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Cooperative coloring of some graph families ☆

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ABSTRACT

In a family $G_1, G_2, \dots, G_m$ of graphs sharing the same vertex set V, a cooperative coloring involves selecting one independent set I_i from G_i for each $i \in \{1, 2, \dots, m\}$ such that $\bigcup_{i=1}^m I_i = V$. For a graph class $\mathcal{G}$, let $m_{\mathcal{G}}(d)$ denote the minimum m required to ensure that any graph family $G_1, G_2, \dots, G_m$ on the same vertex set, where $G_i \in \mathcal{G}$ and $\Delta(G_i) \le d$ for each $i \in \{1, 2, \dots, m\}$, admits a cooperative coloring. For the graph classes $\mathcal{T}$ (trees) and $\mathcal{W}$ (wheels), we find that $m_{\mathcal{T}}(3) = 4$ and $m_{\mathcal{W}}(4) = 5$. Also, we prove that $m_{\mathcal{B}^*}(d) = O(\log_2 d)$ and $m_{\mathcal{L}}(d) = O\left(\frac{\log d}{\log\log d}\right)$, where $\mathcal{B}^*$ represents the class of graphs whose components are balanced complete bipartite graphs, and $\mathcal{L}$ represents the class of graphs whose components are generalized theta graphs.

1. Introduction

All graphs discussed in this paper are finite, undirected, and loopless. For notation or terminology not explicitly defined here, we follow those used in Bondy [4].

In a family $G_1, G_2, \dots, G_m$ (not necessarily distinct) of graphs sharing the same vertex set V, a *cooperative coloring* is a selection of one independent set I_i from G_i for each $i \in [m]$ such that $\bigcup_{i=1}^m I_i = V$, where $[m] = \{1, 2, \dots, m\}$. Let G be an edge-colored multigraph with an edge coloring $\phi : E(G) \to [m]$. An *adapted coloring* on G is a vertex coloring $\sigma : V(G) \to [m]$ in which no edge is assigned the same color as both of its endpoints. Formally, this means that for every edge uv in $E(G)$, we have $\neg\big(\phi(uv) = \sigma(u) = \sigma(v)\big)$. It is worth noting that only the colors used for the edges are available for use in an adapted coloring of G. For each $i \in [m]$, let G_i denote the graph with vertex set $V(G)$ and edge set $\phi^{-1}(i)$. It can be verified that a cooperative coloring of the graph family $G_1, G_2, \dots, G_m$ is equivalent to an adapted coloring of G. The concept of adapted coloring was initially introduced by Kostochka and Zhu [11] and has subsequently been extensively studied [8,10,14,16].

Let $m(d)$ be the minimum m such that for any family $G_1, G_2, \dots, G_m$ of graphs on the same vertex set, where $\Delta(G_i) \le d$ for all $i \in [m]$, there exists a cooperative coloring. For a graph G with $\Delta(G) \le d$, it is possible to partition its vertex set greedily into $d+1$ independent sets. Consequently, cooperative coloring can be achieved for $d+1$ identical copies of G. However, there exist $d+1$ graphs, all with maximum degree d, sharing the same vertex set, yet they do not have a cooperative coloring (Aharoni et al., [3]). Therefore, we have the lower bound $m(d) \ge d+2$. On the other hand, Haxell [9]

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proved that $m(d)$ does not exceed $2d$. When all the graphs within the graph family are locally sparse, Loh and Sudakov [12] showed that the upper bound $2d$ can be improved to $d + o(d)$.

The concept of cooperative coloring can be generalized by relaxing the constraint that all graphs must share the same vertex set. When dealing with a family $G_1, G_2, \ldots, G_m$ of graphs with vertex sets $V_1, V_2, \ldots, V_m$ (not necessarily the same), a *cooperative list coloring* involves selecting one independent set from G_i for each $i \in [m]$ in such a way that their union covers the vertex set $V = \bigcup_{i=1}^{m} V_i$. Bradshaw [5] introduced the notation $l(d)$ to represent the minimum value of l such that every family $G_1, G_2, \ldots, G_m$ of graphs with $\Delta(G_i) \le d$ for $i \in [m]$, and where each vertex v in $\bigcup_{i=1}^{m} V(G_i)$ belongs to at least l graphs in this family, admits a cooperative list coloring. In this definition, the graph family must consist of at least l graphs since each vertex must be part of at least l graphs. Bradshaw [5] summarized results about $m(d)$ and $l(d)$ with the following inequality:

$$d + 2 \le m(d) \le l(d) \le 2d. \tag{1}$$

Even for $d = 3$, the precise value of $m(d)$ remains unknown. To refine understanding of this problem, researchers have delved into its study within specific graph classes. The first of the following definitions was introduced by Aharoni et al. [1], while the latter was proposed by Bradshaw [5].

Definition 1.1. [1] For a graph class $\mathcal{G}$, let $m_{\mathcal{G}}(d)$ be the minimum value of m such that any family $G_1, G_2, \ldots, G_m$ of graphs on the same vertex set, where $G_i \in \mathcal{G}$ and $\Delta(G_i) \le d$ for all $i \in [m]$, admits a cooperative coloring.

Definition 1.2. [5] For a graph class $\mathcal{G}$, let $l_{\mathcal{G}}(d)$ be the minimum value of l such that any family $G_1, G_2, \ldots, G_m$ of graphs, where $G_i \in \mathcal{G}$, $\Delta(G_i) \le d$ for all $i \in [m]$, and each $v \in \bigcup_{i=1}^{m} V(G_i)$ belongs to at least l graphs in this family, admits a cooperative list coloring.

Obviously, $m_{\mathcal{G}}(d) \le l_{\mathcal{G}}(d)$ for any graph class $\mathcal{G}$. A *chordal graph* is defined as a graph in which every cycle of length greater than three contains at least one chord, which is an edge not part of the cycle. Aharoni et al. [1–3] explored the values of $m_{\mathcal{G}}(d)$ when $\mathcal{G}$ is represented by the class of chordal graphs, paths, trees, and bipartite graphs, respectively. In this paper, we specify the default base for logarithms as the natural number e when the base is missing.

Theorem 1.3. [2] *Let $\mathcal{C}$ be the class of chordal graphs. Then $m_{\mathcal{C}}(d) = d + 1$ for $d \ge 1$.*

Theorem 1.4. [3] *Let $\mathcal{P}$ be the class of paths. Then $m_{\mathcal{P}}(2) = 3$.*

Theorem 1.5. [1] *Let $\mathcal{T}$ be the class of trees and $\mathcal{B}$ be the class of bipartite graphs. Then for $d \ge 2$,*

$$\log_2 \log_2 d \le m_{\mathcal{T}}(d) \le (1 + o(1)) \log_{4/3} d,$$
$$\log_2 d \le m_{\mathcal{B}}(d) \le (1 + o(1)) \frac{2d}{\log d}.$$

Let $\mathcal{F}$ be the graph class of forests. Aharoni et al. [1] showed that $m_{\mathcal{T}}(d) = m_{\mathcal{F}}(d)$ for $d \ge 2$. It follows from $\mathcal{T} \subseteq \mathcal{F}$ that $m_{\mathcal{T}}(d) \le m_{\mathcal{F}}(d)$. Conversely, consider $m := m_{\mathcal{T}}(d)$ forests $F_1, F_2, \ldots, F_m$ with the maximum degree d. When $d \ge 2$, we can augment each F_i by adding edges to obtain a tree F'_i that maintains the maximum degree, for every $i \in [m]$. By the definition of $m_{\mathcal{T}}(d)$, the graph family $F'_1, F'_2, \ldots, F'_m$ admits a cooperative coloring. This implies that the graph family $F_1, F_2, \ldots, F_m$ also admits a cooperative coloring, therefore $m_{\mathcal{F}}(d) \le m$.

Bradshaw and Masařík [6] investigated the upper bound on $m_{\mathcal{G}}(d)$ for the class $\mathcal{G}$ of degenerate graphs. Since every tree is a 1-degenerate graph, the following result generalizes Theorem 1.5 at the expense of a constant factor.

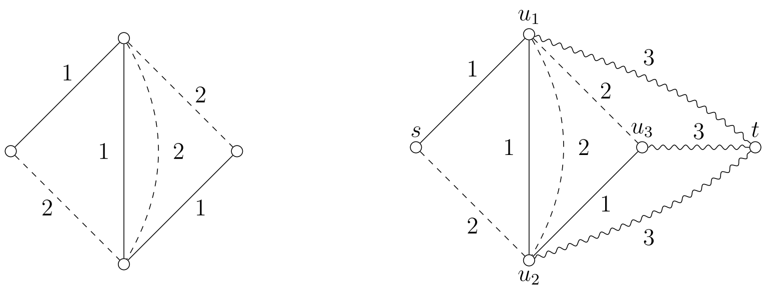
Theorem 1.6. [6] *Let $\mathcal{G}$ be the class of graphs and every graph in $\mathcal{G}$ is at most k-degenerate. Then $m_{\mathcal{G}}(d) \le 13(1 + k \log_2(kd))$.*

Bradshaw [5] also studied the value $m_{\mathcal{S}}(d)$ for the class $\mathcal{S}$ of star forests, which improves the lower bound on $m_{\mathcal{T}}(d)$ in Theorem 1.5.

Theorem 1.7. [5] *Let $\mathcal{S}$ be the class of star forests. Then,*

$$m_{\mathcal{S}}(d) \ge (1 + o(1)) \frac{\log d}{\log \log d}.$$

In Section 2, we study $m_{\mathcal{G}}(d)$ for small values of d within specific graph classes $\mathcal{G}$. For the class of trees, denoted as $\mathcal{T}$, the exact value of $m_{\mathcal{T}}(d)$ remains undetermined, except for the cases where $d \le 2$. Specifically, when $d = 3$, it can be



(a) H_1 (b) H_2

Fig. 1. In our construction illustrating that $m_{\mathcal{T}}(3) \geq 4$, these two graphs are elemental subgraphs. It is easy to check that H_1 does not admit an adapted coloring using color set $\{1, 2\}$.

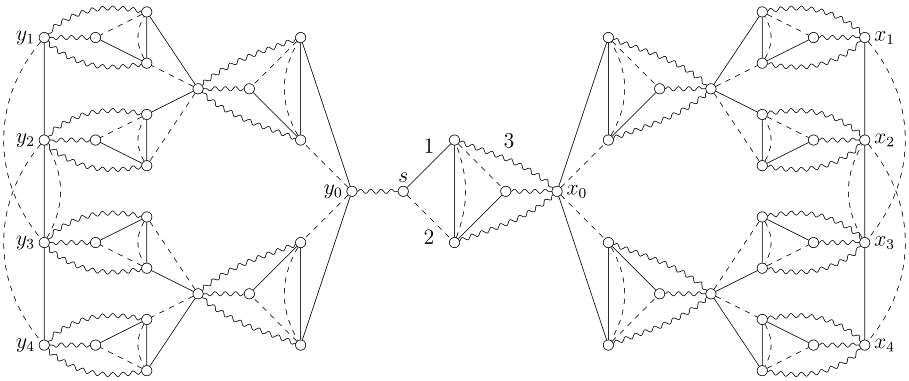


Fig. 2. This is a graph that has been edge-colored with three colors and does not admit an adapted coloring.

deduced from Theorems 1.3 and 1.4 that $3 \leq m_{\mathcal{T}}(3) \leq 4$ since $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{C}$. Although the proof of Theorem 1.7 in [5] gave a better construction to improve the lower bound of $m_{\mathcal{T}}(d)$, this construction method cannot prove that $m_{\mathcal{T}}(3) \geq 4$. We show that $m_{\mathcal{T}}(3) \geq 4$, leading to the conclusion that $m_{\mathcal{T}}(3) = 4$. Furthermore, we prove that $m_{\mathcal{W}}(4) = 5$, in which $\mathcal{W}$ is the class of graphs whose components are wheels. In Section 3, we study $m_{\mathcal{B}^*}(d)$ and $m_{\mathcal{L}}(d)$, where $\mathcal{B}^*$ represents the class of graphs whose components are balanced complete bipartite graphs, and $\mathcal{L}$ represents the class of graphs whose components are generalized theta graphs. Specifically, we show that $m_{\mathcal{B}^*}(d) = O(\log_2 d)$ and $m_{\mathcal{L}}(d) = O\left(\frac{\log d}{\log \log d}\right)$.

2. Trees and wheels

For the sake of clarity in our discussion, most of the proofs are presented in terms of adapted coloring. By showing that the edge-colored graph in Fig. 2 does not admit an adapted coloring, we can establish the lower bound $m_{\mathcal{T}}(3) \geq 4$.

Lemma 2.1. *Let σ be an adapted coloring of the edge-colored multigraph H_2 depicted in Fig. 1(b). If $\sigma(s) \neq 3$, then $\sigma(t) \neq 3$.*

Proof. The subgraph $H_2[s, u_1, u_2, u_3]$ depicted in Fig. 1(b) does not admit an adapted coloring when using the color set $\{1, 2\}$. Consequently, color 3 must be assigned to one of the vertices in $\{s, u_1, u_2, u_3\}$. If $\sigma(s) \neq 3$, then at least one of $\{u_1, u_2, u_3\}$ must be colored with 3. By the definition of adapted coloring, we can deduce that $\sigma(t) \neq 3$. $\square$

Theorem 2.2. *Let $\mathcal{T}$ be the class of trees. Then $m_{\mathcal{T}}(3) \geq 4$.*

Proof. Let (G, ϕ) be the edge-colored graph using colors from $\{1, 2, 3\}$ in Fig. 2. Observe that each monochromatic induced subgraph of (G, ϕ) is a forest of maximum degree 3. Suppose there exists an adapted coloring σ of (G, ϕ).

If $\sigma(s) \neq 3$, then $\sigma(x_i) \neq 3$ for $1 \leq i \leq 4$ by Lemma 2.1. Since $G[x_1, x_2, x_3, x_4]$ is isomorphic to H_1 in Fig. 1(a), it is evident that $G[x_1, x_2, x_3, x_4]$ does not admit an adapted coloring using colors from $\{1, 2, 3\} \backslash \{3\}$, a contradiction. If $\sigma(s) = 3$, then $\sigma(y_0) \neq 3$. Therefore, $\sigma(y_i) \neq 3$ for $1 \leq i \leq 4$, by Lemma 2.1. Similarly, $G[y_1, y_2, y_3, y_4]$ does not admit an adapted coloring using colors from $\{1, 2, 3\} \backslash \{3\}$, a contradiction. $\square$

It follows from Theorems 1.3 and 1.4 that $3 \leq m_{\mathcal{T}}(3) \leq 4$ since $\mathcal{P} \subseteq \mathcal{T} \subseteq \mathcal{C}$. Together with Theorem 2.2, we have the following conclusion.

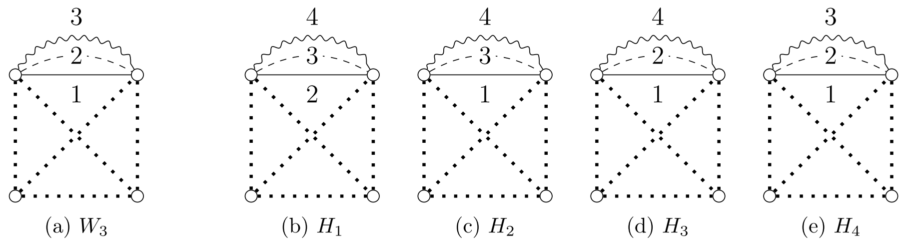


Fig. 3. In the graph depicted, each dotted line represents three edges that are colored differently. For instance, in W_3, every pair of vertices is connected by three edges, each assigned a distinct edge color from $\{1, 2, 3\}$. The graphs labeled as H_1, H_2, H_3, and H_4 are copies of W_3 with edge colorings that have been shifted.

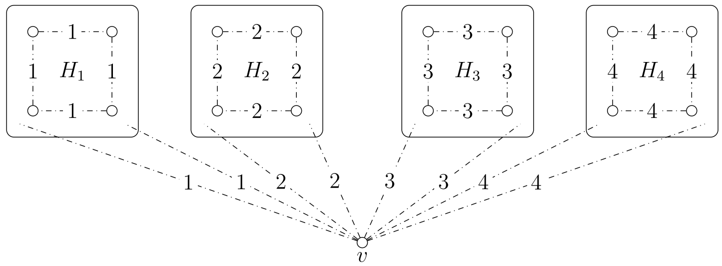


Fig. 4. An edge-colored multigraph using four colors, where vertex v is connected to each vertex in H_i by an edge colored i for each $1 \le i \le 4$. This graph does not admit an adapted coloring.

Corollary 2.3. $m_{\mathcal{T}}(3) = 4$.

A *wheel* W_n is the join of a cycle C_n and a single vertex, i.e., $W_n \cong C_n \vee K_1$, where $n \ge 3$. It is worth mentioning that the construction in the proof of the lower bound of Theorem 2.4 is similar to the construction presented in Theorem 1 of [5].

Theorem 2.4. *Let $\mathcal{W}$ be the class of graphs whose components are wheels. Then,*

$$m_{\mathcal{W}}(4) = 5.$$

Proof. First, we show that $m_{\mathcal{W}}(4) \le 5$. Given a graph family $G_1, G_2, \dots, G_5$ of $\mathcal{W}$ on the same vertex set, where $\Delta(G_i) \le 4$ for $i \in [5]$, it can be checked that each G_i is either a $C_3 \vee K_1 (\cong K_4)$ or a $C_4 \vee K_1 (\subseteq K_5 - e)$. Both K_4 and $K_5 - e$ are chordal graphs. Thus each G_i is a subgraph of a chordal graph G_i^* with the maximum degree at most 4. It follows from Theorem 1.3 that the graph family $G_1^*, G_2^*, \dots, G_5^*$ admits a cooperative coloring. Consequently, the graph family $G_1, G_2, \dots, G_5$ admits a cooperative coloring.

Next, we construct an edge-colored graph G with a color set $\{1, 2, 3, 4\}$ such that its maximal monochromatic subgraphs are graphs in $\mathcal{W}$ with the maximum degree at most 4. Afterward, we show that G does not admit an adapted coloring. By the relation of adapted coloring and cooperative coloring, we get $m_{\mathcal{W}}(4) \ge 5$.

We denote the edge-colored multigraph depicted in Fig. 3(a) by W_3, and denote its edge-coloring be ϕ. It can be verified that (W_3, ϕ) does not admit an adapted coloring using the color set $\{1, 2, 3\}$. For $1 \le i \le 4$, the shift function $\psi_i : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ is defined as:

$$\psi_i(x) = \begin{cases} x, & 1 \le x \le i - 1, \\ x + 1, & i \le x \le 3. \end{cases}$$

By applying these shift functions, we can generate four disjoint copies of W_3 denoted as H_1, H_2, H_3, and H_4 (as shown in Fig. 3). Each H_i is edge-colored using the function $\psi_i \circ \phi$. It is easy to see that $(H_i, \psi_i \circ \phi)$ is isomorphic to (W_3, ϕ) as an edge-colored graph. Therefore, $(H_i, \psi_i \circ \phi)$ does not admit an adapted coloring using the color set $\{1, 2, 3, 4\} \setminus \{i\}$.

The edge-colored multigraph G (see Fig. 4) is obtained by the following construction. First, add a 4-cycle C_i in H_i for each $i \in [4]$. At the same time, add a new vertex v and join v with each vertex of $\cup_{i=1}^4 H_i$. Next, assign color i to edges between to v and H_i for $i \in [4]$. In this construction, each component of the monochromatic subgraphs of G is either a W_3 or a W_4. In any adapted coloring of G using color set $\{1, 2, 3, 4\}$, at least one vertex of H_i must be colored with i. It implies that no color in $\{1, 2, 3, 4\}$ is available at v. Therefore, G does not admit an adapted coloring using color set $\{1, 2, 3, 4\}$. $\square$

3. Bipartite graphs and generalized theta graphs

Aharoni et al. [1] investigated cooperative coloring of the class of bipartite graphs and showed that $m_{\mathcal{B}}(d) \geq \log_2 d$ (as stated in Theorem 1.5). A complete bipartite graph $K_{m,n}$ is called *balanced* if $m = n$. Let $\mathcal{B}^*$ be the class of graphs in which each component is a balanced complete bipartite graph, and we can get $m_{\mathcal{B}^*}(d) \geq \log_2 d$ by the constructive proof of lower bound on bipartite graphs in [1]. Now we show that $\log_2 d$ is asymptotically the best possible for $m_{\mathcal{B}^*}(d)$. The following famous Lovász Local Lemma first appears in a weaker form in [7] and can be found in many textbooks, including for example [15, Chapter 4].

Theorem 3.1. [15] *Let $\mathcal{B}$ be a set of bad events. Suppose that each event $B \in \mathcal{B}$ occurs with probability at most p, and suppose further that each event $B \in \mathcal{B}$ is independent with all but at most d other events $B' \in \mathcal{B}$. If $ep(d+1) < 1$, then with positive probability, no bad event in $\mathcal{B}$ occurs.*

The proof of the following theorem could be proceeded using the idea of choosing independent sets in [13]. Nevertheless, by noting that each graph in $\mathcal{B}^*$ is a bipartite graph and each part of its components is an independent set, we present here a bit different but more straightforward and concise proof.

Theorem 3.2. *For $d \geq 2$, it holds that $m_{\mathcal{B}^*}(d) \leq (1 + o(1)) \log_2 d$.*

Proof. Consider a family $G_1, G_2, \ldots, G_m$ of graphs on the same vertex set V, where the components of each G_i are balanced complete bipartite graphs and $\Delta(G_i) \leq d$ for $1 \leq i \leq m$.

Let X_i be a set of vertices by choosing one part from each complete bipartite component uniformly at random in G_i for each $1 \leq i \leq m$. It is evident that X_i is an independent set in G_i. We now show that with a positive probability, $X_1, X_2, \ldots, X_m$ collectively constitute a cooperative coloring of the graph family $G_1, G_2, \ldots, G_m$. For each vertex $v \in V$, let B_v be the event that $v \notin \cup_{i=1}^m X_i$. Then we get $\Pr(v \notin \cup_{i=1}^m X_i) = (1/2)^m$ since $\Pr(v \in X_i) = \frac{1}{2}$. It follows from the degree of v is at most d that B_v is independent with all but fewer than $2md$ other events. By applying the Lovász Local Lemma (Theorem 3.1), if

$$e \times \left(\frac{1}{2}\right)^m \times 2md \leq 1, \tag{2}$$

then no B_v occurs with positive probability, meaning that the sets $X_1, X_2, \ldots, X_m$ form a cooperative coloring. The inequality holds when $m \geq (1 + o(1)) \log_2 d$. $\square$

The *generalized theta graph* $\theta_{s_1,\ldots,s_k}$ consists of a pair of vertices joined by k internally disjoint paths of lengths $s_1, \ldots, s_k$, where each $s_i \geq 1$. Let $\mathcal{L}$ be the class of graphs whose components are generalized theta graphs. To study the value of $m_{\mathcal{L}}(d)$, we first introduce some definitions and a lemma. Given a rooted tree T with a root r, the *height* of a vertex v in T is the distance from v to r, and the height of T is the maximum height achieved over all vertices $v \in V(T)$. Given integers $q \geq 1$ and $h \geq 1$, a *q-ary tree of height h* is a rooted tree in which every vertex of height at most $h-1$ has exactly q children. Given an integer $k \geq 1$, we write $\log^{(k)} d = \underbrace{\log\log\ldots\log}_{k\text{ times}} d$.

Lemma 3.3. [5] *Let $q \geq 2$ and $h \geq 1$ be fixed integers. If $\mathcal{H}$ is a family of graphs with no q-ary tree of height h as a subgraph, then*

$$l_{\mathcal{H}}(d) \leq (1 + o_{q,h}(1)) \frac{\log d}{\log^{(h)} d} + O_q(1).$$

Theorem 3.4. *For $d \geq 2$, $m_{\mathcal{L}}(d) = (1 + o(1)) \frac{\log d}{\log\log d}$.*

Proof. Since no generalized theta graph contains a 3-ary tree of height 2 as a subgraph, we have $m_{\mathcal{L}}(d) \leq l_{\mathcal{L}}(d) \leq (1 + o(1)) \frac{\log d}{\log\log d}$ by Lemma 3.3. In the following, we discuss the lower bound of $m_{\mathcal{L}}(d)$.

For each $t \geq 1$, we construct a graph G_t whose edges are colored with $\{1, 2, \ldots, t\}$ using some function ϕ_t. In this graph, for each $i \in [t]$, every component of the monochromatic subgraph induced by the edges of color i is a $K_{2,s}$ with $s \geq 2$, which is a generalized theta graph. We then translate the edge-colored graph (G_t, ϕ_t) into a graph family that demonstrates our lower bound. It is worth mentioning that the construction here uses ideas from [5].

We construct the edge-colored graphs (G_t, ϕ_t) recursively, as illustrated in Fig. 5 for (G_1, ϕ_1) and (G_2, ϕ_2). Let (G_1, ϕ_1) be a $K_{2,2}$ whose edge is colored with the color 1. Assume that we have constructed G_t along with an edge-coloring $\phi_t : E(G_t) \to \{1, 2, \ldots, t\}$, and assume that (G_t, ϕ_t) does not admit an adapted coloring with the color set $\{1, \ldots, t\}$. For $1 \leq i \leq t+1$, we define a shift function $\psi_i : \{1, 2, \ldots, t\} \to \{1, 2, \ldots, t+1\}$ such that

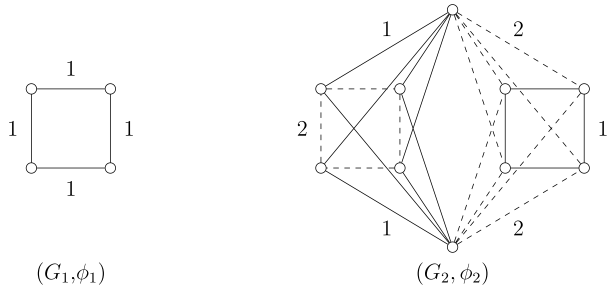


Fig. 5. The first two graphs of the recursively constructed graphs (G_t, ϕ_t).

$$\psi_i(x) = \begin{cases} x, & 1 \le x \le i-1, \\ x+1, & i \le x \le t. \end{cases}$$

We begin to construct (G_{t+1}, ϕ_{t+1}) as follows: first create $t+1$ disjoint copies, $H_1, H_2, \ldots, H_{t+1}$, of G_t, where each H_i is edge-colored using the function $\psi_i \circ \phi_t$. Note that $(H_i, \psi_i \circ \phi_t)$ is isomorphic to (G_t, ϕ_t) as an edge-colored graph. Therefore, $(H_i, \psi_i \circ \phi_t)$ does not admit an adapted coloring with colors from $\{1, 2, \ldots, i-1, i+1, \ldots, t+1\}$. Next, we add two new vertices, u and v, and add an edge of color i joining each vertex of $\{u, v\}$ and each vertex of H_i for $1 \le i \le t+1$. We denote the resulting graph as G_{t+1} and its edge coloring as ϕ_{t+1}. With this construction, every component of the monochromatic subgraph induced by the edges of each color in (G_{t+1}, ϕ_{t+1}) is a $K_{2,s}$ with $s \ge 2$. If (G_{t+1}, ϕ_{t+1}) admits an adapted coloring with the color set $[t+1]$, then H_i must contain a vertex colored i, and no color from $[t+1]$ is available at either u or v. Consequently, (G_{t+1}, ϕ_{t+1}) does not admit an adapted coloring.

Now, we determine the maximum degree of each monochromatic subgraph in G_t. We define V_t as the number of vertices in G_t and Δ_t as the maximum degree in all monochromatic subgraphs of G_t. For G_1, we have $V_1 = 4$ and $\Delta_1 = 2$. Moving on to the recursive case for $t \ge 2$:

$$V_t = tV_{t-1} + 2, \quad \Delta_t = V_{t-1}.$$

Solving the recurrence, we have that

$$\begin{aligned} V_t &= V_1 \cdot t! + 2 \cdot \frac{t!}{2!} + 2 \cdot \frac{t!}{3!} + \cdots + 2 \cdot \frac{t!}{(t-2)!} + 2 \cdot \frac{t!}{(t-1)!} + 2 \\ &= 2 \cdot t! \cdot \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(t-2)!} + \frac{1}{(t-1)!} + \frac{1}{t!}\right) \\ &= \big(2e + o(1)\big) \cdot t!. \end{aligned}$$

Moreover, $\Delta_t = V_{t-1} = \big(2e + o(1)\big)(t-1)!$.

Now, consider a value d, and choose t so that $\Delta_t \le d < \Delta_{t+1}$. We construct (G_t, ϕ_t) as above, and let G_t^i be the monochromatic subgraph of (G_t, ϕ_t) with color i for $1 \le i \le t$. Note that each component of G_t^i can be regarded as a generalized theta graph with the maximum degree at most d. Since (G_t, ϕ_t) does not admit an adapted coloring, the graph family $G_t^1, G_t^2, \ldots, G_t^t$ does not admit a cooperative coloring.

To complete the proof, it suffices to show that $d \le (2e + o(1))t!$ implies $t \ge (1 + o(1)) \frac{\log d}{\log \log d}$. Let

$$d = (2e + o(1))t! = (2e + o(1))\sqrt{2\pi t}\left(\frac{t}{e}\right)^t,$$

and we have

$$\begin{aligned} \log d &= \log(2e + o(1)) + \frac{1}{2}\log 2\pi t + t\log t - t = t(\log t - 1 + o(1)), \\ \log\log d &= \log t + \log\big(\log t - 1 + o(1)\big). \end{aligned}$$

It follows that

$$t \cdot \frac{\log\log d}{\log d} = \frac{\log t + \log\big(\log t - 1 + o(1)\big)}{\log t - 1 + o(1)} = 1 + o(1),$$

as required. □

4. Remark

Let $\mathcal{W}^*$ be the class of graphs whose components are wheel graphs ($C_m \vee K_1$) or fan graphs ($P_n \vee K_1$), where $m \geq 3$ and $n \geq 1$. A *caterpillar* is defined as a tree in which, upon removing all the pendant vertices, it results in a path. A *ring star* is a graph that can be decomposed into a cycle (or ring) and a set of vertices each of them not belonging to the cycle but adjacent to it. Let $\mathcal{M}$ be the class of caterpillar graphs, and $\mathcal{R}$ be the class of ring star graphs. Observe that each graph in $\mathcal{W}^*$, $\mathcal{M}$ and $\mathcal{R}$ satisfies the property that their high-degree vertices induce a low-degree subgraph. In particular, for each graph in $\mathcal{W}^*$, $\mathcal{M}$ and $\mathcal{R}$, there is no ternary tree of height 2 as a subgraph. Let $\mathcal{H} \in \{\mathcal{W}^*, \mathcal{M}, \mathcal{R}\}$. Then, Lemma 3.3 implies that $m_{\mathcal{H}}(d)$ is at most $(1+o(1))\frac{\log d}{\log\log d}$. Furthermore, $m_{\mathcal{H}}(d) \geq m_{\mathcal{S}}(d)$ because any star forest F of $\mathcal{S}$ is a subgraph of $\mathcal{H}$ of maximum degree $\Delta(F)$ for $d \geq 2$. Therefore, it follows from Theorem 1.7 that $m_{\mathcal{H}}(d) \geq (1+o(1))\frac{\log d}{\log\log d}$. Hence, we obtain the following corollary.

Corollary 4.1. *For $d \geq 2$, $m_{\mathcal{H}}(d) = (1+o(1))\frac{\log d}{\log\log d}$, where $\mathcal{H} \in \{\mathcal{W}^*, \mathcal{M}, \mathcal{R}\}$.*

Since determining the value of $m(d)$ can be quite challenging even when d is small, it is highly meaningful to further investigate cooperative colorings of special classes of graphs for small values of d. Therefore, we propose the following problem.

Problem 4.2. Determine the precise values of $m_{\mathcal{G}}(3)$ and $m_{\mathcal{T}}(4)$ where $\mathcal{G}$ represents the class of bipartite graphs and $\mathcal{T}$ represents the class of trees.

Declaration of competing interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled "Cooperative coloring of some graph families".

Data availability

No data was used for the research described in the article.

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