use **H** to encode since one may solve for the parity bits sequentially from the equation $\mathbf{cH}^T = \mathbf{0}$ starting with the top row of **H** and moving on downward.

- Decoding RA codes: 采用BCJR的串行译码或基于因子图的SPA
- Despite their simple structure, they were shown to provide good performance and, more importantly, they paved a path toward the design of efficiently encodable LDPC codes.

4.8.8.2 **Irregular RA Codes**

The RA encoder structure presented above can be generalized to an irregular repeat-accumulate structure [Jin-McEliece2000]. In this code there are k input bits. Instead of repeating all k bits an equal number of times, we choose fractions $f_2, f_3, ..., f_q$ such that

$$\sum_{i=2}^{q} f_i = 1$$

Then the first block of $f_2 k$ bits is repeated two times, the next block of $f_3 k$ bits is repeated three times, and so forth. Furthermore, the parity check nodes are generalized to connect to a+1 nodes, of which a are variable nodes. This structure is illustrated in Figure 4.8.25. From this general structure, assignment of the code parameters can be optimized using density evolution, so that the decoding SNR threshold can be minimized.

Irregular RA codes provide several advantages over RA codes. They allowing both flexibility in the choice of the repetition rate for each information bit so that high rate codes may be designed and capacity is more easily approached.

While an irregular RA code is simply a special case of an irregular LDPC code, it has an important advantage: it can be very efficiently encoded. The RA codes thus provide an instance of a code which has linear encode complexity and linear decode complexity, while

still achieving excellent decoding performance.

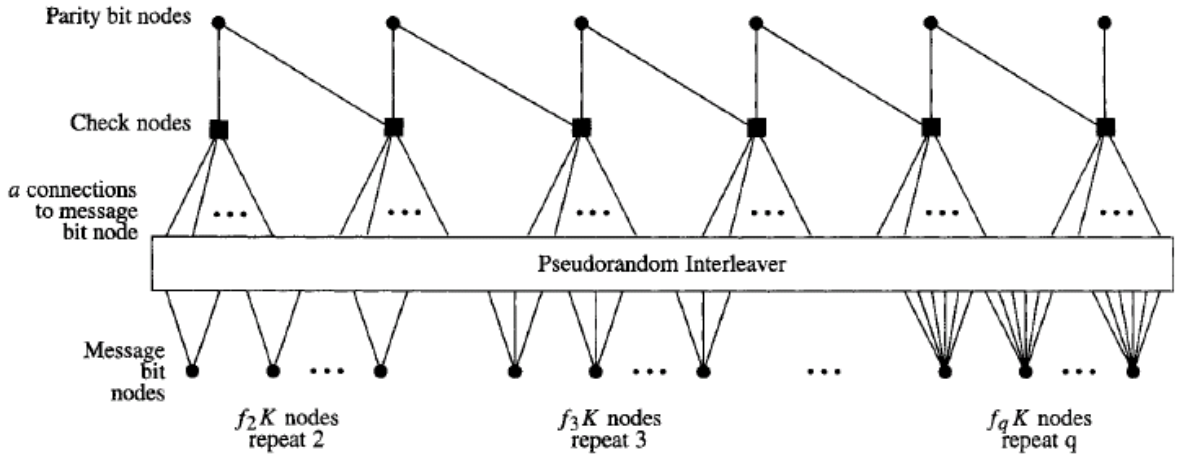

Figure 4.8.25 Factor graph for IRA codes [Moon05]

4.8.8.3 基于 RA 结构的准循环 LDPC 码

基本思想：先设计一个没有小环的RA码矩阵作为基矩阵$\mathbf{H}_b$，然后再用循环置换矩阵替换$\mathbf{H}_b$中的非零元素得到$\mathbf{H}$。

Given the code rate, length, and degree distributions, an IRA code is defined entirely by the matrix $\mathbf{H}_u$. While a random-like $\mathbf{H}_u$ would generally give good performance, it is problematic for both encoder and decoder implementations.

Tanner was the first to consider structured RA codes, more specifically, quasi-cyclic RA codes, which require tail-biting in the accumulator. Simulation results in [Tanner99] demonstrate that the QC-RA codes compete well with random-like RA codes and surpass their performance at high SNR values. Similar ideas were applied to IRA codes in [Mansour04][Zhang-Ryan07]. In [Zhang-Ryan07], IRA codes with quasi-cyclic structure are called *structured IRA* (S-IRA) codes.

Let $\mathbf{P}$ be an $L\times J$ array of $Q\times Q$ circulant permutation matrices ,且没有长为4的环 (Conditions for designing $\mathbf{P}$ to avoid 4-cycles, etc., are described in [Zhang-Ryan07].) Then $\mathbf{H}_u = \Pi^T\mathbf{P}$ and

$$\mathbf{H}_a = \left[\Pi^T\mathbf{P} \quad \mathbf{H}_p\right]$$

where $\mathbf{H}_p$ represents the tail-biting accumulator. Note that $m = L \times Q$ and $k = J \times Q$. We now choose choose Π to be a standard deterministic "row-column" interleaver so that the row $lQ+q$ in $\mathbf{P}$ becomes row $qL+l$ in $\Pi^T\mathbf{P}$, for all $0 \le l < L$ and $0 \le q < Q$. Next, we permute the rows of $\mathbf{H}_a$ by Π^{-T} to obtain

$$\mathbf{H}_b = \Pi^{-T}\mathbf{H}_a = \left[\mathbf{P} \quad \Pi\mathbf{H}_p\right]$$

where we have used the fact that $\Pi^{-T} = \Pi$. Finally, we permute only the columns corresponding to the parity parity part of $\mathbf{H}_b$, which gives

$$\mathbf{H} = \left[\mathbf{P} \quad \Pi\mathbf{H}_p\Pi^T\right]$$

It is easily shown that the parity part of $\mathbf{H}$ (i.e., $\Pi\mathbf{H}_p\Pi^T$) is exactly in QC form.

$$\begin{bmatrix} \mathbf{I}_0 & & & & \mathbf{I}_1 \\ \mathbf{I}_0 & \mathbf{I}_0 & & & \\ & \ddots & \ddots & & \\ & & \mathbf{I}_0 & \mathbf{I}_0 & \\ & & & \mathbf{I}_0 & \mathbf{I}_0 \end{bmatrix}$$

where $\mathbf{I}_0$ is the $Q \times Q$ identity matrix and $\mathbf{I}_1$ is obtained from $\mathbf{I}_0$ by cyclically shifting all of its rows leftward. Therefore, $\mathbf{H}$ corresponds to a quasi-cyclic IRA code since $\mathbf{P}$ is also an array of $Q \times Q$ circulant permutation matrices. Observe that, except for a re-ordering of the parity bits, $\mathbf{H}$ describes the same code as $\mathbf{H}_a$ and $\mathbf{H}_b$.

Simulation results show that the S-IRA codes perform as well as IRA codes in the waterfall region and possess very low error floors. As an example, Fig. 4.8.26 depicts the performance of a rate-1/2 (2044, 1024) S-IRA code.

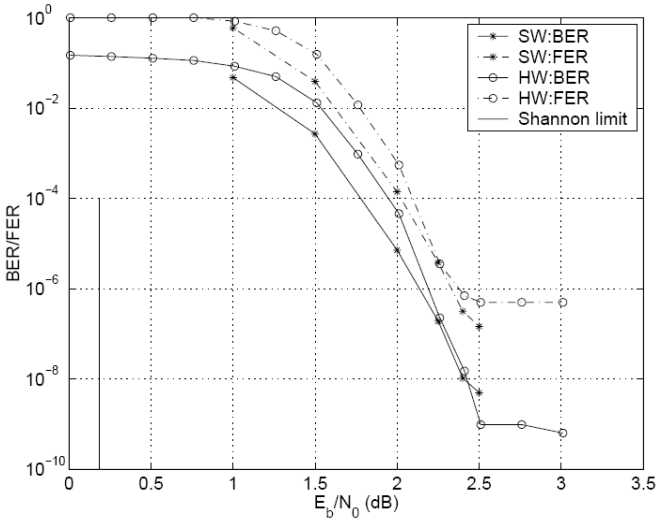

Fig. 6. Performance of a (2044,1024) S-IRA code on the BI-AWGNC. HW=hardware simulator. SW=software simulator.

Figure 4.8.26

4.8.8.4 ARA codes

For accumulate-repeat-accumulate (ARA) codes, introduced in [Abbasfar04], an accumulator is added to precode a subset of the information bits of an IRA code. The primary role of this second accumulator is to improved the decoding threshold of a code, that is, to shift the BER waterfall region leftward. ARA codes are a subclass of LDPC codes and Fig. 4.8.27 presents a generic ARA Tanner graph in which punctured variable nodes are highlighted. The sparseness of the ARA graph is achieved at the price of these punctured variable nodes which act as auxiliary nodes that enlarge the H used by the decoder. The iterative graph-based ARA decoder thus has to deal with a redundant representation of the code, implying a larger $\mathbf{H}$ matrix than the nominal $(n-k)\times n$. This issue, together with the presence of a large number of degree-1 and degree-2 variable nodes, results in slow decoding convergence.

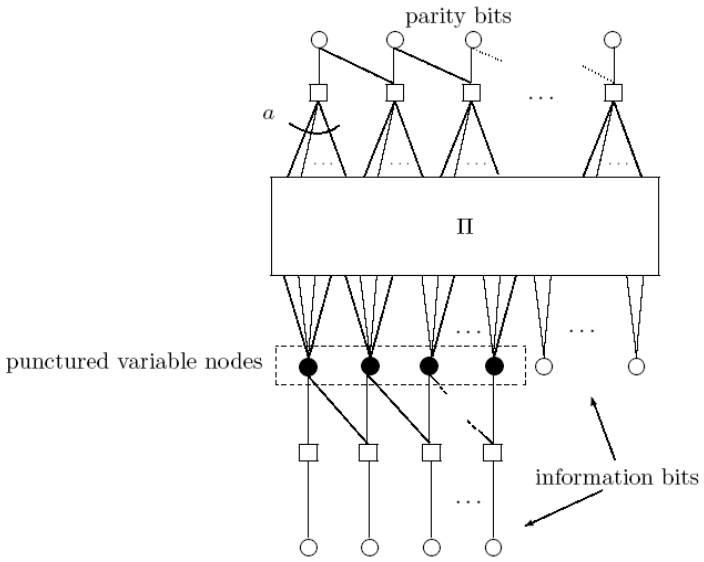

Figure 4.8.27

The ARA codes presented in [Abbasfar-comm07] relies on very simple protographs. Several modified ARA protographs have been introduced in [Divsalar05], leading to ARA and ARA-like code families with excellent performance in both the waterfall and floor regions of the codes' performance curves. The protograph of a rate-1/2 ARA code ensemble with repetition rate 4, denoted AR4A, is depicted in Fig. 4.8.28. The dark circle corresponds to a state-variable node, and it is associated with the precoded fraction of the information bits. As emphasized in the figure, such a protograph is the serial concatenation of an accumulator protograph and an IRA protograph. Half (node 2) of the information bits are sent directly to the IRA encoder, while the other half (node 1) is first precoded by the outer accumulator. This encoding procedure corresponds to a systematic code.

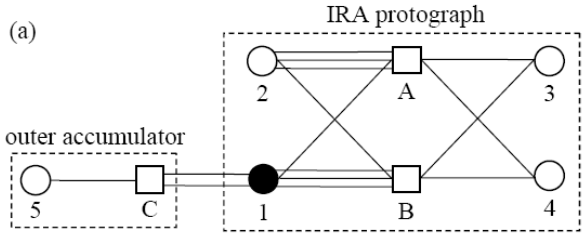

Figure 4.8.28

References

1. Y. Zhang and W. E. Ryan, "Structured IRA Codes: Performance Analysis and Construction," *IEEE Trans. Commun.*, vol.55, no.5, pp.837-844, May 2007.
2. W. E. Ryan and Shu. Lin, 信道编码：经典与现代. 白宝明，马啸译. 北京: 电子工业出版社，2017。