State Key Laboratory of Integrated Services Networks

Polar Codes

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Outline

- Introduction
- Channel Polarization
- Construction
- Encoding & Decoding
- Performance
- Open Problems

Introduction

- Polar codes, invented by Erdal Arikan in 2008, are the first codes to provably achieve capacity.

- 特点：
  - Capacity-achieving for symmetric binary-input memoryless channels (包括BI-AWGN, BSC, BEC)
  - Low encoding and decoding complexity: $O(N \log N)$
  - Block error probability is roughly $O(2^{-\sqrt{N}})$

    And this performance guarantee is analytical.
  - For symmetric channels, code construction is deterministic. That is, the above statements are true not only for ensembles of codes, but also for individual polar codes.

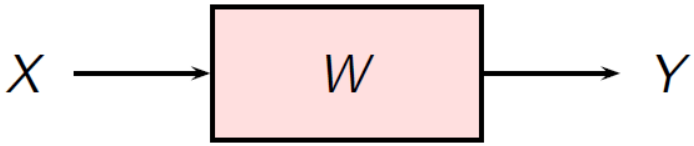
核心思想

- **Channel Polarization**
  - Create extreme channels: Either noiseless or useless.
- Channel coding problem trivial for these two types of channels
  - Perfect channels: $C(W)=1$, the output Y determines the input X
  - Useless channels: $C(W)=0$, the output Y is independent of the input X
- Channel polarization is a technique to convert any binary-input channel into a mixture of binary-input extreme channels

- 极化方法
  - 信道合并与分裂（由编译码共同完成）

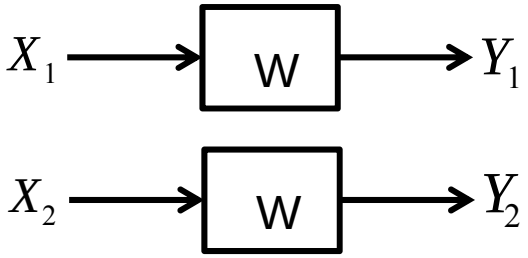
The Channel

- Let $W: \mathcal{X} \to \mathcal{Y}$ be a binary-input discrete-memoryless channel



  - Input alphabet: $\mathcal{X} = \{0,1\}$
  - Transition probabilities: $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$

- Two independent uses of the channel *W*

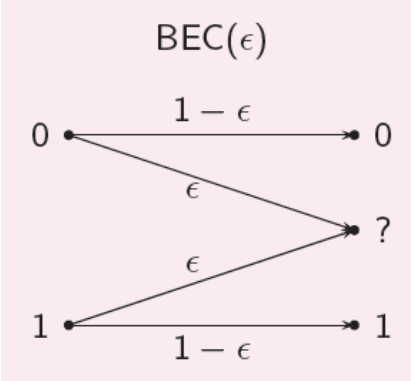


- *N* independent copies of *W*: $W^N: X^N \to Y^N$

$$W^N(y_1^N \mid x_1^N) = \prod_{i=1}^{N} W(y_i \mid x_i)$$

Symmetry assumption and Capacity

- Assume that the channel has "input-output symmetry." (Example: BSC, BEC)



- (Symmetric) Capacity

  For channels with I/O symmetry, the capacity is given by

$$C(W) \triangleq I(X;Y) \qquad \text{with } X \sim \text{unif. } \{0,1\}$$

$$= \sum_{y \in Y} \sum_{x \in X} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}$$

Use base-2 logarithms:

$$0 \le C(W) \le 1$$

Symmetry assumption and Capacity

- Bhattacharyya parameter

$$Z(W) \triangleq \sum_{y \in Y} \sqrt{W(y|0)W(y|1)}$$

- The parameters *C*(W) and *Z*(W) are used as measures of rate and reliability.

- Define the decision region for message *m*, $\mathcal{D}_m = \left\{ \mathbf{y} \in \mathcal{A}_Y^N \mid f(\mathbf{y}) = m \right\}$
- For codes with two codewords, the error probability, when message 2 is transmitted, is $P_B(e|2) = \sum_{\mathbf{y} \in \mathcal{D}_1} P_N(\mathbf{y}|\mathbf{x}_2)$
- For $\mathbf{y} \in \mathcal{D}_1$, $P_N(\mathbf{y}|\mathbf{x}_1) \geq P_N(\mathbf{y}|\mathbf{x}_2)$ for ML decoding
- 经过简单处理，有 $P_B(e|2) \leq \sum_{\mathbf{y} \in \mathcal{D}_1} \sqrt{P_N(\mathbf{y}|\mathbf{x}_1)P_N(\mathbf{y}|\mathbf{x}_2)}$
- 进而 $P_B(e|m) \leq \sum_{\mathbf{y}} \sqrt{P_N(\mathbf{y}|\mathbf{x}_1)P_N(\mathbf{y}|\mathbf{x}_2)}$

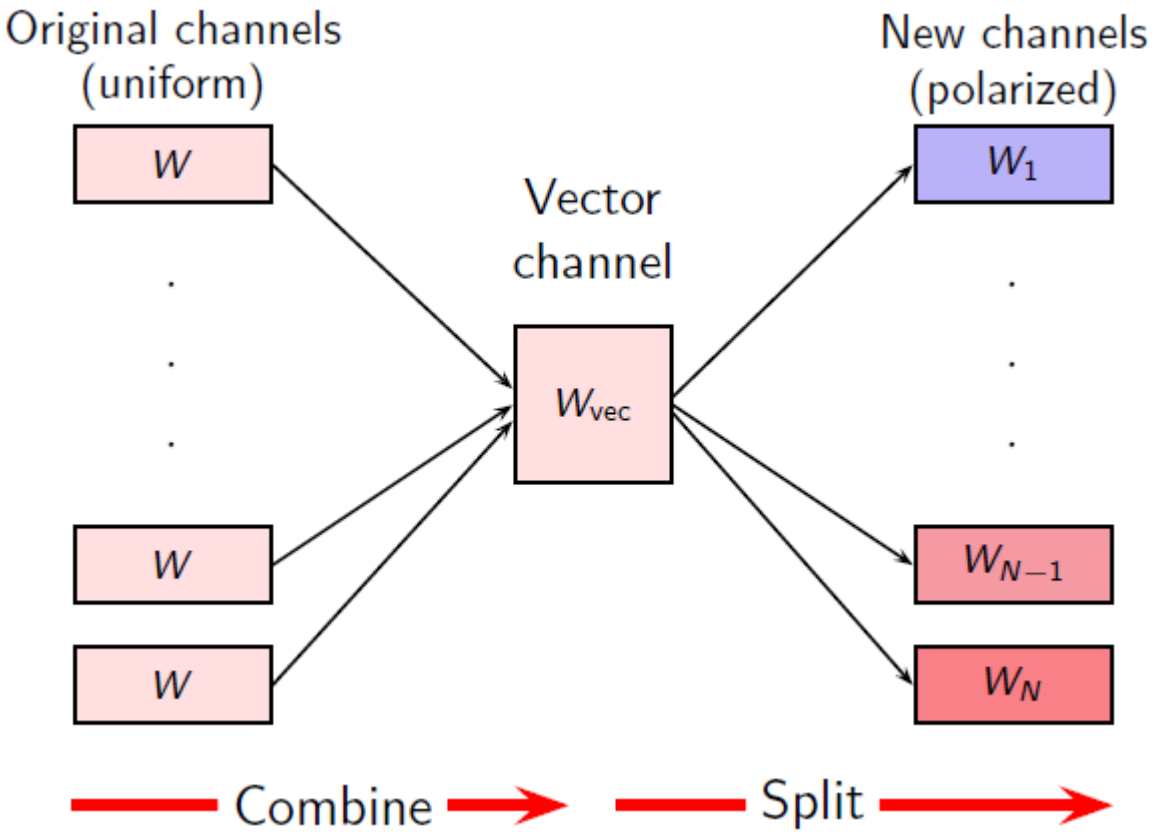
known as the *Bhattacharyya bound* on error probability

Main idea for channel polarization

- Channel coding problem trivial for two types of channels
  - Perfect: $C(W) = 1$
  - Useless: $C(W) = 0$
- Transform ordinary W into such extreme channels
  - Perfect channels: C(W)=1, the output Y determines the input X
  - Useless channels: C(W)=0, the output Y is independent of the input X
- Channel polarization is a technique to convert any binary-input channel into a mixture of binary-input extreme channels

极化方法

- Aggregate and redistribute capacity：信道合并与分裂



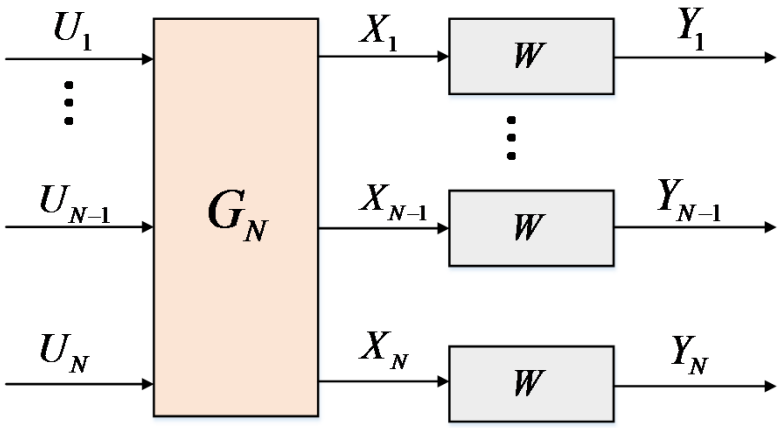
Channel Combining

- Begin with *N* copies of *W*
  - Use a 1-1 mapping

$$G_N : \{0,1\}^N \to \{0,1\}^N$$

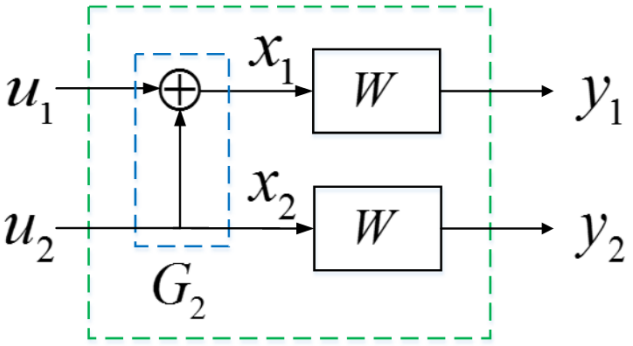
  - to create a vector channel

$$W_N : U^N \to Y^N$$



- Combining operation is lossless:
  - Let $U_1,...,U_N$ i.i.d. $\sim$ Unif. $\{0,1\}$ then $X_1,...,X_N$ i.i.d. $\sim$ Unif. $\{0,1\}$
  - and

$$C(W_N) = I(U^N;Y^N)$$
$$= I(X^N;Y^N)$$
$$= NC(W)$$



Channel Splitting

- $W_N \rightarrow W_N^{(i)} \qquad (1 \le i \le N)$

$$C(W_N) = I(U^N; Y^N)$$

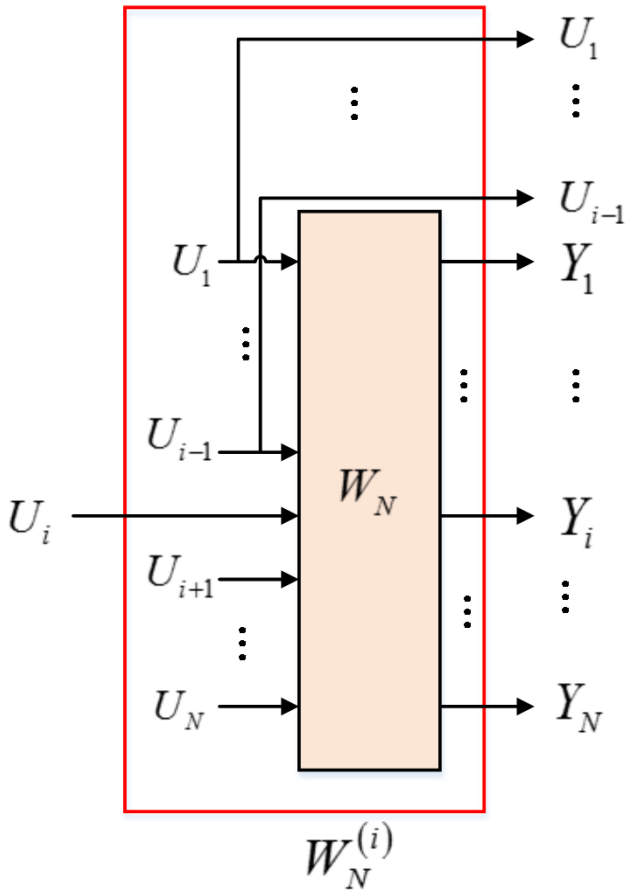
$$= \sum_{i=1}^{N} I(U_i; Y^N, U_1^{i-1})$$

$$= \sum_{i=1}^{N} C(W_N^{(i)})$$

- Define bit-channels

$$W_N^{(i)} : U \rightarrow Y^N \times U^{i-1}$$

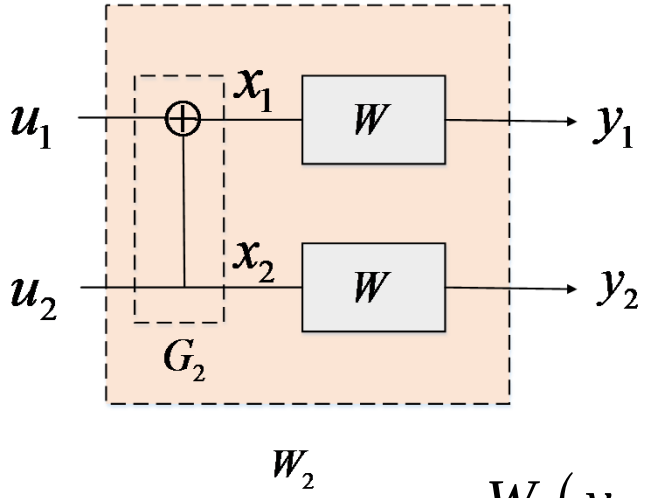
$$W_N^{(i)}(y_1^N, u_1^{i-1} \mid u_i) = \sum_{u_{i+1}^N \in X^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N \mid u_1^N)$$



- Successive-cancellation (SC)译码结构建立信道分裂

How to polarize: Basic module

- Combine two copies of *W*



$$G_2 \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$x_1^2 = u_1^2 \boldsymbol{G}_2$$

$$W_2(y_1, y_2 \mid u_1, u_2) = W(y_1 \mid u_1 \oplus u_2) W(y_2 \mid u_2)$$

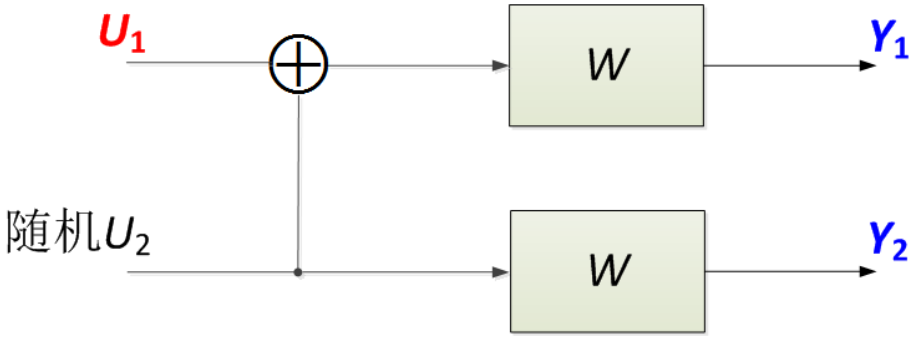
- and split to create two bit-channels

$$W^- : U_1 \to (Y_1, Y_2)$$
$$W^+ : U_2 \to (Y_1, Y_2, U_1)$$

- The bad channel W^-

$$W^-: U_1 \to (Y_1, Y_2)$$

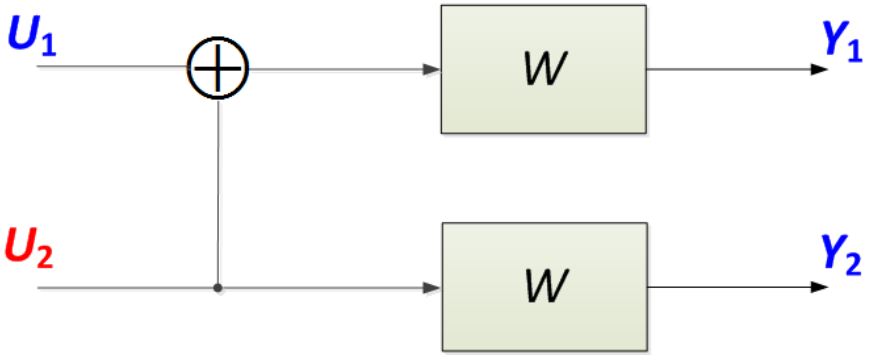
$$C(W^-) = I(U_1; Y_1, Y_2)$$



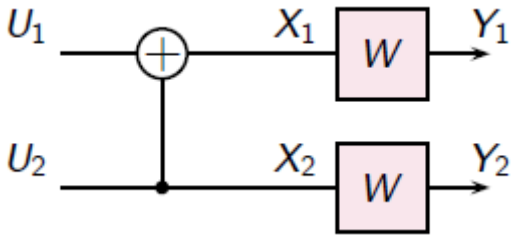
- The good channel W^+

$$W^+: U_2 \to (Y_1, Y_2, U_1)$$

$$C(W^+) = I(U_2; Y_1, Y_2, U_1)$$



- 信道总容量保持不变，但是被不均等分配了



- Conservation:

$$2C(W) = I(X_1X_2;Y_1Y_2) = I(U_1U_2;Y_1Y_2)$$
$$= I(U_1;Y_1Y_2) + I\left(U_2;Y_1Y_2\,|U_1\right)$$
$$= I(U_1;Y_1Y_2) + I\left(U_2;Y_1Y_2U_1\right) = C(W^-) + C(W^+)$$

- Extremization:

$$C(W^-) \le C(W) \le C(W^+)$$

with equality iff C(W) = 0 or 1.

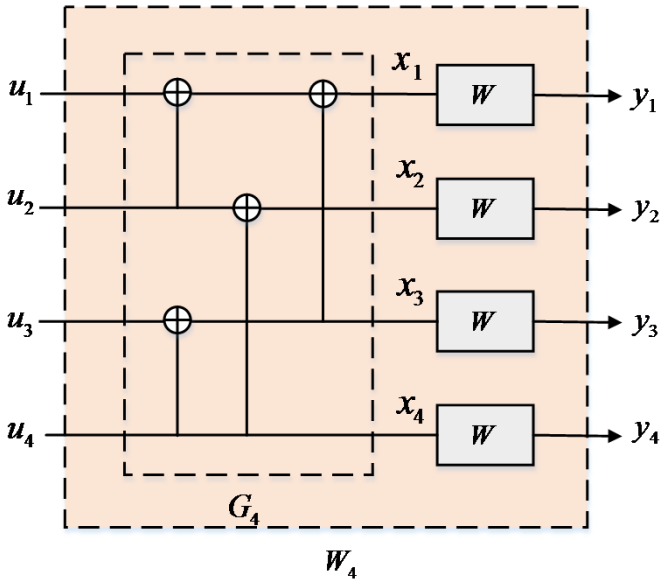
- **Problem**
  - The channel $W^+ : U_2 \to (Y_1, Y_2, U_1)$ poses the problem: Y1Y2 is observed by the receiver, but U1 is not observed by the receiver.
  - We can synthesize W+ by first processing the output of W- to estimate U1. We have $\hat{U}_1$ instead of U1 for a implementable receiver.
- **Implementable receiver**

$$\hat{U}_1 = \phi_1(Y^2)$$

$$\hat{U}_2 = \phi_2(Y^2 \hat{U}_1)$$

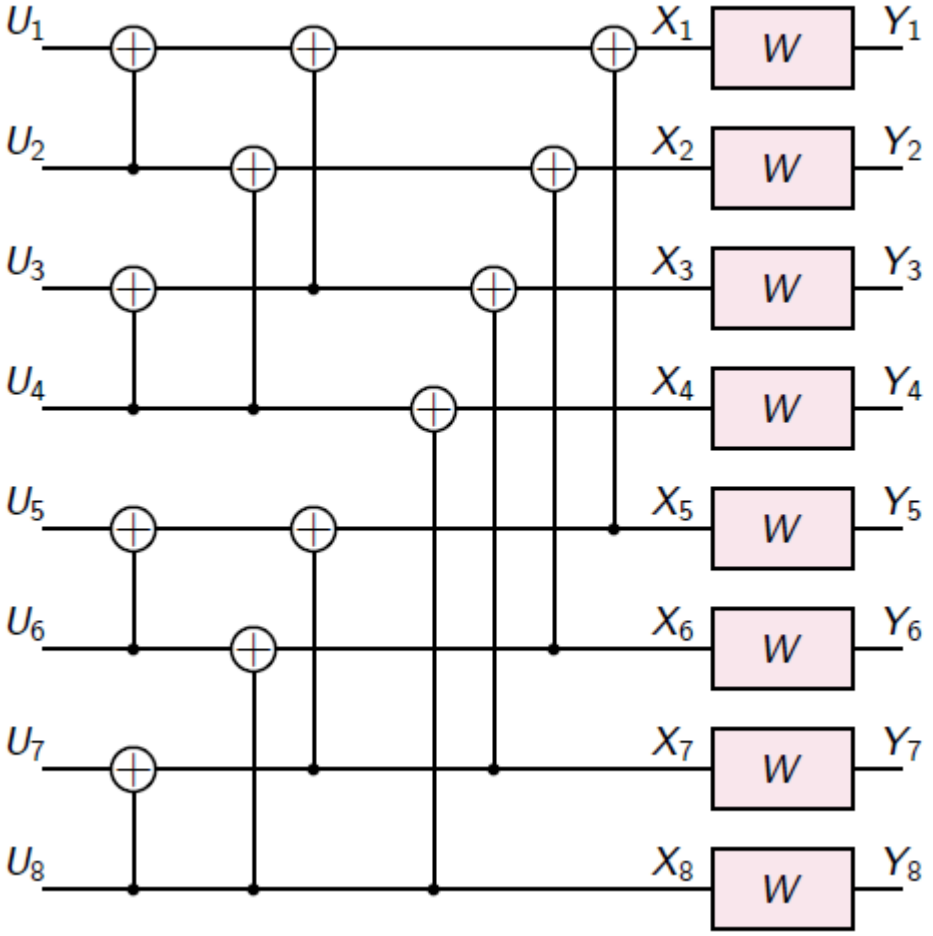
  - Successively decode U1, U2.

Size 4 and 8 constructions

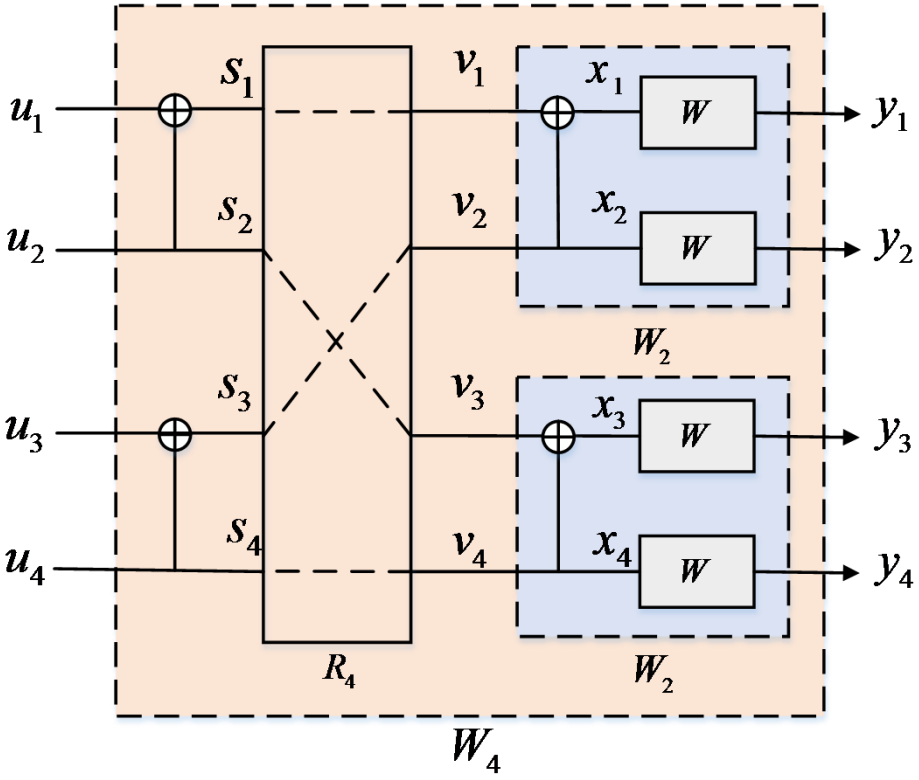


$$G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$X_1^4 = U_1^4 G_4$$



Size 4 construction：奇偶分离



$$G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$x_1^4 = u_1^4 G_4$$

$$W_4(y_1^4 \mid u_1^4) = W_2(y_1^2 \mid u_1 \oplus u_2, u_3 \oplus u_4) W_2(y_3^4 \mid u_2, u_4)$$

Recursive construction of W_N

$$x_1^N = u_1^N G_N \qquad N = 2^n$$

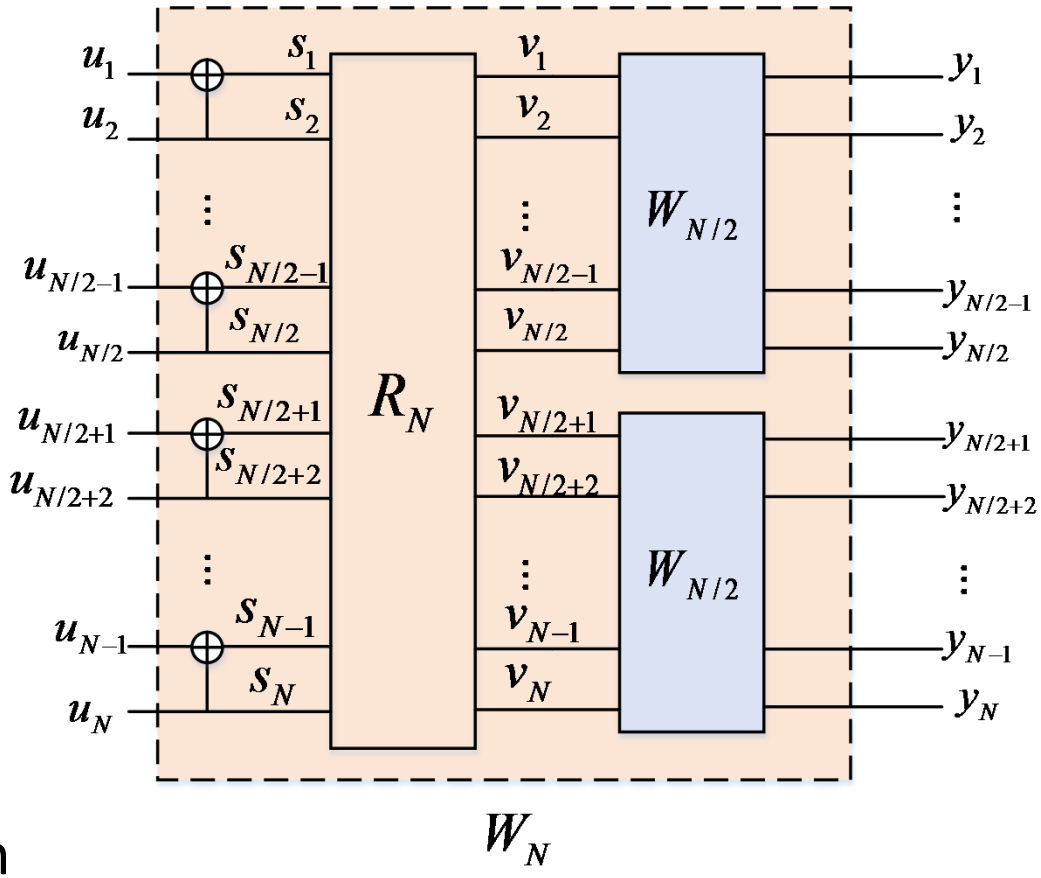
$$G_N = (I_{\frac{N}{2}} \otimes F) R_N (I_2 \otimes G_{\frac{N}{2}})$$

$$= B_N F^{\otimes n}$$

- here

$$F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\otimes$ **Kronecker Product**



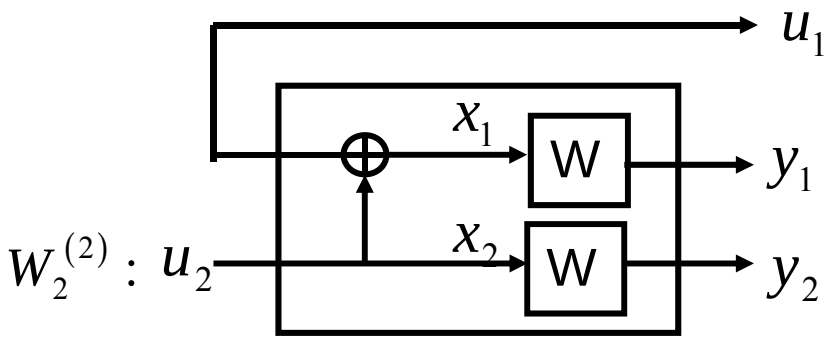
R_N : reverse shuffle operation

B_N : bit-reversal operator

$$\text{if } v^N = u^N B_N, \text{ then } v_{b_1 \ldots b_n} = u_{b_n \ldots b_1}$$

（奇偶）信道的转移概率

$$W_N^{(i)} : U \to Y^N \times U^{i-1}$$



W_2

$$W_N^{(i)}(y_1^N, u_1^{i-1} \mid u_i) = \sum_{u_{i+1}^N \in X^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N \mid u_1^N)$$

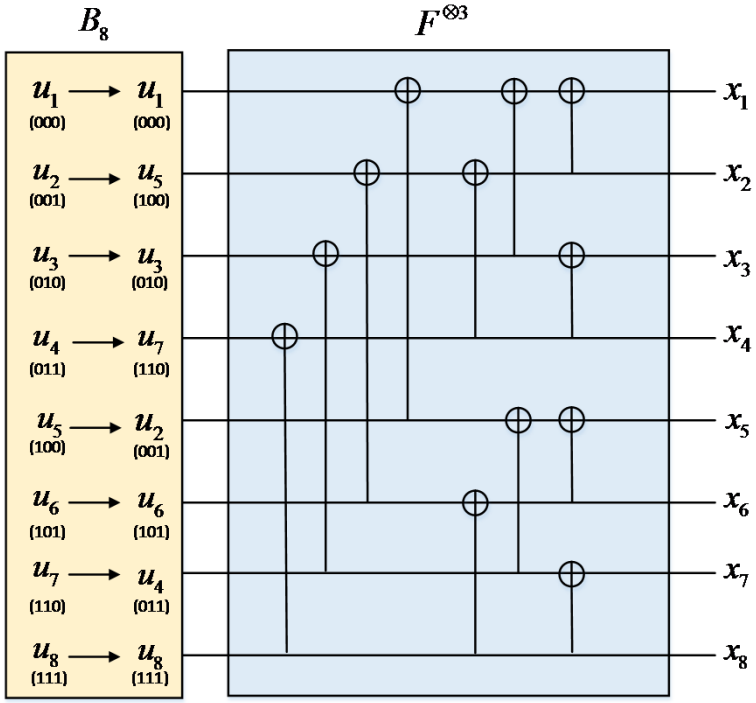
$$W_{2N}^{(2i-1)}(y_1^{2N}, u_1^{2i-1} \mid u_{2i-1}) = \sum_{u_{2i}} \frac{1}{2} W_N^{(i)}(y_1^N, u_{1,o}^{2i-2} \oplus u_{1,e}^{2i-2} \mid u_{2i-1} \oplus u_{2i}) \cdot$$

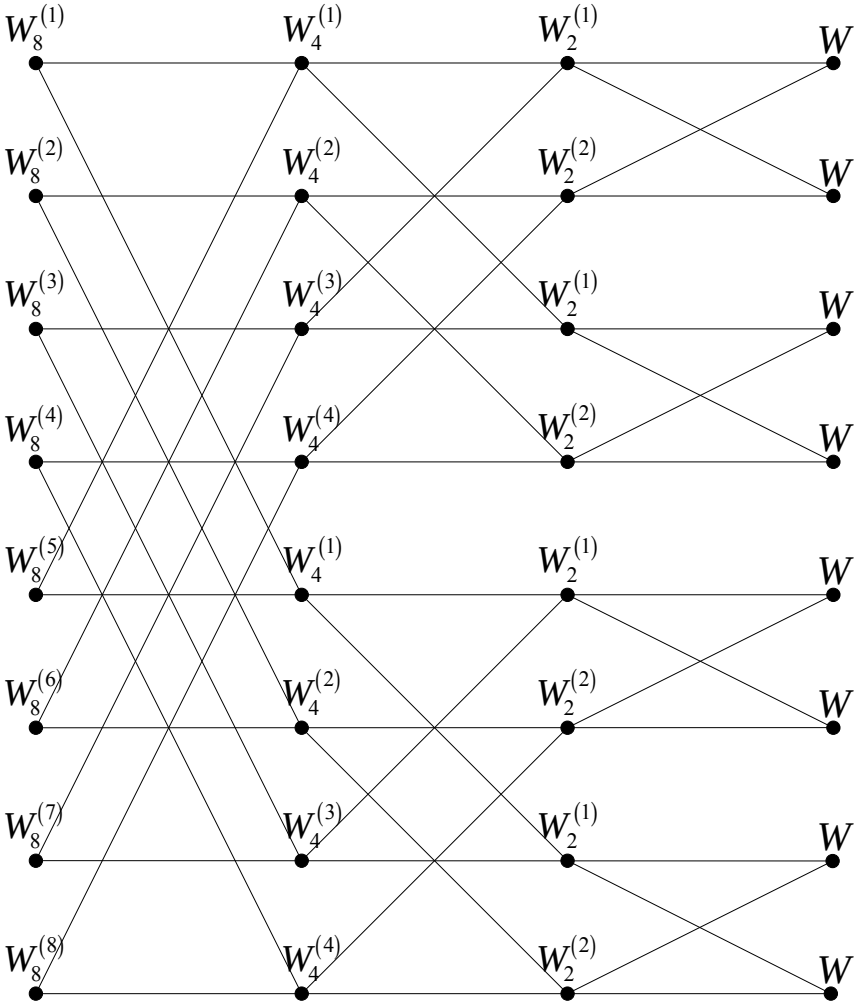
$$W_N^{(i)}(y_{N+1}^{2N}, u_{1,e}^{2i-2} \mid u_{2i})$$

$$W_{2N}^{(2i)}(y_1^{2N}, u_1^{2i-1} \mid u_{2i}) = \frac{1}{2} W_N^{(i)}(y_1^N, u_{1,o}^{2i-2} \oplus u_{1,e}^{2i-2} \mid u_{2i-1} \oplus u_{2i}) \cdot$$

$$W_N^{(i)}(y_{N+1}^{2N}, u_{1,e}^{2i-2} \mid u_{2i})$$

一般表示： Butterfly结构





$$G_N = B_N F^{\otimes n} \qquad F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

B_N: bit-reversal operator

$$\text{if } v^N = u^N B_N, \text{ then } v_{b_1 \ldots b_n} = u_{b_n \ldots b_1}$$

Channel polarization

- When $N \to \infty$
  - the fraction of indices i for $C(W_N^{(i)}) \in (1-\delta, 1] \to C(W)$
  - the fraction of indices i for $C(W_N^{(i)}) \in [0, \delta) \to 1 - C(W)$
- Theorem (Polarization, A. 2007)

  Under the above polarization scheme, almost all channels become either perfect or completely useless as the construction size grows. For any $\delta \in (0,1)$, as the construction size N grows,

  $$[\text{fraction of (good) channels with } C(W_i) > (1-\delta)\,] \to C(W)$$

  and

  $$[\text{fraction of (bad) channels with } C(W_i) < \delta\,] \to (1 - C(W)).$$

  Theorem (Rate of polarization, A. and Telatar (2008))

  Above theorem holds with $\delta \approx 2^{-\sqrt{N}}$.

Example: BEC

- Assume that W is a BEC with the erasure probability $\varepsilon = 0.5$

- Because of the recursive construction of the BEC channels, the parameters $C(W_N^{(i)})$ can be computed through the recursion

$$C(W_N^{(2i-1)}) = \left(C(W_{N/2}^{(i)})\right)^2$$

$$C(W_N^{(2i)}) = 2\left(C(W_{N/2}^{(i)})\right) - \left(C(W_{N/2}^{(i)})\right)^2$$

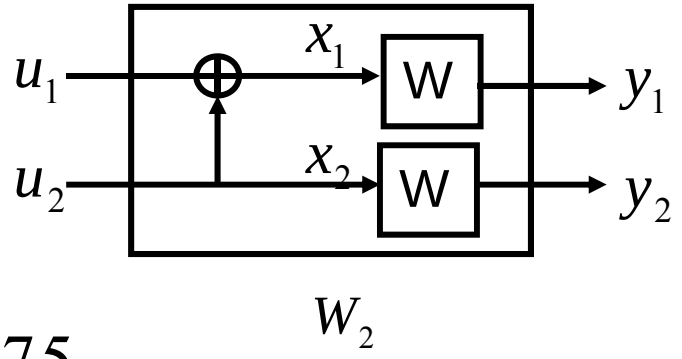
BEC

- For N=2,

$$C(W_1^{(1)}) = 1-\varepsilon = 0.5$$

$$C(W_2^{(1)}) = (1-\varepsilon)^2 = 0.25$$

$$C(W_2^{(2)}) = 2(1-\varepsilon)-(1-\varepsilon)^2 = 0.75$$



Example: BEC

- For N=4,

$$C(W_1^{(1)}) = 1-\varepsilon = 0.5$$

$$C(W_2^{(1)}) = C(W_1^{(1)})^2 = 0.25$$

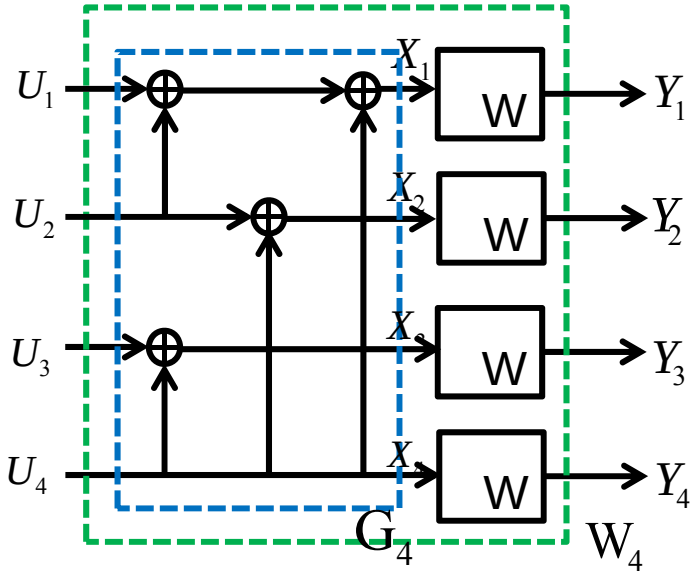
$$C(W_2^{(2)}) = 2C(W_1^{(1)}) - C(W_1^{(1)})^2 = 0.75$$

$$C(W_4^{(1)}) = C(W_2^{(1)})^2 = 0.0625$$

$$C(W_4^{(2)}) = 2C(W_2^{(1)}) - C(W_2^{(1)})^2 = 0.4375$$

$$C(W_4^{(3)}) = C(W_2^{(2)})^2 = 0.5625$$

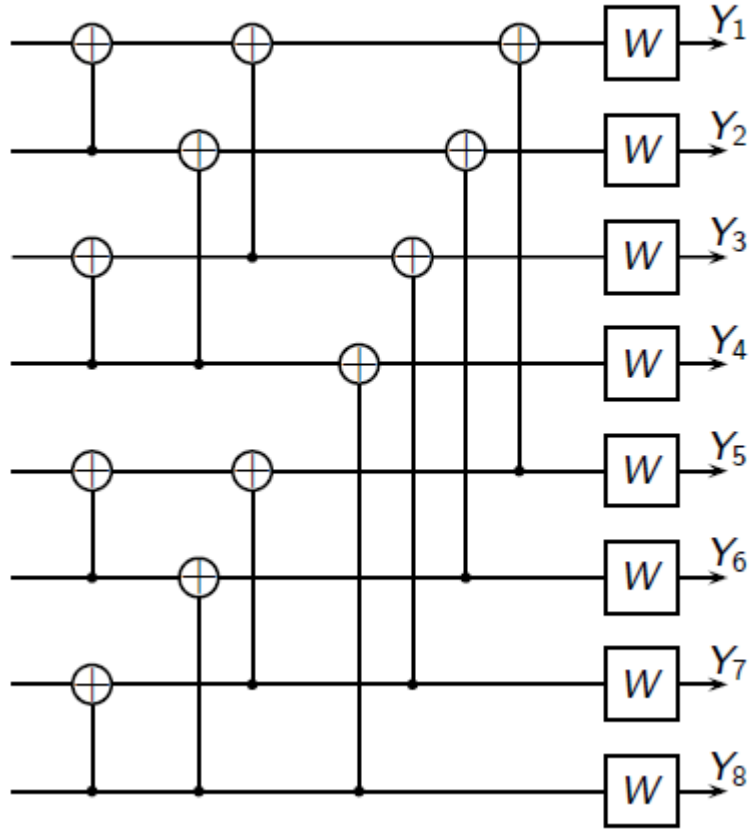
$$C(W_4^{(4)}) = 2C(W_2^{(2)}) - C(W_2^{(2)})^2 = 0.9375$$



BEC

- For N=8, rate=1/2

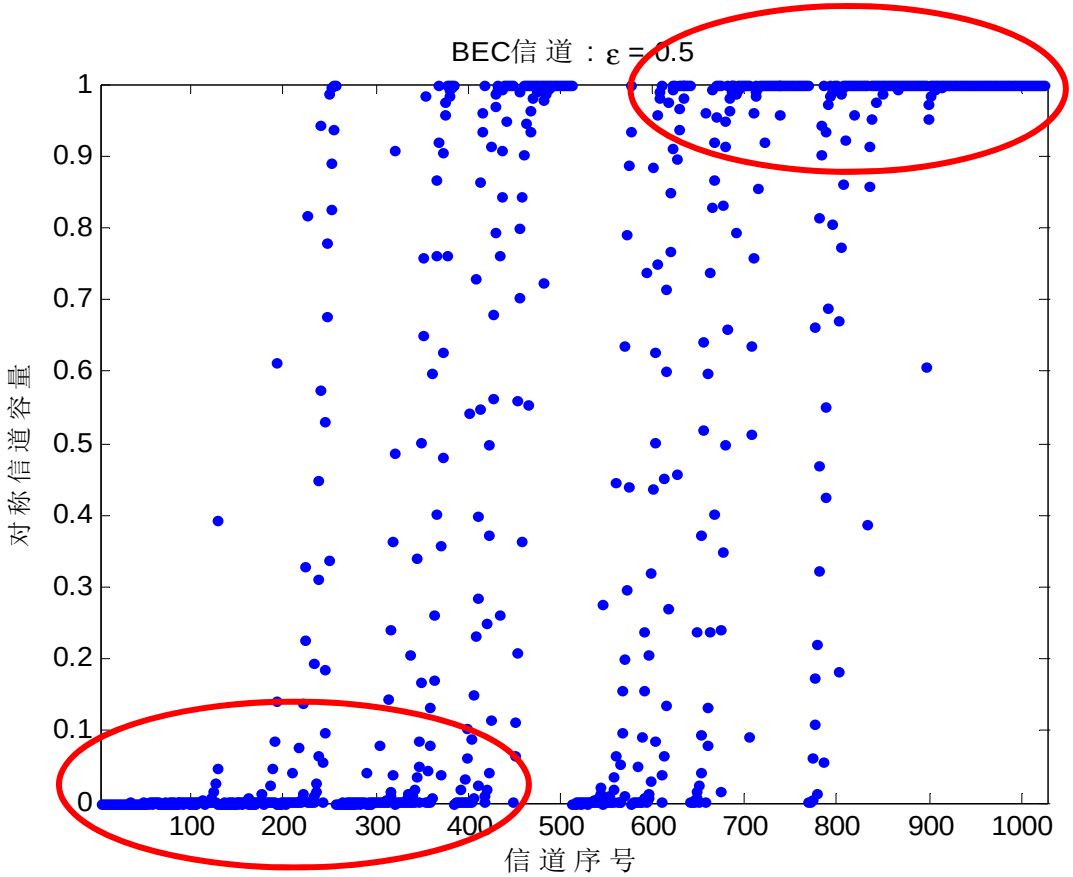
$C(W_i)$	Rank		
0.0039	8	frozen	0
0.1211	7	frozen	0
0.1914	6	frozen	0
0.6836	4	data	U_4
0.3164	5	frozen	0
0.8086	3	data	U_6
0.8789	2	data	U_7
0.9961	1	data	U_8



BEC

- ***N*=1024**

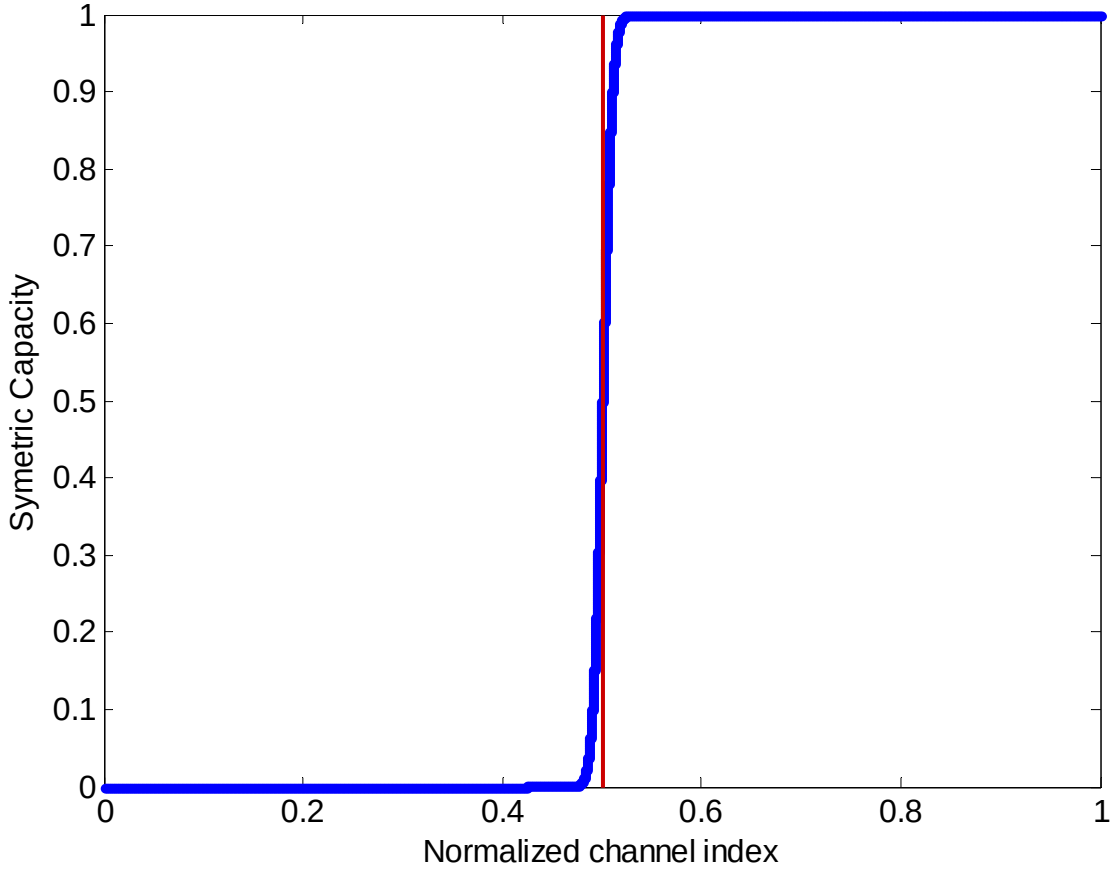
Information bits



Frozen bits

BEC

$N=2^{20}$


Symetric Capacity
Normalized channel index
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

Encoding

Notation

$$x_1^N = u_1^N G_N \qquad G_N\text{-coset code}$$

$$x_1^N = u_{\mathrm{A}} G_N(\mathrm{A}) \oplus u_{\mathrm{A}^c} G_N(\mathrm{A}^c)$$

Codeword: x_1^N

Information bits: $u_{\mathrm{A}} \in \mathcal{X}^K$ **Frozen bits:** $u_{\mathrm{A}^c} \in \mathcal{X}^{N-K}$

Information set: $\mathrm{A} \subseteq \{1, 2, \cdots, N\}$

Code dimension: $K = |\mathrm{A}|$ **Code rate:** $R = K/N$

Individual G_N - coset code will be identified by a parameter vector $(N, K, \mathrm{A}, u_{\mathrm{A}^c})$

Complexity: $O(N \log N)$

Encoding

- For example

For symmetric channels, any choice for u_{A^c} is as good as any other.

$$(N, K, \mathrm{A}, u_{A^c}) = (4, 2, \{2, 4\}, (1, 0))$$

$$G_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$x_1^4 = u_1^4 G_4$$

$$x_1^N = (u_2, u_4)\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} + (1, 0)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$(u_2, u_4) = (1, 1)$$

$$x_1^4 = (1, 1, 0, 1)$$

Encoding

- **In practice,**

$$G_N = B_N F^{\otimes n} = F^{\otimes n} B_N \qquad F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\left(G_N\right)_{b_1 \ldots b_n, a_1 \ldots a_n} = \prod_{i=1}^{n} (1 \oplus a_i \oplus b_{n-i} a_i)$$

B_N : bit-reversal permutation

实现1：输入比特置换 $\tilde{\mathbf{u}} = \mathbf{u} B_N, \mathbf{x} = \tilde{\mathbf{u}} F^{\otimes n}$

实现2：输出比特置换 $\tilde{\mathbf{x}} = \mathbf{u} F^{\otimes n}, \mathbf{x} = \tilde{\mathbf{x}} B_N$

Construction

- **How to determine the frozen set ?**

  In principle, it can be solved by computing all the parameters $\left\{Z\left(W_N^{(i)}\right) \text{ or } C(W_N^{(i)}): 1 \le i \le N\right\}$ and sorting them.

- **For BEC**

$$C(W_1^{(1)}) = 1 - \varepsilon$$

$$C(W_N^{(2i-1)}) = \left(C(W_{N/2}^{(i)})\right)^2$$

$$C(W_N^{(2i)}) = 2\left(C(W_{N/2}^{(i)})\right) - \left(C(W_{N/2}^{(i)})\right)^2$$

可根据删除概率$\mathcal{E}$精确求出各**bit-channel**的信道容量，经排序后确定**information set and frozen set**

Construction

For AWGN

- **Bhattacharyya Bounds (Heuristic BEC近似)**

  采用容量相等准则，变换到BEC近似计算，低复杂度，低精度；

- **Monte-Carlo Method**

  类似于发送训练序列的方法，高复杂度，精度跟传输遍数有关；

- **Density Evolution**

  跟踪bit-channel LLR概率分布，高复杂度，高精度；

- **Tal & Vardy's Estimation**

  夹逼近似计算bit-channel LLR概率分布，高复杂度，高精度；

- **Gaussian Approximation**

  将bit-channel LLR分布近似为高斯分布，只跟踪LLR分布均值，低复杂度，中高精度。

Construction

- Density evolution construction

  - Density evolution can be used to compute the probability density functions (PDF) $p_i(x)$ *of* $L_N^{(i)}(y_i^N, u_1^{i-1})$ from the PDF of $L_1^{(i)}(y_i)$.
  - The main drawback of the density evolution construction method is its high computational complexity.

$$L_N^{(i)}(y_1^N, \hat{u}_1^{i-1}) = \log \frac{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1} \mid u_i = 0)}{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1} \mid u_i = 1)}$$

$$L_N^{(2i-1)}(y_1^N, \hat{u}_1^{2i-2}) = 2\tanh^{-1}\left(\tanh\left(\frac{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2})}{2}\right) \times \tanh\left(\frac{L_{N/2}^{(i)}(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2})}{2}\right)\right)$$

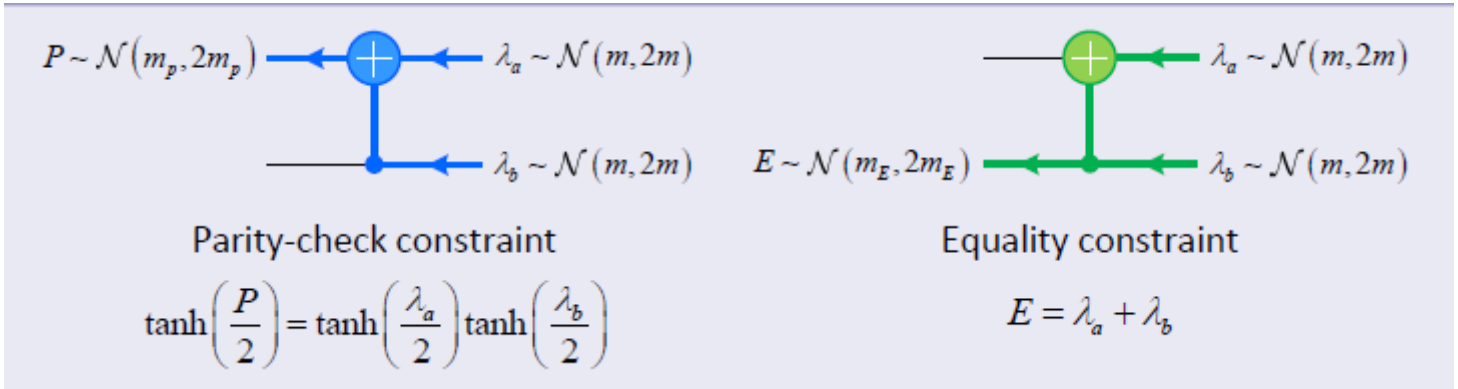
$$L_N^{(2i)}(y_1^N, \hat{u}_1^{2i-1}) = (-1)^{\hat{u}_{2i-1}} L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) + L_{N/2}^{(i)}(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2})$$

[1] R. Mori and T. Tanaka, “Performance of polar codes with the construction using density evolution,” *IEEE Communications Letters, vol. 13,* no. 7, July 2009.

Construction

Gaussian approximation construction

- As for the high complexity of density evolution, the most practically important case corresponding to the AWGN channel is considered. Assume that the all-zero codeword is transmitted. Then $y \sim \mathcal{N}(1, \sigma^2)$. In this scenario, $L_1^{(i)}(y_i) \sim \mathcal{N}(\frac{2}{\sigma^2}, \frac{4}{\sigma^2})$
- So $L_N^{(i)}$ can be considered as Gaussian random variables with $D[L_N^{(i)}] = 2E[L_N^{(i)}]$, where E and D are the mean and variance, respectively.
- This enables one to compute only the expected value of $L_n^{(i)}$.



[1] P. Trifonov, “Efficient design and decoding of polar codes,” *IEEE Transactions on Communications*, vol. 60, no. 11, pp. 1–7, November 2012.

Construction

Gaussian approximation construction

- For SPC constraint:

$$\mathbb{E}\left[\tanh\left(\frac{P}{2}\right)\right]=\mathbb{E}\left[\tanh\left(\frac{\lambda_a}{2}\right)\right]\cdot\mathbb{E}\left[\tanh\left(\frac{\lambda_b}{2}\right)\right]$$

- Then we have $m_p=\phi^{-1}\left(1-(1-\phi(m))^2\right)$, where

$$\phi(x)=\begin{cases}1-\frac{1}{\sqrt{4\pi x}}\int_{\mathbb{R}}\tanh\left(\frac{u}{2}\right)e^{-\frac{(u-x)^2}{4x}}du & x>0,\\ 1 & x=0.\end{cases}$$

- Using a two-segment approximation function [Chung *et al*. IT2001], GA construction is

$$E\left[L_N^{(2i-1)}\right]=\phi^{-1}\left(1-\left(1-\phi\left(E\left[L_{N/2}^{(i)}\right]\right)\right)^2\right)$$

$$E\left[L_N^{(2i)}\right]=2\,E\left[L_{N/2}^{(i)}\right]$$

$$\phi(x)\approx\varphi(x)=\begin{cases}e^{-0.4527x^{0.86}+0.0218} & 0<x<10,\\ \sqrt{\frac{\pi}{x}}e^{-\frac{x}{4}}\left(1-\frac{10}{7x}\right) & x\geq 10.\end{cases}$$

Construction

Gaussian approximation construction

The basic idea is to estimate all the log-likelihood ratios (LLR's) at intermediate stages as Gaussian variables. Only calculate the mean and variance of the LLRs at each stage of decoding.

(rely on construction SNR)

$S = 10^{EdB/10}$ and $n = \log_2 N$

$\mathbf{z}^{(3)} \in \mathbb{R}^N$, initialize $\mathbf{z}^{(3)}[0] = 4S$

for $j = 1 : n$ **do** ▷ for each stage in Fig. 1, right-to-left

  $u = 2^j$

  for $t = 0 : \frac{u}{2} - 1$ **do**

    $T = \mathbf{z}^{(3)}[t]$

    $\mathbf{z}^{(3)}[t] = \phi^{-1}\left(1 - (1 - \phi(T))^2\right)$ ▷ Upper channel

    $\mathbf{z}^{(3)}[u/2 + t] = 2T$ ▷ Lower channel

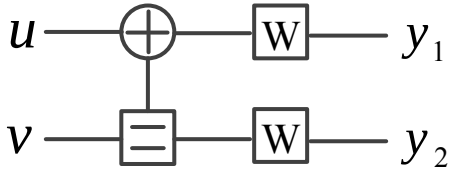
  end

end

$$\phi(x) \triangleq \begin{cases} \exp(-0.4527x^{0.86} + 0.0218) \text{ if } 0 < x \leq 10 \\ \sqrt{\frac{\pi}{2}}(1 - \frac{10}{7x})\exp(-x/4), \text{ if } x > 10 \end{cases}$$

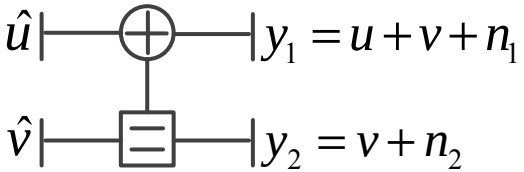
Polar Decoding

- 典型译码算法: **SC, SC-List, CRC-adied SCL,** BP, sphere dec
- **Successive Cancellation Decoder**

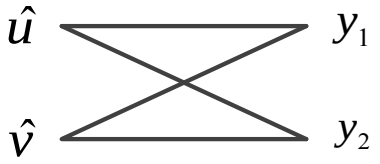


译码时从右往左递归

因子图



蝶形图

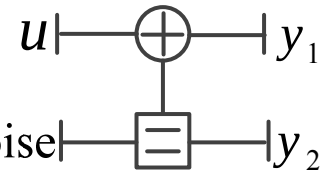


先译左上
再译左下

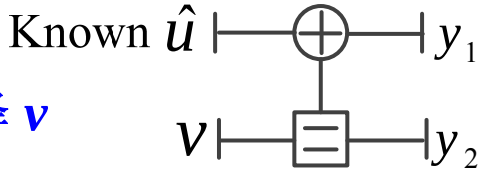
SC译码

1) 译 *u*

(将*v*看作噪声)



2) 译 *v*



- **Initialization**

$$L(y_1) = \frac{W(y_1 \mid 0)}{W(y_1 \mid 1)}, L(y_2) = \frac{W(y_2 \mid 0)}{W(y_2 \mid 1)}$$

- **Calculation of LR**

$$(1)\ L(u) = L(y_1^2) = \frac{L(y_1)L(y_2)+1}{L(y_1)+L(y_2)} \quad (v\text{看作噪声})$$

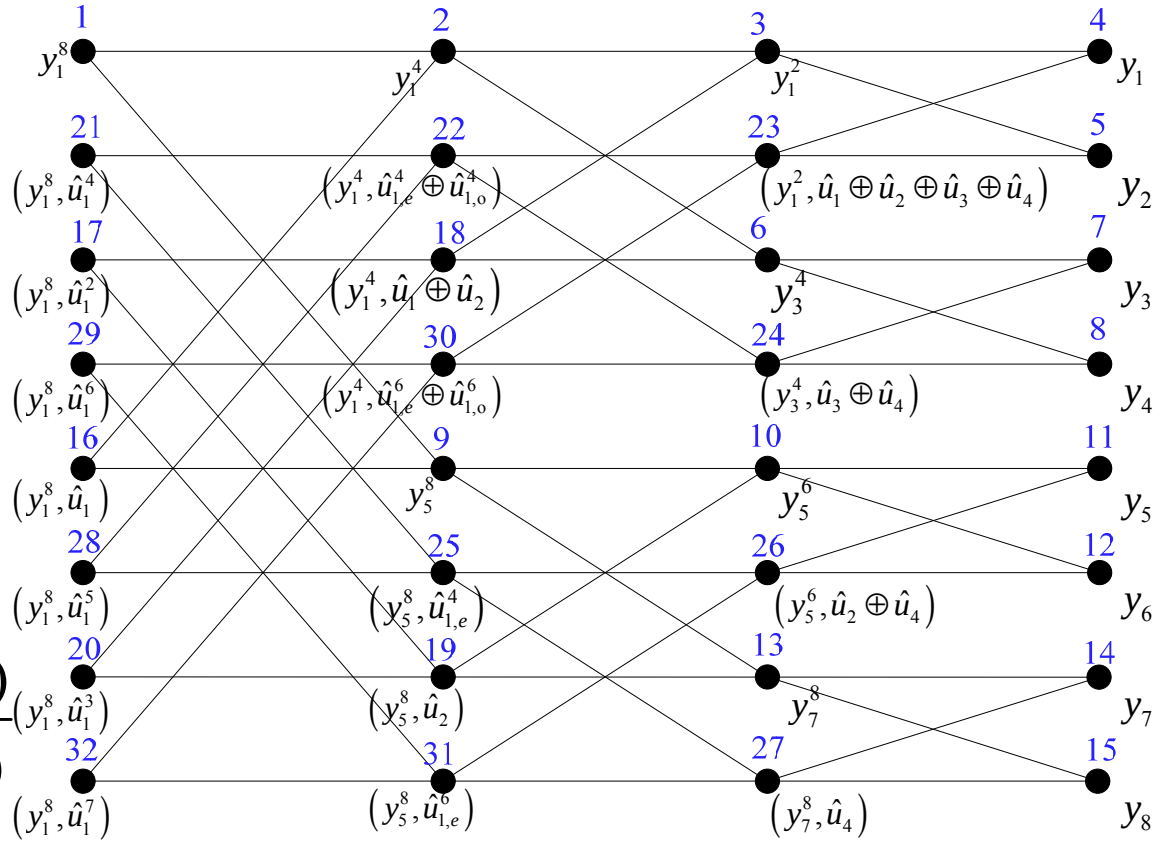
$$(2)\ L(v) = L(y_1^2, \hat{u}) = L(y_1)^{(1-2\hat{u})} \cdot L(y_2) \quad (\text{已知 } \hat{u})$$

Polar Decoding

Successive Cancellation Decoder

SC译码的核心是假设前一比特已成功译出，基于以前比特译当前比特。

译码结构如右图,最右边为接收向量，通过从右往左递归得到估计信息，即最左边为译码后的估计信息。



算法:

- 计算似然比

$$L(u_i)=\frac{W_N^{(i)}(y_1^N,\hat{u}_1^{i-1}\mid u_i=0)}{W_N^{(i)}(y_1^N,\hat{u}_1^{i-1}\mid u_i=1)}$$

- 判决

$$\tilde{u}_i=\begin{cases}0, & \text{if } L(u_i)\geq 1\\ 1, & \text{otherwise}\end{cases}\qquad \hat{u}_i=\begin{cases}u_i, & \text{if } i\in\mathcal{A}^c\\ \tilde{u}_i, & \text{if } i\in\mathcal{A}\end{cases}$$

Complexity: $O(N\log N)$

SC Decoder in the LR Domain

- **Initialization**

$$L_1^{(1)}(y_i) = \frac{W(y_i \mid 0)}{W(y_i \mid 1)}$$

- **Recursion**
  - For “模2和”节点：

$$L_N^{(2i-1)}(y_1^N, \hat{u}_1^{2i-2}) = \frac{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) L_{N/2}^{(i)}(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2}) + 1}{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) + L_{N/2}^{(i)}(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2})} \quad (1)$$

  - For “重复”节点：

$$L_N^{(2i)}(y_1^N, \hat{u}_1^{2i-1}) = [L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2})]^{1-2\hat{u}_{2i-1}} \cdot L_{N/2}^{(i)}(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2}) \quad (2)$$

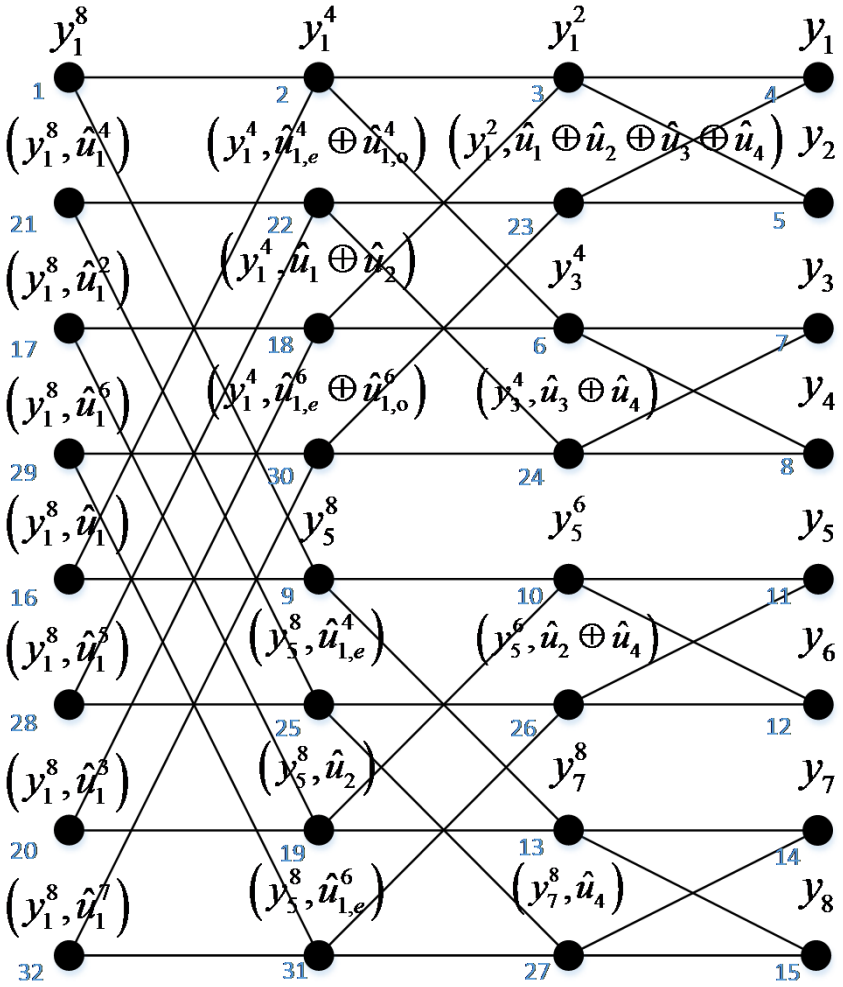
SC Decoder

Decoding structure

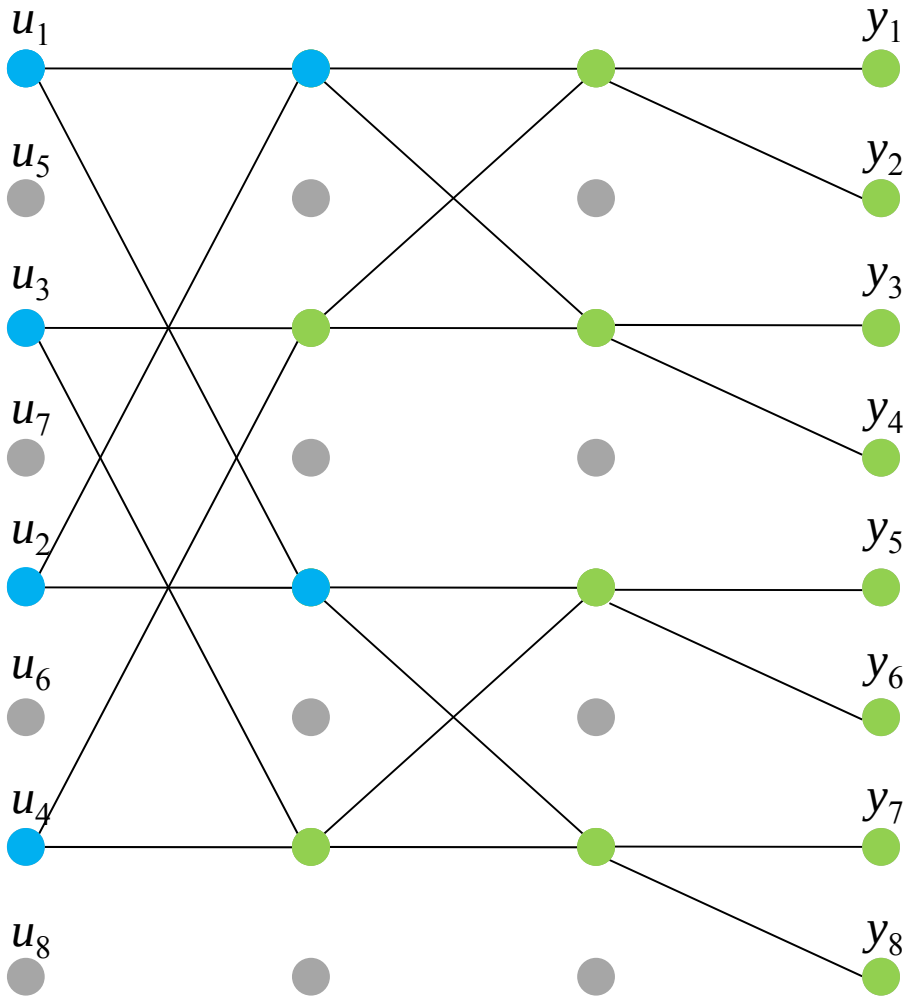
译码从左边开始，左边节点按自然顺序激活译码

从右往左递归，蝶形左上用公式**(1)**，左下用公式**(2)**。

树图：**1--(2,9)==(3,6;10,13)**



Recursive Decoding



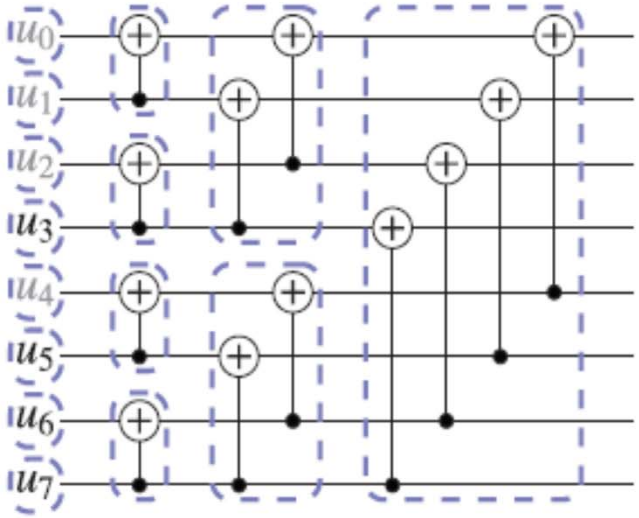
SC Decoding in the LLR Domain

- 定义

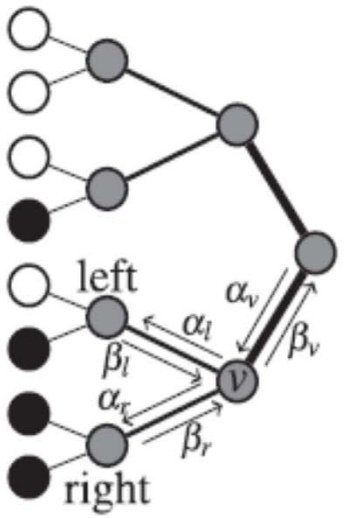
$$\lambda_{u_i} = \ln \frac{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1} \mid u_i = 0)}{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1} \mid u_i = 1)}$$

- **“模2加”节点：box-plus运算（min-sum）**
- **“等号”节点：LLR相加**

$$\begin{cases} f(\lambda_a, \lambda_b) & = & sign(\lambda_a.\lambda_b).\min(|\lambda_a|, |\lambda_b|) \\ g(\lambda_a, \lambda_b, \hat{s}_a) & = & (1-2\hat{s}_a)\lambda_a + \lambda_b \\ h(\hat{s}_a, \hat{s}_b) & = & (\hat{s}_a \oplus \hat{s}_b, \hat{s}_b) \end{cases}$$

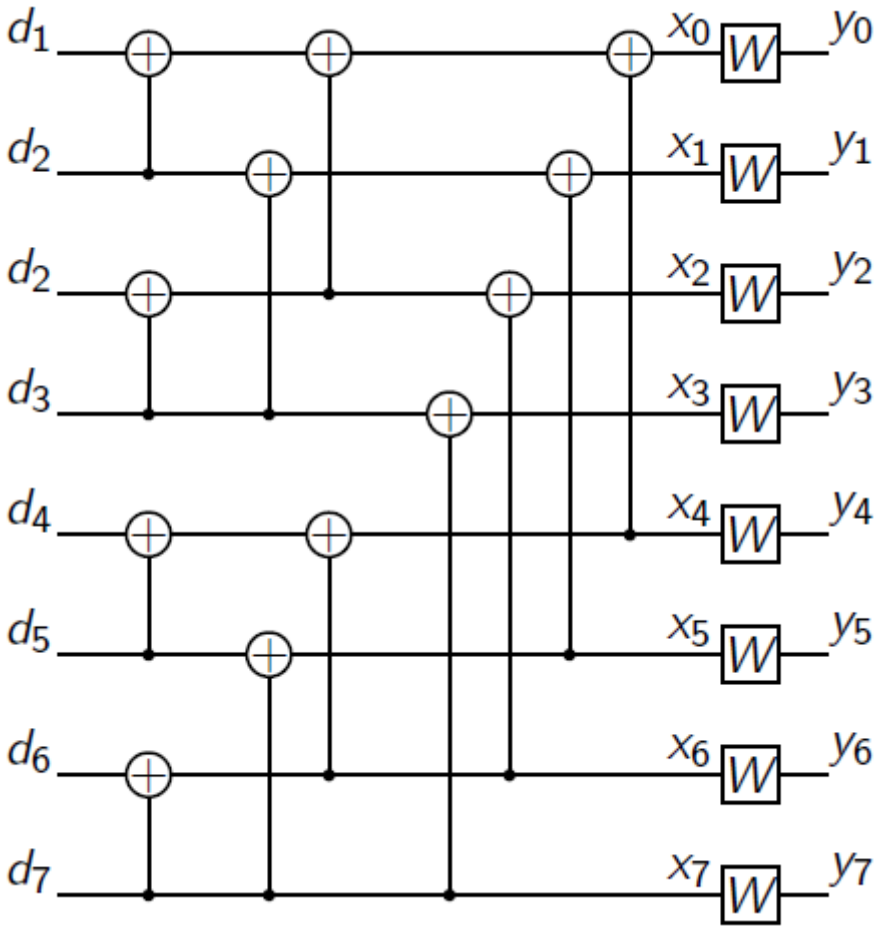


(a) Graph

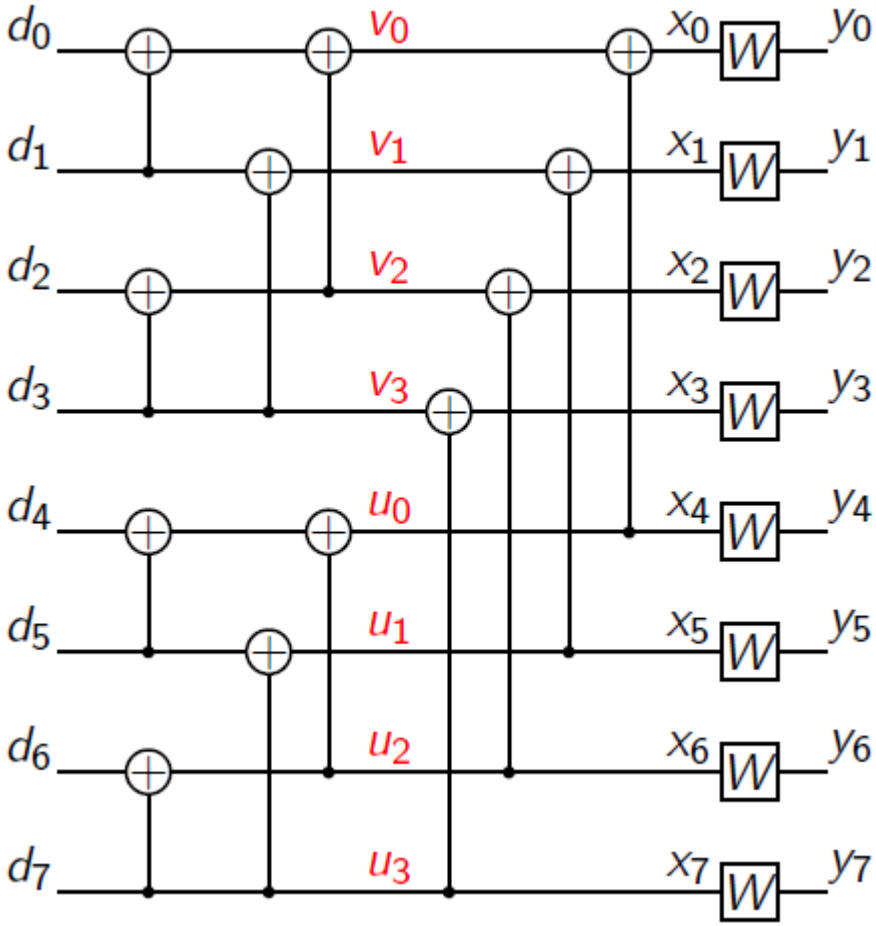


(b) Decoder tree

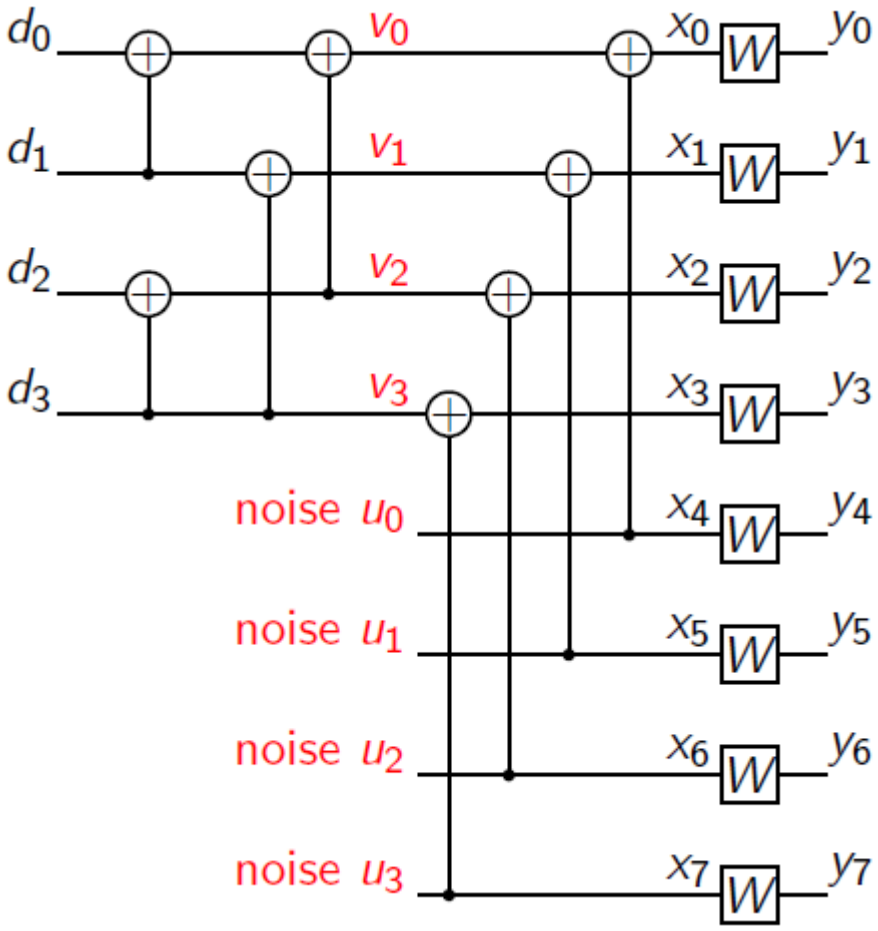
- Recursive decoding example: n = 8



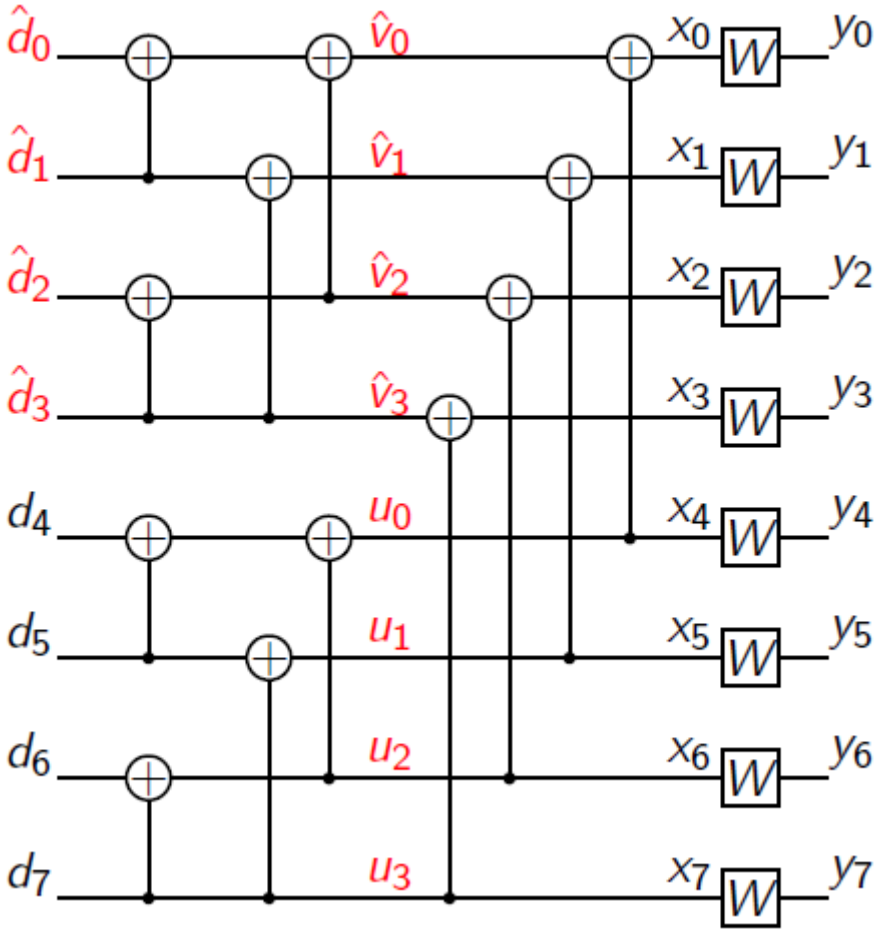
- Exploiting x = |u|u + v|



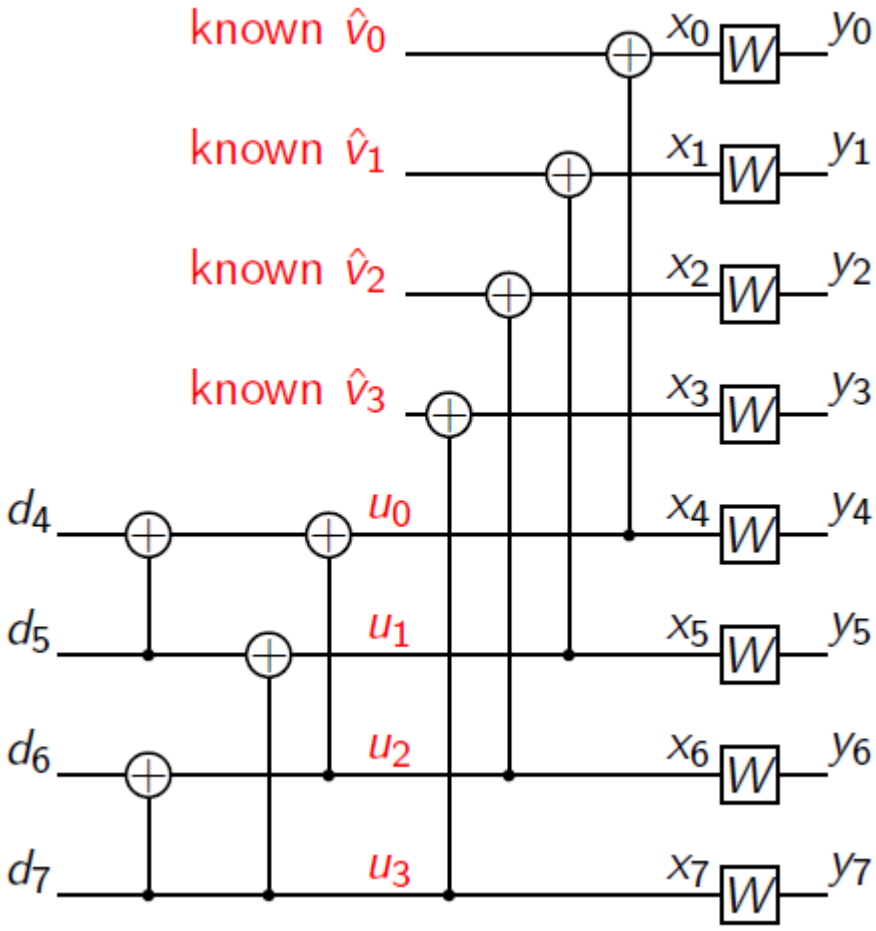
- First phase: Treat u as pure noise, decode for v



- State of decoder at the end of first phase

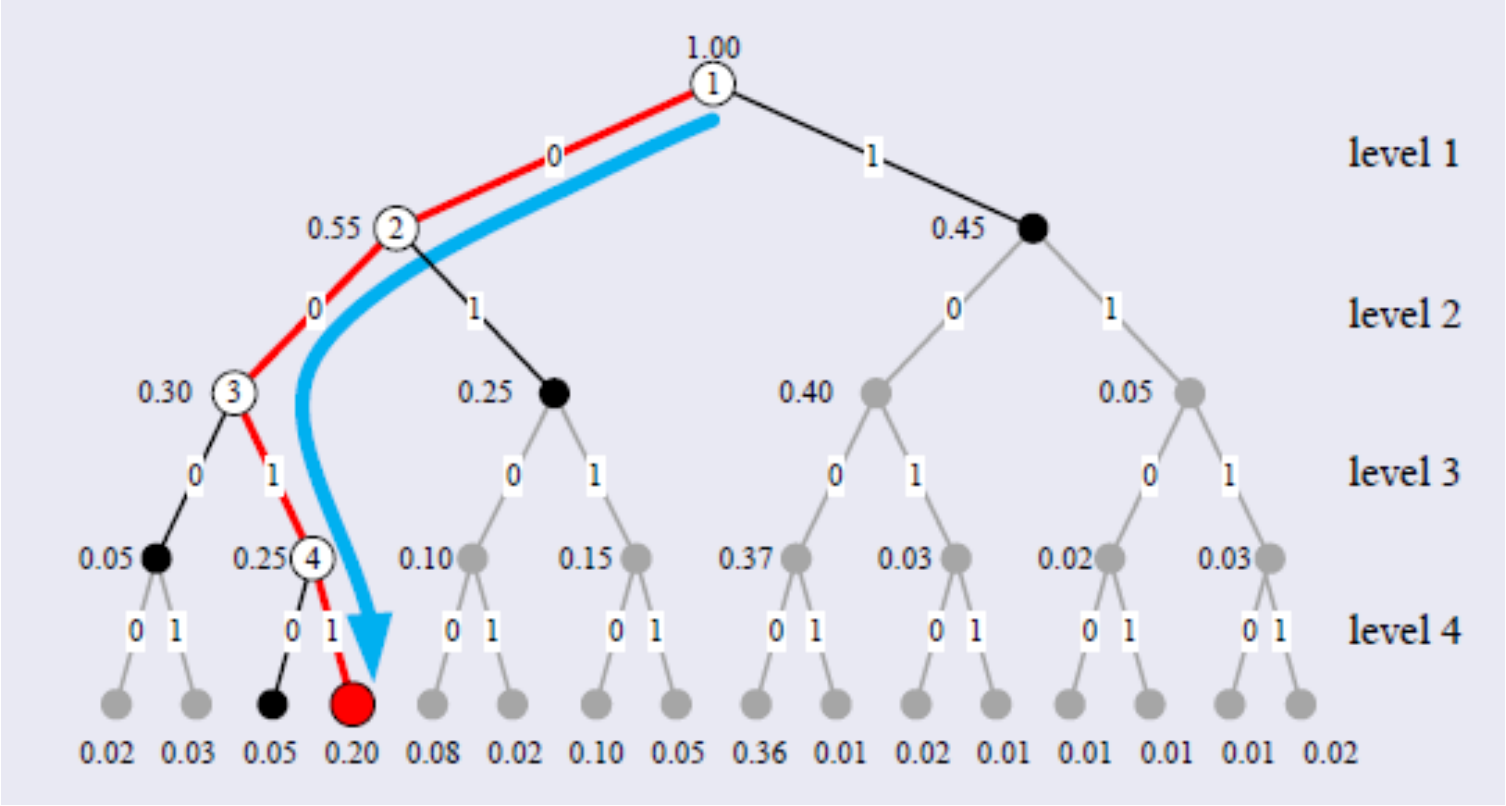


- Second phase: Treat ˆv as correct, decode for u



SCL Decoding

- **Decoding tree of SC decoder**
  - The codeword can be represented using a perfect binary tree with depth N + 1
  - A path from root to one of the leaf nodes determines a **u**



- SCL decoding is a breadth-first search algorithm with limiting branch.
  - First proposed by Tal and Vardy for polar codes [ISIT2011, IT2015]
  - List decoding was used earlier by Dumer and Shabunov (2006) for RM codes.
  - Complexity grows as O(LN logN) for a list size L

SCL Decoding

SC List Decoding

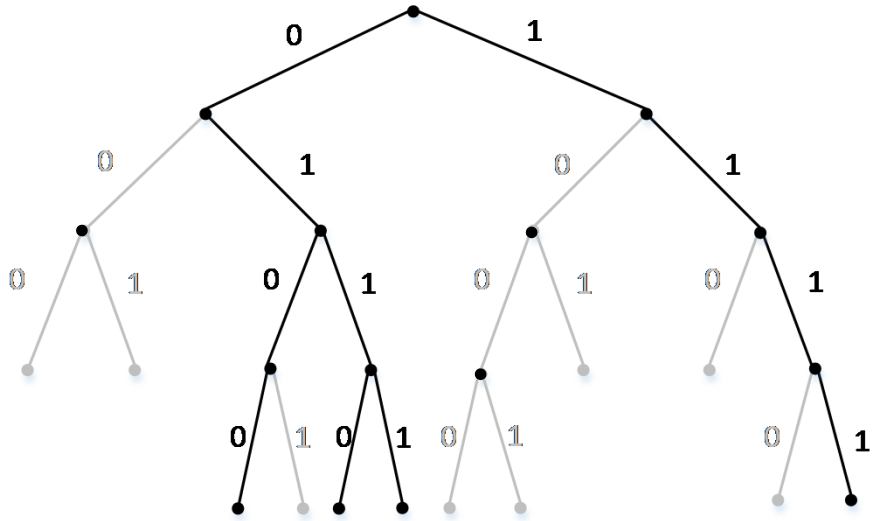
即树形译码，计算每一节点对应的概率，且在每一节点处并不直接判定，而是保留L个最可能的节点，相应的这L个节点对应L条路径。对于bit 级上的树形译码，对其L个节点扩展后，会产生2L个新节点(路径)，通过Bubble排序删掉可靠度低的L个节点，从而保证译码树的每一层上最多保留L个节点。

扩展到最后一层时，选取概率最大的那个节点对应的路径作为最终译码路径，通过回溯得到译码序列。

当译码位为frozen bit时，由于该 bit 收发双发已知，因此只复制先前路径不进行扩展。

宽度优先搜索

Complexity: $O(LN\log N)$



L=4

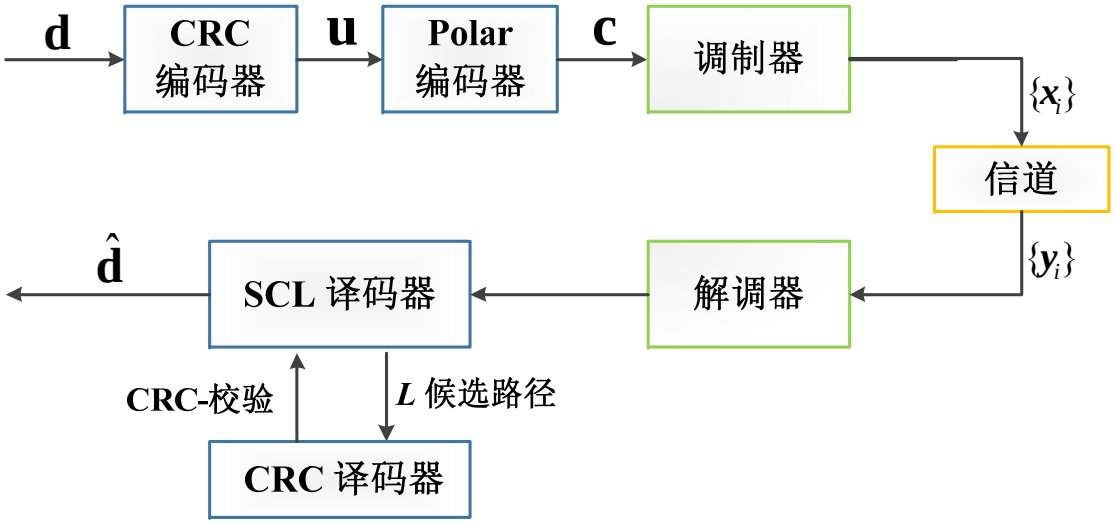
SC List + CRC Decoding

在最后一层，利用CRC进行路径筛选。

SC Stack Decoding

深度优先搜索

CRC-aided List decoding

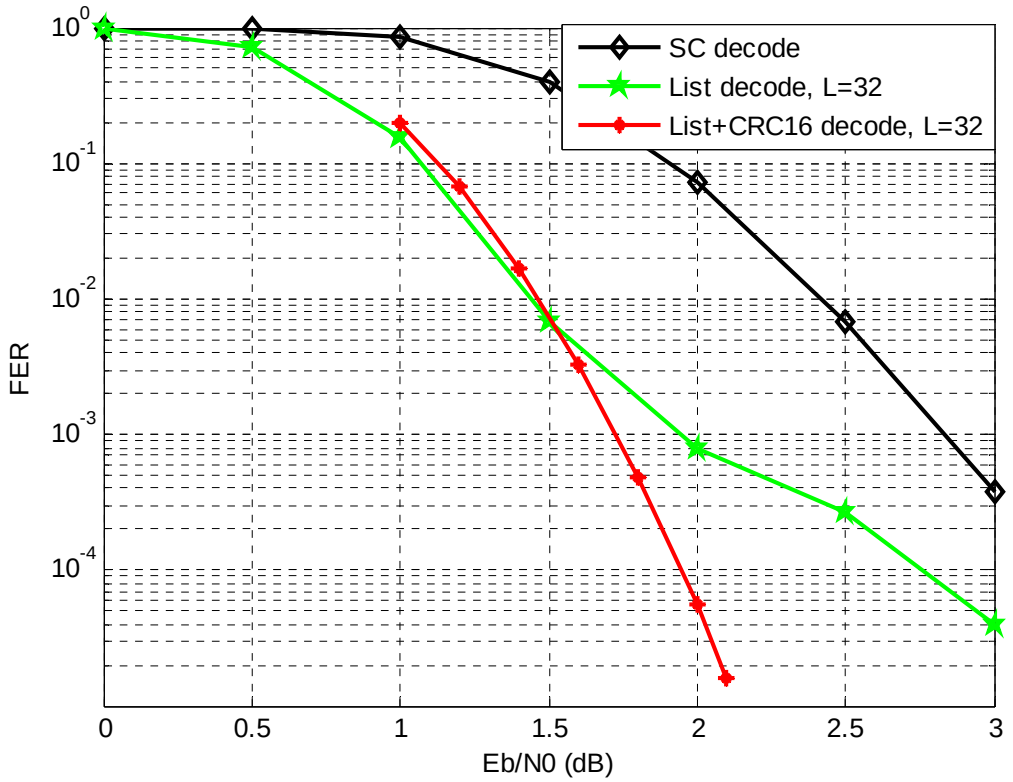


➢ 信源序列 **d** 经过 CRC 编码器产生序列 **u**，送入 Polar 编码器。此时**u**在Polar编码器上可根据比特子信道可靠度大小放置。

➢ 译码时，CRC 比特当做信息比特译码，在SCL 译码结束时CRC译码器将对L条候选路径进行校验，通过CRC校验的路径上的序列作为最终输出。

性能

- **Polar code performance with list decoder**
  - Suggested by Tal-Vardy (2011), based on similar ideas developed earlier by Dumer for Reed-Muller codes.
  - Length n = 2048, rate R = 0.5, BPSK-AWGN channel, list-size L.



Polar code performance with list decoder

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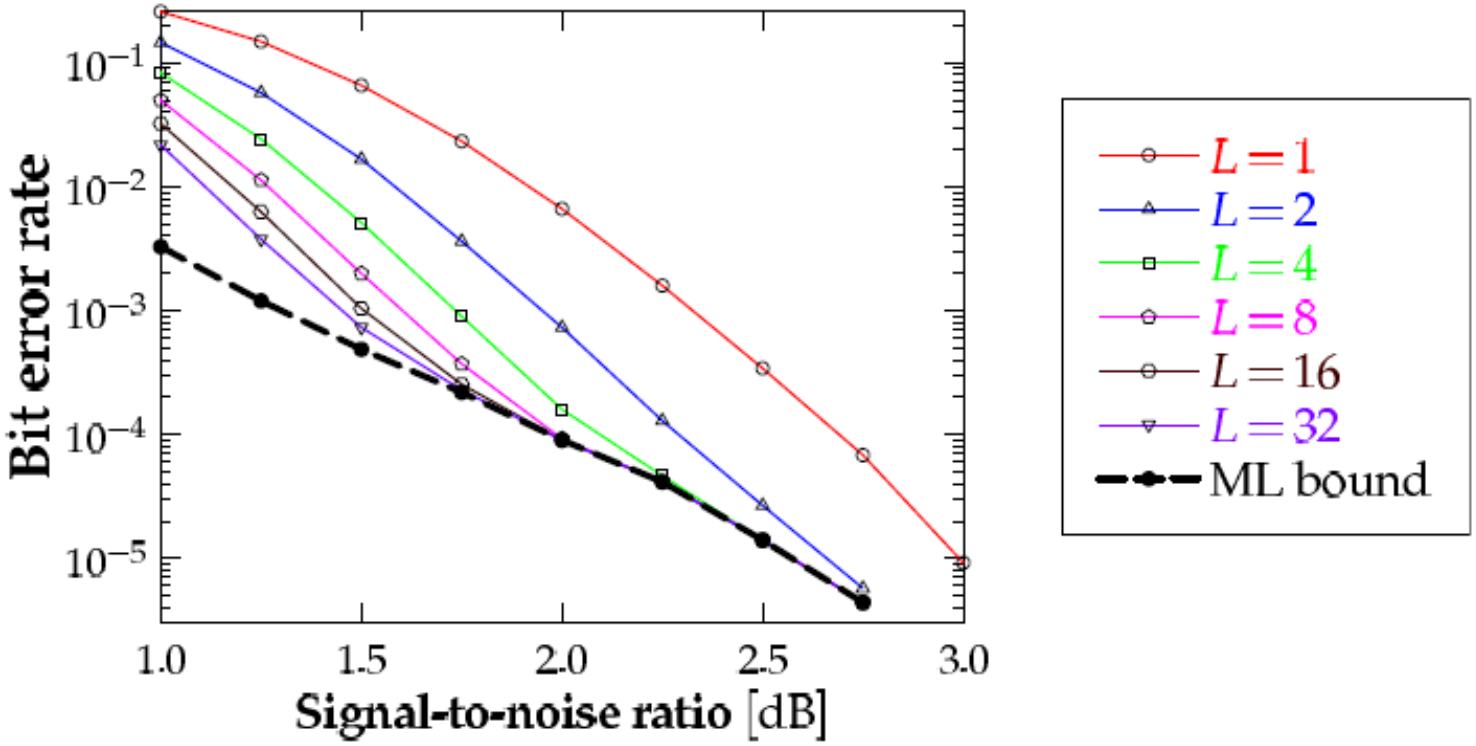
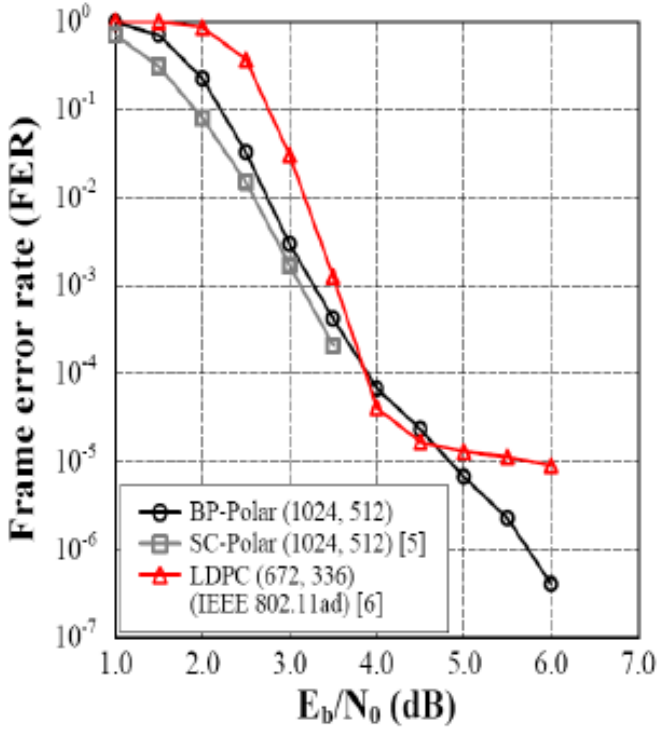


Polar codes vs IEEE 802.11ad LDPC codes

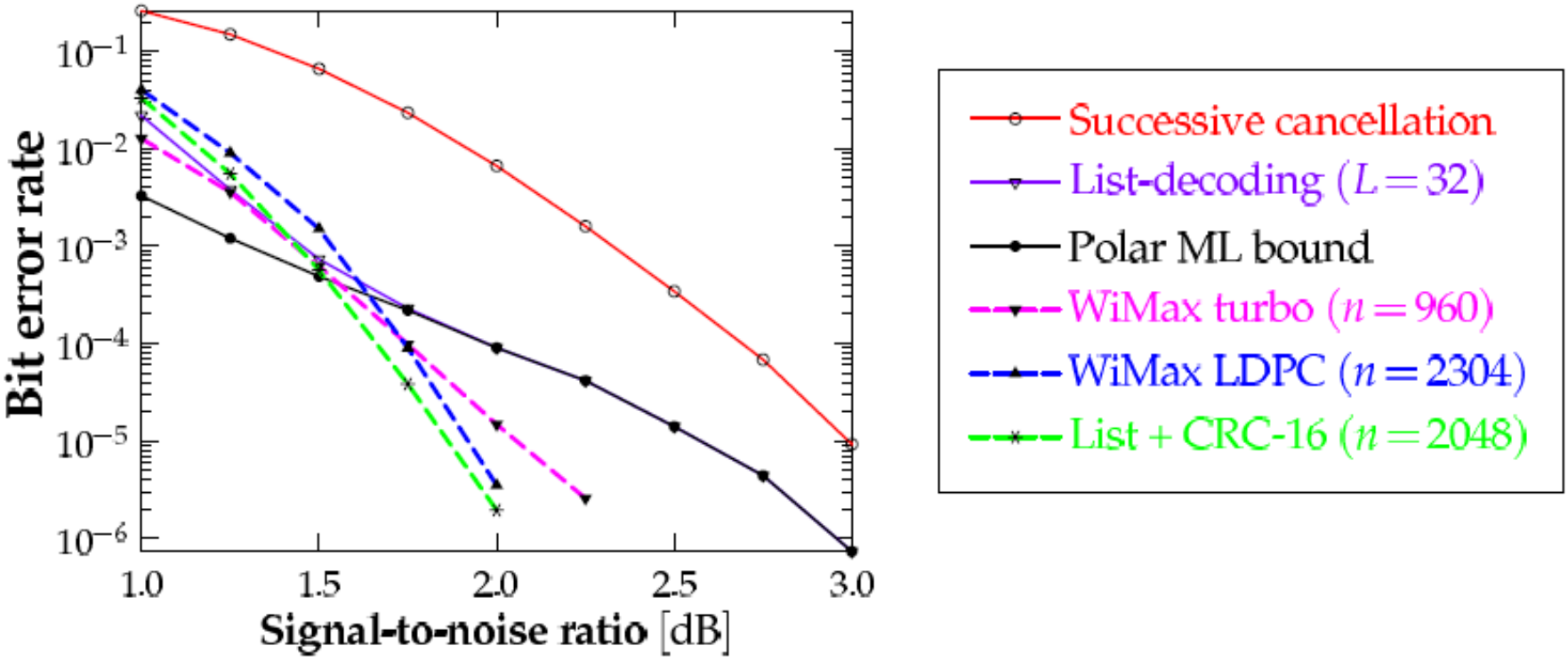
- Park (2014) in a recent PhD thesis gives the following performance comparison.



Source: Youn Sung Park, "Energy-Effcient Decoders of Near-Capacity Channel Codes," PhD Dissertation, The University of Michigan, 2014

Polar codes vs WiMAX Turbo and LDPC codes

Length n = 2048, rate R = 0.5, BPSK-AWGN channel, list-size L



5G Polar Codes for Uplink Control Information

- **UL code construction: （#90）**

Agreement: CRC-aided-Polar

Uplink Control Information Parameters	
Mother code length $N_{max,UCI}$	1024
CRC length	$(n_{FAR} + 3)$
CRC Polynomial	at least CRC11: $D^{11}+ D^{10}+ D^{9}+ D^{5}+ 1$
Minimum code rate	1/8
FAR performance	$1.5 \times 2^{-n_{FAR}}$
Information length K and n_{FAR}	$12 \le K \le 18$, $n_{FAR} = 4$ $19 \le K \le 248$, $n_{FAR} = 8$ $K > 248$, $n_{FAR} = 16$

5G Polar Codes for Downlink Control Information

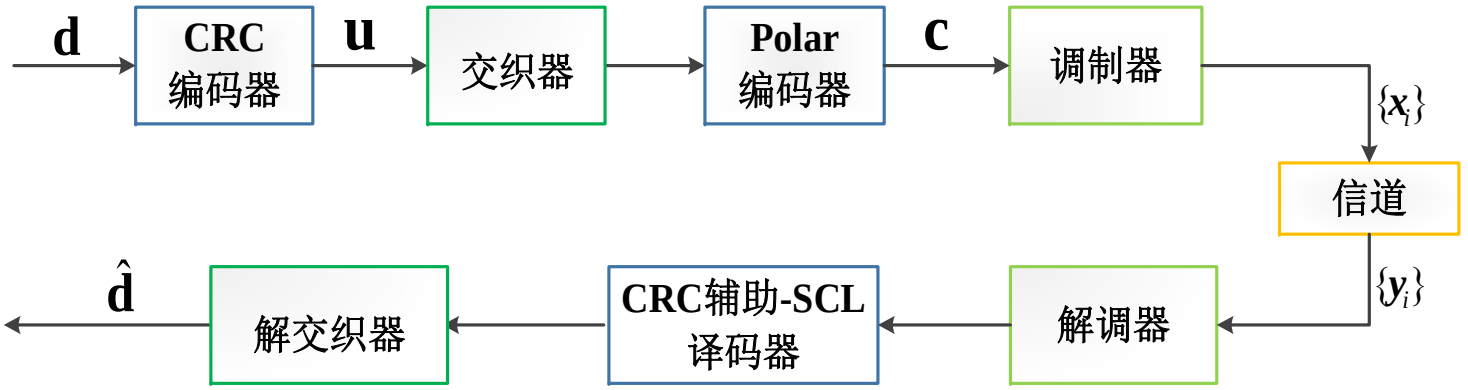
DL and PBCH code construction: （#90）

Agreement: DCRC-Polar

Downlink Control Information Parameters	
Mother code length $N_{max,UCI}$	512
CRC length	$(n_{FAR} + 3)$
CRC Polynomial	CRC24: $D^{24}+D^{23}+D^{21}+D^{20}+D^{17}+D^{15}+D^{13}+D^{12}+D^{8}+D^{4}+D^{2}+D+1$
Minimum code rate	1/8
FAR performance	$1.5 \times 2^{-n_{FAR}}$
Information length K and n_{FAR}	K_{max} = max(140, max DCI payload size in Rel-15 + 20)
Interleaver Pattern	Pattern for n_{FAR}= 21 in R1-1712167

5G Polar 编码候选方案

Distributed CRC-Polar (DCRC-Polar)



- 根据 CRC 编码矩阵及信息长度产生交织器 pattern。
- 交织器重新排列 **u** 中的信息比特和 CRC 比特的位置，使得每一个 CRC 比特的位置都大于它校验的所有信息比特的位置。此时 **u** 在 Polar 编码器上需按照非 frozen 集合的自然顺序放置。
- SCL 译码时，每一次译完 CRC 校验位都需进行 CRC 校验，不通过该比特 CRC 校验的路径需被提前终止，若译出的L条路径都不满足 CRC 校验，则提前终止译码。此方案支持 Early Termination (ET) 。

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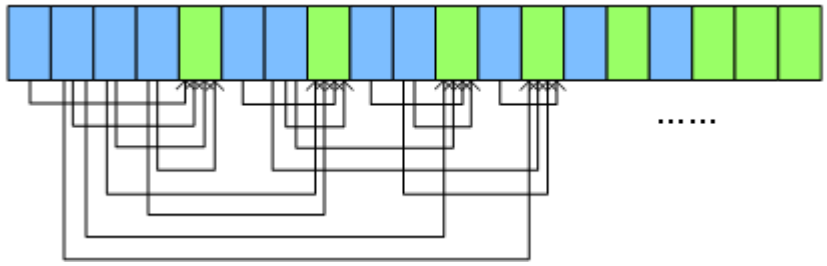
Interleaver pattern: [10, 9, 8, 4, 11, 0, 5, 12, 6, 1, 13, 2, 14, 3, 15, 7, 16, 17, 18]

$$G = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&1&1
\end{bmatrix}$$

$$\Rightarrow \quad G' = \begin{bmatrix}
1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&1\\
0&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&1&\\
0&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&1&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{bmatrix}$$

up-triangle matrix

information bits CRC bits



Summary

Given W, $N = 2^n$, and $R < C(W)$, a polar code with these parameters has

- construction complexity $\mathcal{O}(N)$,
- encoding complexity $\approx N \log N$,
- decoding complexity $\approx N \log N$,
- and frame error probability $P_e(N, R) \approx o(2^{-\sqrt{N}})$

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Thank You !