Counting cliques in 1-planar graphs

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ABSTRACT

The problem of maximising the number of cliques among *n*-vertex graphs from various graph classes has received considerable attention. We investigate this problem for the class of 1-planar graphs where we determine precisely the maximum total number of cliques as well as the maximum number of cliques of any fixed size. We also precisely characterise the extremal graphs for these problems.

1. Introduction

A 1-planar graph is a graph which can be drawn in the plane such that each edge is crossed by at most one other edge (see Section 2 for a formal definition). This class forms a natural extension of the class of planar graphs, and was first introduced by Ringel in 1965 [1]. His motivation was the problem of simultaneously colouring vertices and faces of a plane graph so that each face and vertex gets a distinct colour from all neighbouring faces and vertices. Such a colouring corresponds to a vertex colouring of a 1-planar graph, and Ringel made progress on this problem by showing

that every 1-planar graph is 7-colourable. The problem was later fully answered by Borodin [2], who showed that six colours are sufficient to colour every 1-planar graph (the triangular prism has no such 5-vertex-face-colouring).

Since its introduction, the class of 1-planar graphs has proven to be a topic of much interest, as reflected by the large body of research in the area (some results on the topic are surveyed in [3–5]). One reason for interest in the class is that it is closely related to the class of planar graphs, while having a number of important qualitative differences. Most notably, whereas the classical results of Kuratowski [6] and Wagner [7] tell us that planar graphs can be defined as exactly those graphs with no $K_{3,3}$ or K_5 (topological) minor, the class of 1-planar graphs has no such characterisation. In fact it is easy to see that every graph can be made 1-planar by subdividing its edges sufficiently many times. Another key distinction is that planar graphs can be recognised in time linear in the number of vertices [8], but the recognition problem for 1-planar graphs is NP-complete [9,10]. On the other hand, 1-planar graphs are only a linear number of edges away from being planar. In fact, whenever an n-vertex graph is drawn in the plane so that each edge is crossed at most once and adjacent edges do not cross, the total number of crossings is at most $n - 2$ [11].

In extremal graph theory, determining the maximum number of edges in an n-vertex graph from a given graph class is a fundamental problem. The classical theorem of Turán [12] asserts that in the class of K_r-free graphs, the maximum number of edges is attained (uniquely) by the complete multipartite graph with $r - 1$ parts each of size $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$. In the class of planar graphs it is a simple consequence of Euler's formula that an n-vertex graph contains at most $3n - 6$ edges. The bound on the number of crossings in 1-planar drawings implies that n-vertex 1-planar graphs have at most $4n - 8$ edges, although this was first shown by Bodendiek, Schumacher and Wagner [13]. In the same paper, they characterised the n-vertex 1-planar graphs which achieve this bound as exactly the graphs obtained from 3-connected planar quadrangulations by adding a pair of crossing edges into every face.

Going beyond maximising edges it is natural to consider maximising the number of cliques of a given size t among n-vertex graphs from a class of graphs $\mathcal{G}$. Zykov [14] and Erdős [15] independently determined this number for the class of K_r-free graphs for all positive integers t and n. Hakimi and Schmeichel [16] determined the maximum number of triangles in an n-vertex planar graph (see Theorem 2.4), and the maximum number of cliques of size four in an n-vertex planar graph was determined independently by Alon and Caro [17], and Wood [18]. We determine the maximum number of cliques of any fixed size in an n-vertex 1-planar graph. Note that for $n \leq 6$, the clique K_n is 1-planar, and contains the maximum possible number of cliques of any fixed size. It follows from Borodin's result [2] that 1-planar graphs are 6-colourable that no 1-planar graph contains a clique of size 7 or greater. All remaining cases are covered by the following theorems.

Theorem 1.1.   *Given integers $k \geq 2$ and $s \in \{0, 1, 2\}$, the maximum number $f_3(3k + s)$ of subgraphs isomorphic to K_3 in a 1-planar graph with $3k + s$ vertices is given by*

$$f_3(3k + s) = \begin{cases} 32 & \text{if } 3k + s = 8, \\ 19k + 5s - 18 & \text{otherwise.} \end{cases}$$

Theorem 1.2.   *Given integers $k \geq 2$, and $s \in \{0, 1, 2\}$ and $t \in \{4, 5, 6\}$, the maximum number $f_t(3k + s)$ of subgraphs isomorphic to K_t in a 1-planar graph with $3k + s$ vertices is given by*

$$f_t(3k + s) = (k - 1)\binom{6}{t} + \binom{s + 3}{t}.$$

Building on these questions, it is natural to ask for a structural characterisation of the extremal graphs. The planar graphs which maximise the number of triangles are the planar graphs which can be formed from K_3 by iteratively pasting copies of K_4 on facial triangles. These graphs are called Apollonian networks, and they are also the planar graphs with the maximum number of cliques of size four. We provide such a structural characterisation for 1-planar graphs, determining precisely which graphs attain the bounds in Theorems 1.1 and 1.2. When the number of vertices is divisible by three, these extremal graphs are analogous to Apollonian networks; they are the 1-planar graphs
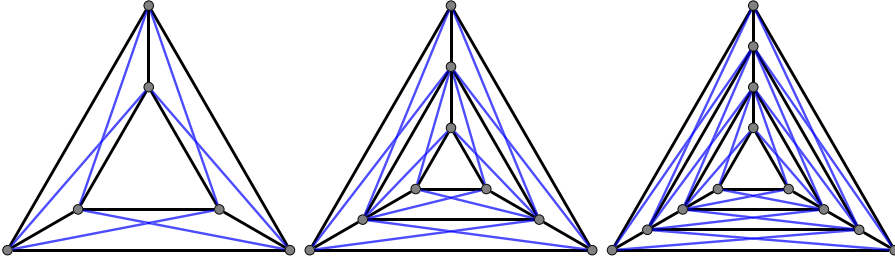
Fig. 1.1. Examples of extremal graphs for Theorems 1.1, 1.2 and 1.3.

formed from K_3 by iteratively pasting copies of K_6 on facial triangles [6] (see Fig. 1.1). We give a formal description of all extremal graphs in Section 3.

In the class of planar graphs, the n-vertex graphs with the maximum number of cliques of size four also have the maximum number of triangles and the maximum number of edges [14], and therefore have the maximum total number of cliques. In fact this was Wood's motivation for counting cliques of size four in planar graphs [18]. The same is true of the class of K_r-free graphs, where the graphs with the maximum number of cliques of size t are exactly those with the maximum number of edges for all t between 2 and r. In the case of 1-planar graphs, the situation is not as simple. In fact, for n at least 9 and t between 3 and 6, every n-vertex 1-planar graph which maximises the number of cliques of size t has strictly fewer than $4n - 8$ edges (i.e. the maximum number of edges in an n-vertex 1-planar graph). Furthermore, when n is at least 7 and not divisible by 3, there is no n-vertex 1-planar graph which simultaneously maximises the number of triangles and cliques of size five. We nevertheless determine the maximum total number of cliques in n-vertex 1-planar graphs. We also characterise the 1-planar graphs attaining this bound, which we describe in Section 3. When n is divisible by three, these are the previously described graphs which maximise the number of cliques of all fixed sizes between 3 and 6 (see Fig. 1.1).

Theorem 1.3. *Given integers $k \geq 1$ and $s \in \{0, 1, 2\}$, the maximum number $f(3k + s)$ of cliques in a 1-planar graph with $3k + s$ vertices is given by*

$$f(3k + s) = 56(k - 1) + 2^{s+3}.$$

We summarise our results in the context of the extremal functions for counting cliques in 1-planar graphs in Table 1.

1.1. Related extremal results

The problem of maximising cliques in a given graph class has long been an important topic in extremal combinatorics. Motivated by the famous problem of Erdős of maximising pentagons in triangle-free graphs (solved by Hatami et al. [19] and Grzesik [20]), the dual problem of maximising triangles in the class of pentagon-free graphs was introduced by Bollobás and Győri [21], and has still not been resolved. There are now many graphs H for which the problem of maximising triangles in H-free graphs has been studied, see for example [22,23]. Determining the maximum number of cliques in a regular graph was a famous open problem of Alon and Kahn (phrased in terms of independent sets). Kahn [24] resolved the problem under the assumption that the graph is bipartite. The conjecture was finally settled by Zhao [25] who was able to deduce the general case from the bipartite case. The maximum number of cliques in the class of graphs with bounded clique or independence number was determined by Cutler and Radcliffe [26]. If only the number of edges and vertices are fixed, then Wood [18] determined the maximum possible number of cliques.

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[6] We will see in Section 7 that these graphs are exactly the graphs which are isomorphic to the strong product $K_3 \boxtimes P_m$ of a triangle K_3 and a path P_m of length $m - 1$ for some positive integer m.

Table 1

The maximum number $f_t(n)$ of cliques of size t and the maximum number $f(n)$ of cliques in n-vertex 1-planar graphs.

	$n \le 6$	$n \ge 7$					
		$n \equiv 1 \pmod 3$		$n \equiv 2 \pmod 3$		$n \equiv 0 \pmod 3$	
		$n = 7$	$n > 7$	$n = 8$	$n > 8$	$n = 9$	$n > 9$
$f_2(n)$	$\binom{n}{2}$	19	$4n-8$	$4n-8$		27	$4n-8$
$f_3(n)$	$\binom{n}{3}$	$\dfrac{19n-58}{3}$		32	$\dfrac{19n-62}{3}$	$\dfrac{19n-54}{3}$	
$f_4(n)$	$\binom{n}{4}$	$5n-19$		$5n-20$		$5n-15$	
$f_5(n)$	$\binom{n}{5}$	$2n-8$		$2n-9$		$2n-6$	
$f_6(n)$	$\binom{n}{6}$	$\dfrac{n-4}{3}$		$\dfrac{n-5}{3}$		$\dfrac{n-3}{3}$	
$f(n)$	$\sum\limits_{t=0}^{n}\binom{n}{t}$	$\dfrac{56n-176}{3}$		$\dfrac{56n-184}{3}$		$\dfrac{56n-144}{3}$	

A more topological example is the class of graphs embeddable on an arbitrary fixed surface, where Huynh, Joret and Wood [27] determined asymptotically the maximum number of triangles and the maximum number of subgraphs isomorphic to K_4. In the case of K_t for $t \ge 5$ they obtained estimates on the maximum number of subgraphs isomorphic to K_t (which for $t \ge 5$ is constant with respect to n). Dujmović et al. [28] considered the question of maximising the total number of cliques in graphs embeddable on an arbitrary fixed surface, determining the precise asymptotic dependence on n, and giving a structural description of the class of extremal graphs.

Other minor closed classes have also been considered. Wood [29] determined for all t and $r \le 9$ the maximum number of subgraphs isomorphic to K_t in the class of K_r-minor free graphs, as well as the maximum total number of cliques in these classes. While giving bounds on the number of graphs in proper minor closed classes, Norine, Seymour, Thomas and Wollan [30] obtained a bound on the total number of cliques in a graph with no K_r-minor (of the form $(\alpha r\sqrt{\log r})^r n$ for some fixed constant α). A better bound, of the form $2^{O(r\sqrt{\log r})}n$, was obtained by Reed and Wood [31], and used in an algorithmic problem about finding separators. Fomin, Oum and Thilikos [32] improved the bound to $2^{O(r\log(\log(r)))}n$. This bound was further improved to $2^{5r+o(r)}n$ by Lee and Oum [33], resolving a conjecture of Wood [18], and finally to $3^{2r/3+o(r)}n$ by Fox and Wei [34], which is sharp for $n \ge 4r/3$. This result was subsequently strengthened by Fox and Wei [35] to hold in the class of K_r-subdivision free graphs. Kawarabayashi and Wood [36] considered the class of graphs forbidding a given odd-minor and proved a bound of $O(n^2)$ on the number of cliques, which is tight up to a constant factor.

In this paper we emphasise only the extremal problem of bounding the number of cliques in graph classes. However, the algorithmic problem of enumerating the cliques (or other subgraphs) in graphs from a given graph class has also garnered much attention. In particular, Chiba and Nishizeki [37] gave an algorithm to list the cliques in any graph, whose running time is linear in the number of cliques and arboricity of the graph.

1.2. Overview of the proofs

We will prove Theorems 1.1, 1.2 and 1.3 by proving the stronger results Theorems 6.6, 6.9 and 6.11 stated in Section 6, which additionally give a precise characterisation of the extremal graphs. We now give a rough overview of the proof of Theorem 6.6. The proofs of Theorems 6.9

and 6.11 work in a similar way. We proceed by induction on the number of vertices n of an extremal graph G. When n is at most twenty, we determine all possible extremal graphs with some computational assistance.

For larger values of n, we fix a 1-drawing of G (a drawing of G in which each edge is crossed by at most one other edge, see Section 2.2), with some additional useful properties (see Section 4). Our initial strategy will be to iteratively find and delete crossed edges which are contained in at most 3 triangles. If we can reduce G to a planar graph in this way, then we can bound the number of triangles in G in terms of the crossing number and the extremal number of triangles in planar graphs to complete the proof. Otherwise, G contains a subgraph such that every crossed edge is in at least four triangles. Using the properties of the drawing and the simple observation that vertex-disjoint triangles cross an even number of times, we are able to deduce that G contains a subgraph isomorphic to either $K_{2,2,2,2}$ or K_6. It turns out that no 3-connected 1-planar graph contains a proper subgraph isomorphic to $K_{2,2,2,2}$, and it is easy to show that G is 3-connected, so we deduce that G has a subgraph isomorphic to K_6. By analysing the drawing restricted to this subgraph, we find a triangle which separates G. This allows us to inductively deduce the structure of G from the structure of the two sides of the separation, and thus compute the extremal function.

1.3. Structure of the paper

The paper is structured as follows. In Section 2, we formally define drawings, set up the notation which will be used throughout the paper and collect some useful results from the literature. In Section 3, we present the classes of extremal graphs for Theorems 1.1, 1.2 and 1.3 and briefly discuss some properties of these constructions. In Section 4, we describe a class of 1-drawings with some useful properties, and prove interesting facts about them. In Section 5, we compute an upper bound for the number of cliques of any given size in a 1-planar graph with no subgraph isomorphic to K_6. In Section 6, we complete the proofs of Theorems 1.1, 1.2 and 1.3, and additionally we show that the classes of graphs defined in Section 3 are exactly the classes of extremal graphs. In Section 7, we give more precise characterisations of the classes of extremal graphs in terms of tree-decompositions, allowing us to generate the graphs in each class in polynomial time. We conclude in Section 8 with a discussion of some open questions.

2. Preliminaries

2.1. Basic notation

Given an integer k, we denote by $[k]$ the set of all integers i with $1 \leq i \leq k$.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote an edge between vertices v and w by the string vw. For a set $X \subseteq V(G)$, we denote the subgraph of G *induced* on X by $G[X]$. We denote by $G - X$ the subgraph of G induced on $V(G) \setminus X$. We denote by $\overline{G}$ the *complement* of G, which is the graph with vertex set $V(G)$ and edge set $\{vw : v, w \in V(G), vw \notin E(G)\}$. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of G respectively.

Let G and H be graphs. We denote by $G \cap H$ the *intersection* of G and H, which is the graph on $V(G) \cap V(H)$ with the edge set $E(G) \cap E(H)$. We denote by $G \cup H$ the *union* of G and H, which is the graph on $V(G) \cup V(H)$ with the edge set $E(G) \cup E(H)$, and we denote by $G \sqcup H$ the *disjoint union* of G and H, which is the graph on the disjoint union of $V(G)$ and $V(H)$ whose edge set is the disjoint union of $E(G)$ and $E(H)$. We denote by $G + H$ the *graph join* of G and H, which is the union of $G \sqcup H$ with the complete bipartite graph with bipartition classes $V(G)$ and $V(H)$. We denote by $G \boxtimes H$ the *strong product* of G and H, which is the graph whose vertex set is the cartesian product $V(G) \times V(H)$ with an edge between (v, w) and (v', w') if and only if either

- $v = v'$ and $ww' \in E(H)$;
- $vv' \in E(G)$ and $w = w'$; or
- $vv' \in E(G)$ and $ww' \in E(H)$.

We write $H \subseteq G$ if H is a subgraph of G, that is $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$. We write $\mathcal{N}(G, H)$ for the number of subgraphs $H' \subseteq G$ that are isomorphic to H.

For a non-negative integer k, we denote by K_k a fixed clique (complete graph) with k vertices, and if $k \geq 1$ we denote by P_k a fixed path with k vertices, and if $k \geq 3$ we denote by C_k a fixed cycle with k vertices. Given an integer $\ell \geq 2$ and a family $(k_i : i \in [\ell])$ of positive integers, we denote by $K_{k_1, \ldots, k_\ell}$ a fixed complete multipartite graph with ℓ partition classes of sizes $k_1, \ldots, k_\ell$, respectively.

A *separation* of G is a pair (A, B) of subsets of $V(G)$ such that $G[A] \cup G[B] = G$. Its *order* is the size of its *separator* $A \cap B$, and we call a separation of order k for some integer k a k-*separation*. We say a separation (A, B) is *non-trivial* if both $A \setminus B$ and $B \setminus A$ are non-empty. For a positive integer k, we say G is k-*connected* if G has at least $k + 1$ vertices and no non-trivial k-separation. A subset S of $V(G)$ is a *non-trivial* $|S|$-*separator of* G if $G - S$ has at least two components, i.e. there is a non-trivial separation of G with S as its separator. Given subsets S, X and Y of $V(G)$, we say S *separates* X and Y if there is a separation (A, B) whose separator is S with $X \subseteq A$ and $Y \subseteq B$. We call a triangle C in G for which $V(C)$ is a non-trivial 3-separator a *separating triangle of* G.

2.2. Drawings

Given a graph G, we arbitrarily fix a linear ordering $\prec$ of $V(G)$ and define a topological space $|G|$ as follows [7] The ground set is the disjoint union of $V(G)$ together with the cartesian product $E(G) \times (0, 1)$ of $E(G)$ and the open unit interval $(0, 1)$. A subset S of $|G|$ is open if for every edge e, there is an open subset S'_e of $(0, 1)$ with $S \cap (\{e\} \times (0, 1)) = \{e\} \times S'_e$, and for every vertex $v \in S$ and every neighbour w of v, there is some $\varepsilon > 0$ such that $\{vw\} \times (0, \varepsilon) \subseteq S$ if $v \prec w$ and $\{vw\} \times (1 - \varepsilon, 1) \subseteq S$ if $w \prec v$. For an edge vw of G, we refer to $\{v, w\} \cup (\{vw\} \times (0, 1))$ as an *edge of* $|G|$, and to the points in $\{vw\} \times (0, 1)$ as *inner points* of this edge. Note that up to homeomorphism, changing the linear ordering of $V(G)$ does not change $|G|$.

For a surface space $\mathbb{S}$ a *drawing of G in* $\mathbb{S}$ is a map $\varphi \colon |G| \to \mathbb{S}$ with the following properties:

(D1) φ is continuous;

(D2) φ is injective on the vertices of G, i.e. for every vertex v, if $\varphi(v) = \varphi(x)$, then $x = v$;

(D3) there are only finitely many points $x \in \mathbb{S}$ for which $\varphi^{-1}(x)$ has size at least 2; we call these points the *crossings (of φ)*;

(D4) each crossing x of φ satisfies $|\varphi^{-1}(x)| = 2$, and the corresponding points of $|G|$ are inner points of distinct edges; we call these edges the *edges involved in the crossing*;

(D5) for each crossing x of φ there is an open set D with $x \in D$ such that for every open set $D' \subseteq D$ with $x \in D'$ the removal of the image of one of the edges involved in x disconnects D' such that multiple components contain image points of the other edge involved in x.

For two edges involved in a crossing of φ, we say these edges *cross* (with respect to φ). An edge which crosses some other edge is called a *crossed edge* (with respect to φ). We may omit the phrase "with respect to φ" if the drawing we are referring to is clear from context.

Intuitively we think of a drawing as the image of such a map φ in the surface: a representation of a graph in the surface, where the vertices of the graph are represented by distinct points and the edges by Jordan arcs joining the corresponding pairs of points. Properties (D3) and (D4) mean that there are finitely many points where exactly two edges cross. Property (D5) means that for each crossing point the edges really "cross" and not just "touch".

Let φ be a drawing of a graph G (in a surface $\mathbb{S}$). For simplicity, we may refer to the images of vertices or edges simply as vertices or edges, respectively, of the drawing (and similarly for other graph structures). For a subgraph H of G we write $\varphi {\restriction} H$ for the drawing of H in $\mathbb{S}$ which is just the restriction of φ to $|H|$ (considered as a subspace of $|G|$). We write $\varphi(H)$ instead of $\varphi(|H|)$ for the image of $|H|$, and for simplicity, we will write $\varphi(e)$ instead of $\varphi(G[v, w])$ for an edge $e = vw$. Given another drawing φ' of G in $\mathbb{S}$, we say φ and φ' are *equivalent* if there is an automorphism ψ of $\mathbb{S}$ such that $\psi(\varphi'(H)) = \varphi(H)$ for every subgraph H of G. Moreover, we say φ and φ' are *weakly*

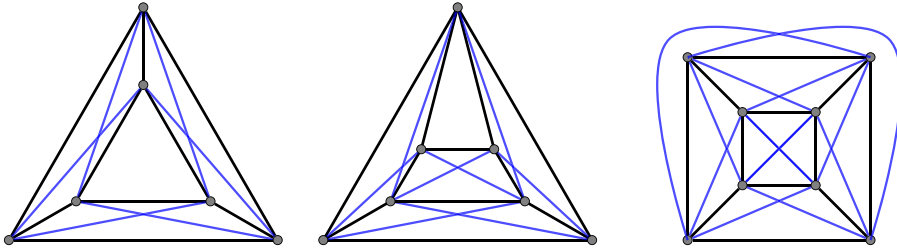[7] This is just the simplicial 1-complex of G.

Fig. 2.1. The unique simple 1-planar drawings (up to weak equivalence) of K_6 on the left, of $K_3 + C_4$ in the middle and of $K_{2,2,2,2}$ on the right.

equivalent if φ is equivalent to the composition of φ' with some automorphism of G. Moreover, we say φ and φ' are *weakly equivalent* if there is an automorphism ϑ of G and a homeomorphism ψ of $\mathbb{S}$ such that $\psi(\varphi'(G[\vartheta(S)])) = \varphi(G[S])$ for every $S \subseteq V(G)$.

A *region* R of $\mathbb{S} \setminus \varphi(G)$ is a connected component of that topological space. For $X \subseteq \mathbb{S}$, we denote by $\mathrm{cl}(X)$ the topological closure of X in $\mathbb{S}$. The *boundary* of a region R is $\mathrm{cl}(R) \setminus R$. If the boundary of R does not contain any crossings of φ, then we call R a *face* of φ. Note that if the boundary of a face contains an inner point of some edge e of G, then it contains $\varphi(e)$. We define the *degree* of a face F to be the number of edges e for which $\varphi(e)$ is contained in the boundary of F, plus the number of edges e for which $\varphi(e)$ is contained in the boundary of F and not contained in the boundary of any other region of $\mathbb{S} \setminus \varphi(G)$. If the boundary of a face is equal to $\varphi(C)$ for a cycle $C \subseteq G$, then we call C a *facial cycle* with respect to φ. Note that if G is 2-connected, then the boundary of each face of φ is a facial cycle with respect to φ. If the drawing is clear from the context, we may refer to the faces of the drawing or facial cycles with respect to the drawing by faces or facial cycles, respectively, of the graph.

We say φ is *simple* if no two adjacent edges cross. Given a non-negative integer k, we say φ is a *k-drawing (of G in $\mathbb{S}$)* if no edge of G is involved in more than k crossings.

Remark 2.1. A graph has a 1-drawing (in $\mathbb{S}$) if and only if it has a simple 1-drawing (in $\mathbb{S}$). □

If φ is a drawing of a graph G in the 2-dimensional sphere $\mathbb{S}^2$, we call φ just a *drawing of G*. From now on we will only consider such drawings.

Note that there is a natural correspondence between drawings in the plane and in $\mathbb{S}^2$.

Given a non-negative integer k, a graph G is called *k-planar* if there is a k-drawing φ of G. Note that the 0-planar graphs in this context are precisely the planar graphs. The minimum integer k for which a graph G is k-planar is the *local crossing number* of G.

A graph G is an *edge-maximal k-planar graph* if G is k-planar and no proper supergraph of G on the same vertex set is k-planar.

Building on earlier work of Schumacher [38], Suzuki [39] studied which edge-maximal 1-planar graphs have unique 1-drawings, up to weak equivalence. We will use the following lemma.

Lemma 2.2 ([39, Lemma 17 and Corollary 4]). *The graphs K_6, $K_3 + C_4$ and $K_{2,2,2,2}$ each have a unique simple 1-drawing up to weak equivalence.*

From Lemma 2.2 and Figs. 2.1 and 2.2 we obtain the following corollary.

Corollary 2.3. *Let G be isomorphic to a graph in $\{K_6, K_3 + C_4, K_{2,2,2,2}\}$ and let φ be a simple 1-drawing of G. The following statements hold.*

(i) Every edge which is not crossed with respect to φ is incident with two edges which cross each other.

(ii) If the boundary of some region R of $\mathbb{S}^2 \setminus \varphi(G)$ contains at least three vertices, then the boundary of R is a facial triangle with respect to φ, and G is not isomorphic to $K_{2,2,2,2}$.
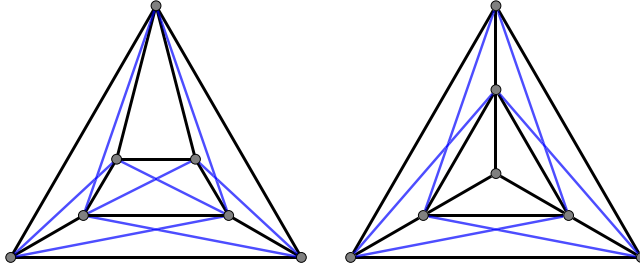
Fig. 2.2. A simple 1-planar drawing of $K_3 + C_4$ on the left and of $K_3 + \overline{K_{1,3}}$ on the right.

(iii) *If G is not isomorphic to $K_{2,2,2,2}$, then exactly two triangles are facial with respect to φ and these are vertex-disjoint.* □

Given a drawing φ of a graph G, we define the *true-planar skeleton* $\mathcal{S}(\varphi)$ of φ to be the subgraph of G containing exactly the edges of G which are not crossed. We will always consider $\mathcal{S}(\varphi)$ to be drawn with the restriction $\varphi{\restriction}\mathcal{S}(\varphi)$, which is a planar drawing of $\mathcal{S}(\varphi)$. For all figures in this paper we use black edges for the true-planar skeletons of the drawings depicted.

2.3. Extremal results for planar and 1-planar graphs

We now collect some results from the literature which will be needed in the upcoming proofs.

A planar graph G is an *Apollonian network* if it is isomorphic to K_3 or it contains a vertex v of degree 3 such that $G - v$ is an Apollonian network.

Theorem 2.4 ([16])**.** *For an integer $n \geq 3$, every n-vertex planar graph has at most $3n - 8$ triangles, with equality if and only if it is an Apollonian network.*

Theorem 2.5 ([17,18])**.** *For an integer $n \geq 4$, every n-vertex planar graph has at most $n - 3$ subgraphs isomorphic to K_4, with equality if and only if it is an Apollonian network.*

Theorem 2.6 ([13])**.** *Given a non-negative integer n, the maximum number $f_2(n)$ of edges in a 1-planar graph with n vertices is given by*

$$f_2(n) = \begin{cases} \binom{n}{2} & \text{if } n \leq 6, \\ 4n - 9 & \text{if } n \in \{7, 9\}, \\ 4n - 8 & \text{otherwise.} \end{cases}$$

Lemma 2.7 ([11])**.** *A simple 1-drawing of an n-vertex graph has at most $n - 2$ crossings.*

Theorem 2.8 ([40])**.** *A 7-vertex graph is 1-planar if and only if it has no subgraph isomorphic to $K_4 + \overline{K_3}$.*

The following corollary is simple to deduce.

Corollary 2.9. *A graph with at most 7 vertices is 1-planar if and only if it is isomorphic to a subgraph of either $K_3 + C_4$ or $K_3 + \overline{K_{1,3}}$.* □

3. Constructing the extremal graphs

In this section, we will construct the extremal graphs for Theorems 1.1, 1.2 and 1.3. In order to do this, we must first give some definitions. The following simple lemma describes a method of combining two 1-planar graphs to form a larger 1-planar graph which is fundamental to our constructions.

Lemma 3.1. *Let G_1 and G_2 be 1-planar graphs such that $G_1 \cap G_2$ is a clique K with at most three vertices. For each $i \in [2]$ let φ_i be a 1-drawing of G_i such that K is contained in $\mathcal{S}(\varphi_i)$ and $\varphi_i(G_i - V(K))$ is contained in a unique region of $\mathbb{S}^2 \setminus \varphi_i(K)$. Then $G := G_1 \cup G_2$ has a 1-drawing φ whose restriction to G_i is equivalent to φ_i for $i \in [2]$, such that no edge of G_1 crosses any edge of G_2.*

Proof. For each $i \in [2]$, let D_i and denote a subset of $\mathbb{S}$ homeomorphic to an open disc such that $\varphi_i(G) \subseteq \mathrm{cl}(D_i)$ and $\varphi_i(G) \setminus D_i = \varphi_i(K)$ (which exists since $K \subseteq \mathcal{S}(\varphi_i)$). Let ψ be an automorphism of $\mathbb{S}$ such that $\psi \circ (\varphi_2{\restriction}K) = \varphi_1{\restriction}K$ and $\psi(D_2) = \mathbb{S}^2 \setminus \mathrm{cl}(D_1)$. The unique 1-drawing of G such that $\varphi{\restriction}G_1 = \varphi_1$ and $\varphi{\restriction}G_2 = \psi \circ \varphi_2$ satisfies the claim. $\quad\square$

This lemma motivates the following definition.

Definition 3.2. Given 1-planar graphs H_1 and H_2, a graph G is a *stitch of H_1 and H_2* if there is a simple 1-drawing φ of G and there are proper subgraphs G_1 and G_2 of G isomorphic to H_1 and H_2 respectively, such that $G_1 \cup G_2 = G$ and $T := G_1 \cap G_2$ is a facial triangle with respect to both $\varphi{\restriction}G_1$ and $\varphi{\restriction}G_2$. We say that (G_1, G_2, T, φ) *witnesses* that G is a stitch of G_1 and G_2.

Lemma 3.1 guarantees that given graphs H_1 and H_2 with 1-drawings φ_1 and φ_2 respectively which both have at least one facial triangle, there is some stitch of H_1 and H_2. Furthermore, every stitch of H_1 and H_2 has a 1-drawing φ such that all but one facial triangle of φ_1 and all but one facial triangle of φ_2 corresponds to a facial triangle of φ. This allows us to iteratively take stitches of copies of a graph H, provided H has a 1-drawing with at least two facial triangles. For example, we observe that the strong product of a triangle and a path is 1-planar by iteratively stitching copies of K_6 together, see Fig. 1.1, where the graph in the middle is a stitch of two copies of the leftmost graph, and the rightmost graph is a stitch of the leftmost graph and the graph in the middle.

On the other hand, we can identify whether a graph is a stitch of two smaller graphs by finding a 1-drawing for which the true-planar skeleton contains a separating triangle, as the following lemma illustrates.

Lemma 3.3. *Given a graph G, there exist graphs H_1 and H_2 such that G is a stitch of H_1 and H_2 if and only if there is a simple 1-drawing φ of G such that $\mathcal{S}(\varphi)$ contains a separating triangle of G.*

Proof. If there exist graphs H_1 and H_2 of which G is a stitch, then the desired 1-drawing and separating triangle can be read off from a witness as in Definition 3.2.

Now suppose G has a simple 1-drawing φ such that $\mathcal{S}(\varphi)$ contains a separating triangle T of G. If T is facial with respect to φ, let (A, B) be any non-trivial separation of G satisfying $A \cap B = V(T)$. Otherwise, let R_1 and R_2 be the two regions of $\mathbb{S}^2 \setminus \varphi(T)$, and let A be the set of all vertices $v \in V(G)$ such that $\varphi(v) \in \mathrm{cl}(R_1)$ and B be the set of all vertices $v \in V(G)$ such that $\varphi(v) \in \mathrm{cl}(R_2)$. In either case, G is a stitch of $G[A]$ and $G[B]$, witnessed by $(G[A], G[B], T, \varphi)$. $\quad\square$

We now define classes $\mathcal{E}_t$ for $t \in \{3, 4, 5, 6\}$ and $\mathcal{E}$, which we will show are the classes of extremal 1-planar graphs for cliques of size t and for the total number of cliques, respectively.

Definition 3.4. We define the classes $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$ and $\mathcal{E}$ recursively as follows. Let $\mathcal{E}_3$ be the class consisting of

- every graph with at most two vertices;
- every graph isomorphic to one of K_3, K_4, K_5, K_6, $K_3 + C_4$ or $K_{2,2,2}$;
- every graph G which is a stitch of some graphs $H_1, H_2 \in \mathcal{E}_3 \cup \{K_2 + \overline{P_6}\}$ such that for each $i \in [2]$ there exist integers $k_i \geq 2$ and $s_i \in \{0, 1, 2\}$ with $s_1 + s_2 \leq 2$ and $|V(H_i)| = 3k_i + s_i$.

Let $\mathcal{E}_4$ be the class consisting of

- every graph with at most three vertices;
- every clique with at most six vertices;
- every graph isomorphic to $K_3 + C_4$;

- every graph G which is a stitch of some graphs $H_1, H_2 \in \mathcal{E}_4$ such that for each $i \in [2]$ there exist integers $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ with $s_1 = 0$ and $|V(H_i)| = 3k_i + s_i$.

Let $\mathcal{E}_5$ be the class consisting of

- every graph with at most four vertices;
- every graph isomorphic to one of K_5 or K_6;
- every graph G which is a stitch of some graphs $H_1, H_2 \in \mathcal{E}_5$ such that for each $i \in [2]$ there exist integers $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ with $s_1 = 0$ and $|V(H_i)| = 3k_i + s_i$.

Let $\mathcal{E}_6$ be the class consisting of

- every graph with at most five vertices;
- every graph isomorphic to K_6;
- every graph G which is a stitch of some graphs $H_1, H_2 \in \mathcal{E}_6$ such that for each $i \in [2]$ there exist integers $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ with $s_1 + s_2 \leq 2$ and $|V(H_i)| = 3k_i + s_i$.

Let $\mathcal{E}$ be the class consisting of

- every clique with at most six vertices;
- every graph isomorphic to $K_3 + C_4$;
- every graph G which is a stitch of some graphs $H_1, H_2 \in \mathcal{E}$ such that for each $i \in [2]$ there exist integers $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ with $s_1 = 0$ and $|V(H_i)| = 3k_i + s_i$.

It is worth noting that since stitches are defined in terms of 1-drawings, the above definitions do not immediately provide an easy way of generating all of the graphs of a given size in any of these classes. In order to generate such a list, one would need a method of determining for each graph in the class which of its triangles are facial with respect to some 1-drawing, and thus which stitches are possible. In Section 7, we discuss in more detail how to generate the n-vertex graphs in these classes, and provide alternative structural descriptions for them.

4. Rich 1-drawings and their true-planar skeletons

Consider the following basic observation about 1-planar graphs.

Lemma 4.1 (*[41, Lemma 3]*). *If G is an edge-maximal 1-planar graph and some edge vw crosses some edge xy in some simple 1-planar drawing of G, then $\{v, w, x, y\}$ induces a K_4 in G.*

This fact motivates the following definition. A 1-drawing φ of a graph G (in a surface $\mathbb{S}$) is called *quasi-rich* if it is simple and for every pair vw and xy of edges which cross, no other edges in $G[\{v, w, x, y\}]$ are crossed, and is called *rich* [8] if it is quasi-rich and for every pair vw and xy of edges which cross, $G[\{v, w, x, y\}]$ is isomorphic to K_4.

Lemma 4.2. *Let φ be a simple 1-drawing of an edge-maximal 1-planar graph G such that G is 4-connected or φ has the minimum possible number of crossings among all 1-drawings of G. Then φ is rich. In particular, every edge-maximal 1-planar graph has a rich 1-drawing.*

Proof. Let vw and xy be a pair of edges which cross. By Lemma 4.1, $\{v, w, x, y\}$ induces a K_4 in G. Suppose for a contradiction that φ is not rich. Without loss of generality, the edge vx is crossed by some edge e. Since vw and xy cross each other and no other edges, there is a region of $\mathbb{S}^2 \setminus \varphi(G)$ whose boundary contains both v and x. Hence the number of crossings could be reduced by redrawing vx through this region, so we may assume that G is 4-connected. However for some endvertex z of e, the set $\{v, x, z\}$ is a 3-separator of G, a contradiction. $\square$

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[8] Note that richness is a property of a 1-drawing, and not a graph. Similar concepts appear in the literature, for example the drawings defining "kite-augmented" 1-planar graphs in [42] or "partial 4-framed" graphs in [43], but there are some technical differences between all of these notions.

Note that not all simple 1-drawings of edge-maximal 1-planar graphs are rich, see for example Fig. 4.1. Lemma 4.2 and Corollary 4.5, which we prove at the end of this section, together imply that for 3-connected edge-maximal 1-planar graphs, rich 1-drawings are exactly the 1-drawings with the fewest crossings. However this is not the case for 1-planar graphs in general. Indeed some 1-planar graphs, such as $K_{3,3}$, contain no cliques of size 4, and hence have no rich 1-drawings. However it is easily seen that deleting crossed edges from a 1-drawing preserves richness, which leads to the following observation.

Remark 4.3. If φ is a rich 1-drawing of a graph G and $\mathcal{S}(\varphi) \subseteq H \subseteq G$, then $\varphi{\restriction}H$ is rich.

The following lemma establishes a strong connection between 3-connected 1-planar graphs with at least five vertices and the true-planar skeletons of their rich 1-drawings, in fact allowing us to completely recover the original graph from the true-planar skeleton.

Lemma 4.4. Let φ be a rich 1-drawing of a 3-connected edge-maximal 1-planar graph G with at least five vertices. The following statements hold.

  (i) $\mathcal{S}(\varphi)$ is 3-connected;
 (ii) no face of $\mathcal{S}(\varphi)$ has degree greater than 4;
(iii) the number of faces of $\mathcal{S}(\varphi)$ of degree 4 is equal to the number of crossings of φ;
(iv) the number of triangles in G is at most the number of non-trivial 3-separators of $\mathcal{S}(\varphi)$, plus four times the number of faces of $\mathcal{S}(\varphi)$ of degree 4, plus the number of faces of $\mathcal{S}(\varphi)$ of degree 3.

Proof. Let $H := \mathcal{S}(\varphi)$. For (i), suppose for a contradiction that there is a non-trivial separation (A, B) of H of order at most 2. Since G is 3-connected, there is an edge in $G - E(H)$ between $A \setminus B$ and $B \setminus A$. Consider such an edge vw, and let xy be the edge which crosses vw in φ. Since φ is rich, $\{vx, vy, wx, wy\} \subseteq E(H)$, so $\{x, y\} = A \cap B$. In particular, vw is the unique edge crossing xy in φ, so (A, B) is a separation of $G - vw$. Since φ is rich, no edge incident with a vertex in $A \setminus B$ crosses any edge incident to a vertex in $B \setminus A$. Also, since $xy \notin E(H)$, no edge incident to x crosses any edge incident to y. Let R be a region of $\mathbb{S}^2 \setminus \varphi(G)$ whose boundary contains x, a segment of an edge e incident to x in $G[A \setminus \{y\}]$ and a segment of an edge e' incident to x in $G[B \setminus \{y\}]$. Note that the boundary of R contains a vertex v' in $A \setminus B$ (either an endvertex of e or an endvertex of an edge which crosses e). Similarly, the boundary of R contains a vertex w' in $B \setminus A$. Since G is edge-maximal, we have $v'w' \in E(G) \setminus E(H)$, and so $v'w'$ crosses xy, and hence $\{v', w'\} = \{v, w\}$. Now x and y lie in different regions of $\mathbb{S}^2 \setminus (R \cup \varphi(vw))$. Hence, every component of $G - \{v, w, x, y\}$ has at most two neighbours in $\{v, w, x, y\}$, contradicting that G is 3-connected.

For (ii), suppose for a contradiction that some facial cycle C of H has length at least 5. Since K_5 is not planar, some pair of vertices of C are not adjacent in H. Since G is maximal, there is a crossed edge between some pair of vertices on C which are not adjacent in H. Since H is 3-connected, this crossed edge is drawn in the face bounded by C. Hence, since φ is rich there is a 4-cycle in H containing only vertices of C. Some edge of this 4-cycle is not in C, and deleting the endvertices of this edge disconnects H, contradicting (i).

For (iii), consider a pair of edges e and f which cross in φ. Since φ is rich, these edges have four distinct endvertices in total which, by (ii), induce a facial cycle of H. Now consider a facial cycle C of H of length 4. Since there is no facial cycle of length 4 with respect to any planar drawing of K_4, there is a pair of vertices of C which are not adjacent in H. As G is maximal, there is a crossed edge between some pair of vertices of C which are not adjacent in H. Since H is 3-connected, this crossed edge is drawn in the face bounded by C. Now, since every face of H has degree at most 4, there is a one-to-one correspondence between crossings of φ and faces of H of degree 4.

For (iv), observe that trivially the number of triangles of G whose vertices all lie on a facial cycle F of H is at most four if F has length 4, and at most one if F has length 3. Consider a triangle T of G whose vertices do not all lie in any facial cycle of H. If no edge of T is crossed in φ, then T is a non-facial triangle of H, and hence a separating triangle of H. Suppose instead that some edge vw of T is crossed by an edge $v'w'$. The vertices v, w, v' and w' are all on the facial cycle of H in which this crossing is drawn, so by assumption neither v' nor w' is a vertex of T. Hence, $\varphi(v')$ and $\varphi(w')$ are

in different regions of $\mathbb{S}^2 \setminus \varphi(T)$. It follows that v' and w' are in separate components of $H - V(T)$. Hence, if T is a triangle of G such that $V(T)$ is not contained in a facial cycle of H, then $V(T)$ is a non-trivial 3-separator of H. $\square$

Corollary 4.5. *A rich* 1-*drawing* φ *of an edge-maximal* 3-*connected* 1-*planar graph* G *on at least* 5 *vertices has exactly* $|E(G)| - 3|V(G)| + 6$ *crossings.*

Proof. Let $H := \mathcal{S}(\varphi)$. By Lemma 4.4, H is a 3-connected planar graph with no face of degree more than 4. Hence, H can be extended to a planar triangulation by adding exactly one edge for each facial cycle of length 4. The result then follows from Lemma 4.4(iii). $\square$

5. Bounding the number of cliques in 1-planar graphs excluding K_6

In this section, we bound the number of cliques of any and all sizes in 1-planar graphs excluding subgraphs isomorphic to K_6. The key tool which allows us to do this is the following lemma. While it only references triangles, it is easy to bound the number of cliques of any size which contain a given edge in terms of the number of triangles which contain it.

Lemma 5.1. *Let* φ *be a rich* 1-*drawing of a graph* G *with at least one crossing. If every crossed edge of* G *is contained in at least four triangles, then* G *has an induced subgraph* H *isomorphic to either* K_6 *or* $K_{2,2,2}$.

Proof. We first observe the following easy fact.

(†) Every 1-drawing of a graph restricted to the union of two vertex-disjoint triangles has 0 or 2 crossings.

Let $v_1 v_2$ and $x_1 x_2$ be two edges which cross, chosen if possible such that v_1, v_2, x_1 and x_2 have a common neighbour. Since φ is rich, $\{v_1, v_2, x_1, x_2\}$ induces a K_4 in G. Note that v_1 and v_2 (and likewise x_1 and x_2) have at least four common neighbours since by assumption the edge between them is in at least four triangles. Choose distinct common neighbours w_1 and w_2 of v_1 and v_2 in $G - \{x_1, x_2\}$, and distinct common neighbours y_1 and y_2 of x_1 and x_2 in $G - \{v_1, v_2\}$, so that $|\{w_1, w_2\} \cap \{y_1, y_2\}|$ is maximised.

First, suppose $\{w_1, w_2\} = \{y_1, y_2\}$. By (†), for some $i, j \in [2]$, the edge $w_1 v_i$ crosses $w_2 x_j$. Hence, since φ is rich, w_1 and w_2 are adjacent. It follows that $H := G[\{v_1, v_2, w_1, w_2, x_1, x_2\}]$ is isomorphic to K_6 (see Fig. 5.1(a)).

Suppose instead that $|\{w_1, w_2\} \cap \{y_1, y_2\}| = 1$. Without loss of generality, we may assume that $w_1 = y_1$. By (†), without loss of generality, $v_2 w_1$ crosses $x_1 y_2$. Since φ is rich, $w_1 x_1$ and $v_2 x_2$ are edges of G which are uncrossed with respect to φ. Now by (†) applied to $w_2 v_1 v_2 w_2$ and $w_1 x_1 x_2 w_1$, for some $i \in [2]$, the edges $v_i w_2$ and $x_2 w_1$ cross with respect to φ. By (†), there is no triangle containing $x_1 y_2$ which is vertex-disjoint from $w_1 v_2 x_2 w_1$. Hence, $x_1 y_2$ is a crossed edge which is contained in at most three triangles (see Fig. 5.1(b)).

Finally, suppose $\{w_1, w_2\} \cap \{y_1, y_2\} = \emptyset$. By (†), for every $i, j \in [2]$ there exist $k, \ell \in [2]$ such that $v_i w_j$ crosses $x_k y_\ell$. Without loss of generality, $x_1 y_1$ crosses $v_1 w_1$. Since φ is rich, $x_1 w_1$, $y_1 v_1$ and $y_1 w_1$ are all uncrossed edges of G. Hence, by our choice of w_1, w_2, y_1 and y_2 we have that y_1 is not adjacent to v_2 and that w_1 is not adjacent to x_2. Applying (†) to the triangles $x_1 x_2 y_2 x_1$ and $v_1 v_2 w_1 v_1$, we find that $w_1 v_2$ crosses $y_2 x_1$, and hence $v_2 y_2$ and $w_1 y_2$ are uncrossed edges of G. Similarly $w_2 v_1$ crosses $y_1 x_2$ and finally $w_2 v_2$ crosses $y_2 x_2$, and so $w_2 y_1$, $w_2 x_2$ and $w_2 y_2$ are all uncrossed edges of G.

Now consider the pair of crossing edges $y_1 x_1$ and $v_1 w_1$. Let w'_1 and w'_2 be the two common neighbours of v_1 and w_1 distinct from x_1 and y_1, and let y'_1 and y'_2 be the two common neighbours of x_1 and y_1 distinct from v_1 and w_1. By our choice of $v_1 v_2$ and $x_1 x_2$, we know that $\{w'_1, w'_2\} \cap \{y'_1, y'_2\} = \emptyset$. Applying (†), there are at least four pairs of crossing edges such that one edge is contained in $A := \{v_1 w'_1, v_1 w'_2, w_1 w'_1, w_1 w'_2\}$ and the other is contained in $B := \{x_1 y'_1, x_1 y'_2, y_1 y'_1, y_1 y'_2\}$. Therefore each edge in A crosses exactly one edge in B, and vice versa. Observe that $v_2 \in \{w'_1, w'_2\}$,

so $v_2w_1 \in A$, which means $x_1y_2 \in B$, so y_2y_1 is an edge in B. Similarly, w_1w_2 is an edge in A, and by process of elimination these edges cross. Now $H := G[\{v_1, v_2, w_1, w_2, x_1, x_2, y_1, y_2\}]$ is isomorphic to $K_{2,2,2,2}$ (see Fig. 5.1(c)). □

Lemma 5.2.   *Let $n \geq 3$ be an integer, let G be a 3-connected n-vertex graph with no subgraph isomorphic to K_6 and let φ be a rich 1-drawing of G with exactly c crossings. Then*

$$\mathcal{N}(G, K_t) \leq \begin{cases} 3c + 3n - 8 & \text{if } t = 3, \\ 3c + n - 3 & \text{if } t = 4, \\ c & \text{if } t = 5. \end{cases}$$

Furthermore, if $t \in \{3, 4\}$ and equality holds, then G has exactly $c + 3n - 6$ edges.

Proof. First suppose that G contains a subgraph H isomorphic to $K_{2,2,2,2}$. By Corollary 2.3(i) no edge in $E(G) \setminus E(H)$ crosses any edge of H.

Suppose for contradiction that $n \geq 9$, and let C be a component of $G - V(H)$. Note that there is some region R of $\mathbb{S}^2 \setminus \varphi(H)$ which contains $\varphi(C)$. Note that every region of $\mathbb{S}^2 \setminus \varphi(H)$ has exactly two vertices in its boundary by Corollary 2.3(ii). The two vertices in the boundary of R separate $V(C)$ from $V(H)$ in G, contradicting the 3-connectivity of G. Therefore $n = 8$, and so $G \cong K_{2,2,2,2}$ by Theorem 2.6. Note that $\mathcal{N}(G, K_3) = 32$, $\mathcal{N}(G, K_4) = 16$ and $\mathcal{N}(G, K_5) = 0$, as required.

Now suppose that G contains no subgraph isomorphic to $K_{2,2,2,2}$. Let $t \in \{3, 4, 5\}$, let $G_0 := G$ and for each $i \in [c]$, let G_i be a graph obtained from G_{i-1} by deleting an edge e_i which is crossed in $\varphi{\restriction}G_{i-1}$ and subject to this is contained in as few triangles of G_{i-1} as possible. By Remark 4.3 and Lemma 5.1, for all $i \in [c]$, the edge e_i is contained in at most three triangles and hence in at most $\binom{3}{t-2}$ copies of K_t in G_{i-1}. It follows that $\mathcal{N}(G_0, K_t) \leq \binom{3}{t-2}c + \mathcal{N}(G_c, K_t)$. Since φ has exactly c crossings, G_c is an n-vertex planar graph and hence contains no copy of K_5, at most $3n - 8$ copies of K_3 by Theorem 2.4, and at most $n - 3$ copies of K_4 by Theorem 2.5. This yields the required bounds for $\mathcal{N}(G, K_t)$. Furthermore, if $t \in \{3, 4\}$ and equality holds, then G_c is an Apollonian network, and thus the number of edges in G is $c + 3n - 6$. □

The following is an immediate corollary of Lemma 4.2, Corollary 4.5, Theorem 2.6 and Lemma 5.2.

Corollary 5.3.   *If $n \geq 5$ and G is a 3-connected edge-maximal 1-planar graph with n vertices and no subgraph isomorphic to K_6, then*

$$\mathcal{N}(G, K_t) \leq \begin{cases} 6n - 14 & \text{if } t = 3, \\ 4n - 9 & \text{if } t = 4, \\ n - 2 & \text{if } t = 5. \end{cases}$$

In particular, G contains at most $16n - 32$ cliques. Furthermore, if G has less than $4n - 8$ edges, then G contains at most $16n - 40$ cliques. □

6. Maximising cliques in 1-planar graphs

In this section we complete the proofs of all of our main theorems. The following lemma is an important tool in each of these proofs, allowing us to apply Lemma 3.3 to extremal graphs which contain subgraphs isomorphic to K_6.

Lemma 6.1.   *If G is a 1-planar graph with a proper subgraph H isomorphic to K_6, then either there is a simple 1-drawing φ of G such that $\mathcal{S}(\varphi)$ contains a separating triangle of G, or there are at least three components of $G - V(H)$ which have at most two neighbours in H.*

Proof. Let G' be an edge maximal 1-planar graph containing G as a subgraph, let φ' be a rich 1-drawing of G' and let φ_0 be the restriction of φ' to G. By Corollary 2.3(i) no edge in $E(G) \setminus E(H)$ crosses any edge of H in φ_0.

Let s be the number of components of $G - V(H)$, and let $\{C_i : i \in [s]\}$ be the set of these components. Note that since H is a proper subgraph of G, we have that $s \geq 1$. Now, for $i \in [s]$, there is a region R of $\mathbb{S}^2 \setminus \varphi_0(H)$ containing $\varphi_0(C_i)$. The vertices in the boundary of R separate $V(C_i)$ from $V(H)$ in G. If the boundary of R is a face of $\varphi_0 \restriction H$ of degree 3, then its facial triangle is a separating triangle of G contained in $\mathcal{S}(\varphi_0)$, as desired. Otherwise, such a component has at most two neighbours in H by Corollary 2.3(ii), so we may assume that $s \leq 2$. For $i \in [s]$, let S_i be the set of neighbours of C_i in H, and let $G_i := G[V(C_i) \cup S_i]$. Now there are at most two edges of H in $\bigcup\{G_i : i \in [s]\}$, so there is a 1-drawing φ^* of H in which each of these edges is in a facial cycle by Corollary 2.3(iii) and the symmetry of K_6. By considering the restriction of φ_0 to G_i, with s applications of Lemma 3.1 we obtain a simple 1-drawing φ of G such that $\varphi \restriction H$ is equivalent to φ^*. Now some facial cycle with respect to φ^* contains S_1, and this is a separating triangle of G. $\square$

6.1. Maximising triangles in 1-planar graphs

Lemma 6.2. *Let G be a graph with at least four vertices and at least $(19|V(G)| - 62)/3$ triangles. If (A, B) is a separation of G of order at most 2, then for some $X \in \{A, B\}$ we have that $G[X]$ has at least $(19|X| - 53)/3$ triangles.*

Proof. Since (A, B) is a separation of order at most 2, we have that

$$\mathcal{N}(G, K_3) = \mathcal{N}(G[A], K_3) + \mathcal{N}(G[B], K_3).$$

Suppose for a contradiction that $\mathcal{N}(G[X], K_3) < (19|X| - 53)/3$ for each $X \in \{A, B\}$. Then

$$\mathcal{N}(G, K_3) < \frac{19(|A| + |B|) - 106}{3} \leq \frac{19(|V(G)| + 2) - 106}{3} < \frac{19|V(G)| - 62}{3}. \quad \square$$

Lemma 6.3. *Let k, s and n be integers such that $k \in \{1, 2\}$, $s \in \{0, 1, 2\}$ and $n = 3k + s$, and let G be a 1-planar graph with n vertices. If G contains at least $19k + 5s - 18$ triangles, then G is isomorphic to one of K_3, K_6, $K_3 + C_4$, $K_2 + \overline{P_6}$ or $K_{2,2,2,2}$. In particular, G contains exactly $19k + 5s - 18$ triangles unless G is isomorphic to $K_{2,2,2,2}$.*

Proof. The cases where $n \leq 6$ are trivial, so we assume $k = 2$ and $s \in \{1, 2\}$. If $s = 1$, the claim follows easily from Corollary 2.9. If $s = 2$, note that no proper subgraph of $K_2 + \overline{P_6}$ or $K_{2,2,2,2}$ has at least 30 triangles (since every edge of $K_2 + \overline{P_6}$ is in a triangle and every edge of $K_{2,2,2,2}$ is in more than two triangles), so we may take G to be an edge-maximal 1-planar graph. If G has a 2-separation (A, B) with $|A| = 7$, then $G[A]$ has at most 22 triangles by Corollary 2.9, and so G has at most 23 triangles. If G has a 2-separation (A, B) with $|A| \in \{5, 6\}$, then the number of triangles in G is at most $\binom{|A|}{3} + \binom{10 - |A|}{3}$, and is therefore at most 24. Hence, G is 3-connected. Let φ be a rich 1-drawing of G. By Lemma 2.7, we know that φ has at most 6 crossings. If φ has at most 4 crossings, then G has at most $3 \cdot 4 + 3 \cdot 8 - 8 = 28$ triangles by Lemma 5.2.

If φ has exactly 5 crossings, then by Lemma 4.4, $\mathcal{S}(\varphi)$ is a 3-connected planar graph with 8 vertices with 5 faces of degree 4, whose other faces each have degree 3. Note that the number of edges of a 3-connected planar graph can be determined from the degrees of its faces, and then the number of vertices can be deduced from Euler's formula. Since G has 8 vertices, we can deduce that $\mathcal{S}(\varphi)$ has exactly two faces of degree 3, and exactly 13 edges. Furthermore, the number of non-trivial 3-separators in $\mathcal{S}(\varphi)$ is at least $30 - 4 \cdot 5 - 2 = 8$, by Lemma 4.4(iv). Using the program *plantri* written by Brinkmann and McKay [44], we find that there is a unique 3-connected 8-vertex planar graph with 13 edges, faces of degree at most 4 and at least eight non-trivial 3-separators. The unique maximal 1-planar graph with this true-planar skeleton is $K_2 + \overline{P_6}$ (see Fig. 3.1), since it is formed by adding edges between all pairs of vertices of $\mathcal{S}(\varphi)$ that share a face.

If φ has exactly 6 crossings, then by Lemma 4.4, $\mathcal{S}(\varphi)$ is a 3-connected planar quadrangulation. The cube is the unique 3-connected planar quadrangulation on 8-vertices, and $K_{2,2,2,2}$ is the unique maximal 1-planar graph with this true-planar skeleton. $\square$
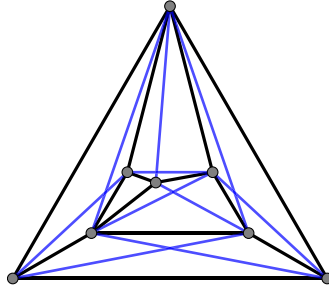
Fig. 3.1. A simple 1-drawing of $K_2 + \overline{P_6}$. We will show in Section 7.2 that this drawing is unique (up to weak equivalence).
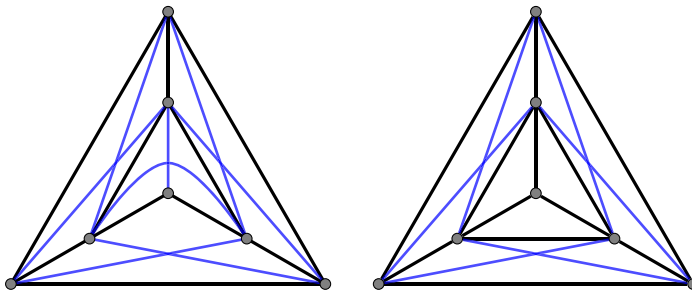

Fig. 4.1. Simple 1-planar drawings of $K_3 + \overline{K_{1,3}}$, the one on the left is not rich, the one on the right is rich.
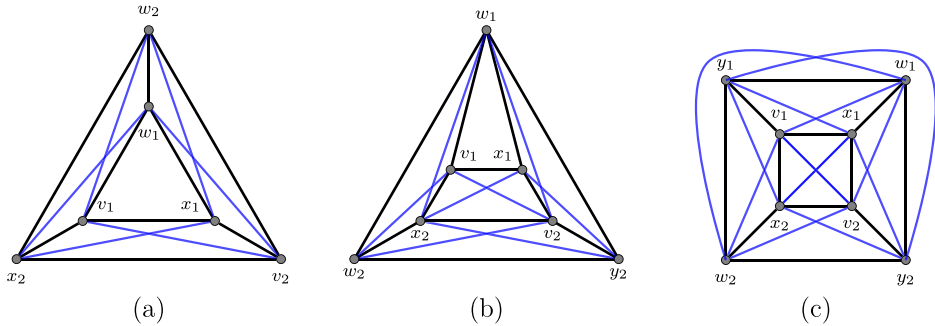

Fig. 5.1. The three cases in the proof of Lemma 5.1.

Lemma 6.4. *Let k, s and n be integers such that $3 \le k \le 6$, $s \in \{0, 1, 2\}$ and $n = 3k + s$, and let φ be a rich 1-drawing of an edge-maximal 1-planar 3-connected graph with n vertices. If G contains at least $19k + 5s - 18$ triangles, then $\mathcal{S}(\varphi)$ contains a separating triangle of G.*

Proof. By Lemma 6.1, we may assume G has no subgraph isomorphic to K_6. Hence, by Lemma 5.2, G has at most $3c + 3n - 8$ triangles, where c is the number of crossings of φ. It follows that $c \ge n - (10 + s - k)/3$. Note that $(10 + s - k)/3 \le 3$ with equality if and only if $n = 11$, and that $c \le n - 2$ by Lemma 2.7. It follows that either $n = 11$ and $c = 8$, or $c = n - 2$.

Suppose that $n = 11$ and $c = 8$. Note that the number of triangles in G is at least 49. By Lemma 4.4, $\mathcal{S}(\varphi)$ has exactly eight faces of degree 4, and the other faces of $\mathcal{S}(\varphi)$ are of degree 3. Thus there are exactly two facial triangles of $\mathcal{S}(\varphi)$, by Euler's formula. By Lemma 4.4(iv), there are at least $49 - 4 \cdot 8 - 2 = 15$ non-trivial 3-separators of $\mathcal{S}(\varphi)$. Using *plantri* [44], we find that there are only three planar graphs[9] satisfying the conditions on $\mathcal{S}(\varphi)$, each of which contains a separating triangle of $\mathcal{S}(\varphi)$. Note that since no edge of $\mathcal{S}(\varphi)$ is crossed with respect to φ, every separating triangle of $\mathcal{S}(\varphi)$ is a separating triangle of G.

Suppose instead that $c = n - 2$. By Lemma 4.4, $\mathcal{S}(\varphi)$ is a 3-connected planar graph with $n - 2$ faces of degree 4, and therefore a 3-connected planar quadrangulation. Furthermore, the number of triangles in G is at most $4(n - 2)$ plus the number of non-trivial 3-separators of $\mathcal{S}(\varphi)$. Again using *plantri* to generate all 3-connected planar quadrangulations on at most 20 vertices, it is quick to computationally verify that no such quadrangulation has the requisite number of non-trivial 3-separators. □

Lemma 6.5. *Let k, s and n be integers such that $k \geq 1$, $s \in \{0, 1, 2\}$ and $n = 3k + s$, and let φ be a rich 1-drawing of a graph G with n vertices and at least $19k + 5s - 18$ triangles. Then G either contains exactly $19k + 5s - 18$ triangles or is isomorphic to $K_{2,2,2,2}$.*

Furthermore, if $k \geq 3$, then for each $i \in [2]$ there exist integers $k_i \geq 2$ and $s_i \in \{0, 1, 2\}$ with $s_1 + s_2 \leq 2$ and a $(3k_i + s_i)$-vertex graph H_i with exactly $19k_i + 5s_i - 18$ triangles such that G is a stitch of H_1 and H_2.

Proof. We proceed by induction on n. For $n \leq 8$ the result follows from Lemma 6.3, so we assume $n \geq 9$.

We first show that G is 3-connected. Suppose for a contradiction that G has a non-trivial separation (A, B) of order at most 2. By Lemma 6.2, we may assume without loss of generality that $G[A]$ has at least $(19|A| - 53)/3$ triangles. Let k' and s' be integers such that $|A| = 3k' + s'$ and $s' \in \{0, 1, 2\}$. Note that $19k' + 5s' - 18$ is strictly less than $(19|A| - 53)/3$, and 32 (the number of triangles in $K_{2,2,2,2}$) is strictly less than $(19 \cdot 8 - 53)/3 = 33$. Hence, by induction, $|A| \leq 2$. It follows that the number of triangles in G equals the number of triangles in $G[B]$. However, $5 \leq |B| \leq n - 1$, so the number of triangles in $G[B]$ is less than

$$(19|B| - 53)/3 \leq \frac{19(3k + s - 1) - 53}{3} < 19k + 5s - 18,$$

a contradiction. Hence, G is 3-connected.

If no subgraph of G is isomorphic to K_6, then $19k + 5s - 18 \leq 6n - 14$ by Lemmas 2.7 and 5.2. It follows that $k \leq 6$, and so there is a separating triangle T of G contained in $\mathcal{S}(\varphi)$ by Lemma 6.4. If G contains a subgraph isomorphic to K_6, then $\mathcal{S}(\varphi)$ contains a separating triangle T of G by Lemma 6.1.

By Lemma 3.3, there are graphs H_1 and H_2 such that G is a stitch of H_1 and H_2. The number of triangles in G is equal to the number of triangles in H_1 plus the number of triangles in H_2 minus one. For $i \in [2]$, let $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ be integers such that $|V(H_i)| = 3k_i + s_i$.

Since T is a facial triangle with respect to $\varphi{\upharpoonright}H_i$, by Corollary 2.3(ii) we have that H_i is not isomorphic to $K_{2,2,2,2}$. It follows that the number of triangles in H_i is at most $19k_i + 5s_i - 18$, by induction. Note that either $s_1 + s_2 = s$ and $k_1 + k_2 = k + 1$, or $s_1 + s_2 = s + 3$ and $k_1 + k_2 = k$. However, in the latter case, the number of triangles in G is at most

$$19(k_1 + k_2) + 5(s_1 + s_2) - 37 = 19k + 5s - 22 < 19k + 5s - 18.$$

Hence $s = s_1 + s_2$ and $k_1 + k_2 = k + 1$, and the number of triangles in G is given by

$$\mathcal{N}(H_1, K_3) + \mathcal{N}(H_2, K_3) - 1 \leq 19(k_1 + k_2) + 5(s_1 + s_2) - 37 = 19k + 5s - 18.$$

Since equality holds, for each $i \in [2]$ applying Lemma 6.3 to H_i implies that $k_i \geq 2$. □

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[9] These graphs are true-planar skeletons of rich 1-drawings of a stitch of K_6 and $K_2 + \overline{P_6}$ and the two possible stitches of $K_3 + C_4$ and itself.

The following theorem immediately implies Theorem 1.1.

Theorem 6.6. *Let n, k and s be non-negative integers with $s \in \{0, 1, 2\}$ and $n = 3k + s$. An n-vertex 1-planar graph G contains the extremal number $f_3(n)$ of triangles if and only if $G \in \mathcal{E}_3$. In particular,*

$$f_3(n) = \begin{cases} \binom{n}{3} & \text{if } n \le 6, \\ 32 & \text{if } n = 8, \\ 19k + 5s - 18 & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on n. The result is trivial when $n \le 6$, and follows from Lemma 6.3 when $n \in \{7, 8\}$, so we may assume that $n \ge 9$.

Consider an n-vertex graph G in $\mathcal{E}_3$. By definition, for each $i \in [2]$ there are integers $k_i \ge 2$ and $s_i \in \{0, 1, 2\}$ with $s_1 + s_2 \le 2$ and a graph $H_i \in \mathcal{E}_3 \cup \{K_2 + \overline{P_6}\}$ with $|V(H_i)| = 3k_i + s_i$ such that G is a stitch of H_1 and H_2. Since $s_1 + s_2 \le 2$ and $|V(G)| = |V(H_1)| + |V(H_2)| - 3$, we have $k_1 + k_2 - 1 = k$ and $s_1 + s_2 = s$. Now $\mathcal{N}(G, K_3) = \mathcal{N}(H_1, K_3) + \mathcal{N}(H_2, K_3) - 1$. Note that $K_2 + \overline{P_6}$ has $19 \cdot 2 + 5 \cdot 2 - 18$ triangles, and that neither H_1 nor H_2 is isomorphic to $K_{2,2,2,2}$ by Corollary 2.3(ii). By induction, we have

$$\mathcal{N}(G, K_3) = 19(k_1 + k_2) + 5(s_1 + s_2) - 37 = 19k + 5s - 18.$$

Now suppose that G is an n-vertex graph containing $f_3(n)$ triangles. To complete the proof, we will show that $G \in \mathcal{E}_3$. Note that $\mathcal{N}(G, K_3) \ge 19k + 5s - 18$ by the previous argument. By Lemma 6.5, for each $i \in [2]$ there exist integers $k_i \ge 2$ and $s_i \in \{0, 1, 2\}$ with $s_1 + s_2 \le 2$ and a $(3k_i + s_i)$-vertex graph H_i with exactly $19k_i + 5s_i - 18$ triangles such that G is a stitch of H_1 and H_2. By the induction hypothesis and Lemma 6.3, we have that k_1 and k_2 are each at least 2 and H_1 and H_2 are isomorphic to graphs in $\mathcal{E}_3 \cup \{K_2 + \overline{P_6}\}$, and hence $G \in \mathcal{E}_3$, as desired. □

6.2. Maximising larger cliques in 1-planar graphs

Lemma 6.7. *Let k, s and t be integers with $k \ge 1$, $s \in \{0, 1, 2\}$ and $t \in \{4, 5, 6\}$ and G be a graph with $3k + s$ vertices, and for $i \in [3]$ let (A_i, B_i) be non-trivial separations of G of order at most 2 such that the sets $A_i \setminus B_i$ are pairwise disjoint and $|\bigcap_{i \in [3]} B_i| \ge 3$. If there do not exist integers $k' \ge 1$ and $s' \in \{0, 1, 2\}$ such that some induced subgraph of G with $3k' + s'$ vertices contains more than $(k' - 1)\binom{6}{t} + \binom{s'+3}{t}$ subgraphs isomorphic to K_t, then G contains fewer than $(k - 1)\binom{6}{t}$ subgraphs isomorphic to K_t.*

Proof. For each $i \in [3]$, we may assume that (A_i, B_i) has order exactly 2. Let $G_i := G[A_i]$, let $G_4 := G[\bigcap_{i \in [3]} B_i]$, and for $i \in [4]$ let k_i and s_i be integers with $k_i \ge 1$ and $s_i \in \{0, 1, 2\}$ such that $|V(G_i)| = 3k_i + s_i$. Let k^* and s^* be integers with $k^* \ge 1$ and $s^* \in \{0, 1, 2\}$ such that $\sum_{i \in [4]} s_i = 3k^* + s^*$ and note that since $3k + s - 6 = \sum_{i \in [4]} (3k_i + s_i)$, we have that $\sum_{i \in [4]} k_i = k + 2 - k^*$. Therefore,

$$\mathcal{N}(G, K_t) \le \sum_{i \in [4]} \left((k_i - 1)\binom{6}{t} + \binom{s_i + 3}{t} \right)$$

$$= (k - 2 - k^*)\binom{6}{t} + \sum_{i \in [4]} \binom{s_i + 3}{t}$$

$$\le (k - 2 - k^*)\binom{6}{t} + \sum_{i \in [4]} \left(\frac{s_i}{2}\binom{5}{t} \right)$$

$$< (k - 2 - k^*)\binom{6}{t} + \frac{3k^* + 3}{2}\binom{5}{t}$$

$$= (k - 1)\binom{6}{t} + (k^* + 1)\left(\frac{3}{2}\binom{5}{t} - \binom{6}{t} \right)$$

$$< (k - 1)\binom{6}{t}. \quad □$$

Lemma 6.8. *Let k, s and t be integers with $k \geq 1$, $s \in \{0, 1, 2\}$ and $t \in \{4, 5, 6\}$, let G be a graph with $3k + s$ vertices, and let (A, B) be a non-trivial separation of order 3 of G. For $X \in \{A, B\}$ let k_X and s_X be integers with $k_X \geq 1$ and $s_X \in \{0, 1, 2\}$ such that $|X| = 3k_X + s_X$. If G contains at least $(k - 1)\binom{6}{t} + \binom{s+3}{t}$ subgraphs isomorphic to K_t, then one of the following holds*

1. *for each $X \in \{A, B\}$, the graph $G[X]$ contains exactly $(k_X - 1)\binom{6}{t} + \binom{s_X+3}{t}$ subgraphs isomorphic to K_t, and either $0 \in \{s_A, s_B\}$, or both $t = 6$ and $s_A = s_B = 1$;*
2. *for some $X \in \{A, B\}$, the graph $G[X]$ contains more than $(k_X - 1)\binom{6}{t} + \binom{s_X+3}{t}$ subgraphs isomorphic to K_t.*

Proof. Assume that (2) fails. Observe that $\mathcal{N}(G, K_t) = \mathcal{N}(G[A], K_t) + \mathcal{N}(G[B], K_t)$. Assume for a contradiction that $s_A + s_B \geq 3$, and hence $k_A + k_B = k$. Now, since $s_A, s_B \leq 2$ and $t \in \{4, 5, 6\}$, we have

$$\mathcal{N}(G[A], K_t) + \mathcal{N}(G[B], K_t) \leq (k_A + k_B - 2)\binom{6}{t} + \binom{s_A + 3}{t} + \binom{s_B + 3}{t}$$
$$< (k - 2)\binom{6}{t} + \binom{6}{t} + \binom{s + 3}{t} \leq \mathcal{N}(G, K_t),$$

a contradiction. So $s_A + s_B < 3$, and hence $k_A + k_B = k + 1$ and $s_A + s_B = s$. Since $t \geq 4$,

$$\mathcal{N}(G[A], K_t) + \mathcal{N}(G[B], K_t) \leq (k_A + k_B - 2)\binom{6}{t} + \binom{s_A + 3}{t} + \binom{s_B + 3}{t}$$
$$\leq (k - 1)\binom{6}{t} + \binom{s_A + s_B + 3}{t} \leq \mathcal{N}(G, K_t).$$

Since equality holds, we deduce that (1) holds. $\square$

The following theorem immediately implies Theorem 1.2.

Theorem 6.9. *Let n, k and s be non-negative integers with $s \in \{0, 1, 2\}$ and $n = 3k + s$ and let $t \in \{4, 5, 6\}$. An n-vertex 1-planar graph G contains the extremal number $f_t(n)$ of subgraphs isomorphic to K_t if and only if $G \in \mathcal{E}_t$. In particular, if $n \geq 3$ then*

$$f_t(n) = (k - 1)\binom{6}{t} + \binom{s + 3}{t}.$$

Proof. We proceed by induction on n. The claim is trivial when $n \leq 6$, so assume $n \geq 7$.

Consider an n-vertex graph G in $\mathcal{E}_t$. If $G \cong K_3 + C_4$, then $t = 4$ and $\mathcal{N}(G, K_4) = 16$, as required. Otherwise, for each $i \in [2]$ there exist integers $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ with $s_1 + s_2 \leq 2$ and a graph $H_i \in \mathcal{E}_t$ with $|V(H_i)| = 3k_i + s_i$ such that G is a stitch of H_1 and H_2, with either $s_1 = 0$ or $t = 6$. Since $s_1 + s_2 \leq 2$ and $|V(G)| = |V(H_1)| + |V(H_2)| - 3$, we have that $k_1 + k_2 - 1 = k$ and $s_1 + s_2 = s$. Now $\mathcal{N}(G, K_t) = \mathcal{N}(H_1, K_t) + \mathcal{N}(H_2, K_t)$ since $t > 3$. Note that $\binom{s_1+3}{t} = 0$, $\binom{s_2+3}{6} = 0$ and $\binom{s_2+3}{t} = \binom{s+3}{t}$. By induction, we have

$$\mathcal{N}(G, K_t) = (k_1 - 1)\binom{6}{t} + \binom{s_1 + 3}{t} + (k_2 - 1)\binom{6}{t} + \binom{s_2 + 3}{t}$$
$$= (k - 1)\binom{6}{t} + \binom{s + 3}{t}.$$

Now suppose that G is an n-vertex graph containing $f_t(n)$ subgraphs isomorphic to K_t. To complete the proof, we will show that $G \in \mathcal{E}_t$. By the previous argument we have that $\mathcal{N}(G, K_t) \geq (k - 1)\binom{6}{t} + \binom{s+3}{t}$.

First, assume that G is 4-connected. By Lemma 6.1, G has no subgraph isomorphic to K_6. This is a contradiction if $t = 6$. By Corollary 5.3, we deduce that $t = 4$ and $n = 7$. By Corollary 2.9, G is

isomorphic to a subgraph of $K_3 + \overline{K_{1,3}}$ or of $K_3 + C_4$. Every proper subgraph of each of these contains strictly fewer subgraphs isomorphic to K_4, so $G \cong K_3 + C_4$, as desired.

Suppose instead that G is not 4-connected, and consider a non-trivial 3-separation of (A, B) of G, not necessarily minimal. By Lemma 6.8 and the induction hypothesis, either $t = 6$ or one of $|A|$, $|B|$ is divisible by 3. Hence G has a subgraph isomorphic to K_6, either because $(k-1)\binom{6}{6} + \binom{s+3}{6} \geq 1$ or by Definition 3.4 and the induction hypothesis. Now by Lemma 6.7 and the induction hypothesis, we obtain from Lemma 6.1 a simple 1-drawing φ of G such that $\mathcal{S}(\varphi)$ contains a separating triangle of G. By Lemma 3.3, there are graphs H_1 and H_2 such that G is a stitch of H_1 and H_2. By the induction hypothesis and Lemma 6.8, it follows that $G \in \mathcal{E}_t$, as desired.   □

6.3. Maximising all cliques in 1-planar graphs

For a graph G, we write $\mathcal{N}_{\text{cliques}}(G)$ for the number of subgraphs $H \subseteq G$ that are cliques.

Lemma 6.10. *Let k and s be integers with $k \geq 1$ and $s \in \{0, 1, 2\}$, let G be a graph with $3k + s$ vertices, and let (A, B) be a non-trivial separation of G of order at most 3. For $X \in \{A, B\}$ let k_X and s_X be integers with $k_X \geq 1$ and $s_X \in \{0, 1, 2\}$ such that $|X| = 3k_X + s_X$. If G contains at least $56(k-1) + 2^{s+3}$ cliques, then one of the following statements holds.*

- *(i) The order of (A, B) is exactly 3, one of s_A or s_B is equal to 0, and for all $X \in \{A, B\}$, the graph $G[X]$ contains exactly $56(k_X - 1) + 2^{s_X+3}$ cliques;*
- *(ii) the order of (A, B) is exactly 3 and $G[A \cap B]$ is not isomorphic to a triangle;*
- *(iii) for some $X \in \{A, B\}$, the graph $G[X]$ contains more than $56(k_X - 1) + 2^{s_X+3}$ cliques.*

Proof. Assume that (iii) fails. Let k^* and s^* be integers with $k^* \geq -1$ and $s^* \in \{0, 1, 2\}$ such that $s_A + s_B - |A \cap B| = 3k^* + s^*$ and note that we have $k_A + k_B = k - k^*$ and $s = s^*$ since $3k + s = 3(k_A + k_B) + s_A + s_B - |A \cap B|$. Observe that for all $s' \in \{0, 1, 2\}$, we have $2^{s'+3} \in \{12s' + 4, 12s' + 8\}$. Now

$$\begin{aligned}
\mathcal{N}_{\text{cliques}}(G) &= \mathcal{N}_{\text{cliques}}(G[A]) + \mathcal{N}_{\text{cliques}}(G[B]) - \mathcal{N}_{\text{cliques}}(G[A \cap B]) \\
&\leq 56(k_A + k_B - 2) + 2^{s_A+3} + 2^{s_B+3} - |A \cap B| \\
&\leq 56(k - k^* - 2) + 12(s_A + s_B) + 16 - |A \cap B| \\
&\leq 56(k - k^* - 2) + 12(3k^* + s + |A \cap B|) + 16 - |A \cap B| \\
&\leq 56(k - 1) + 12s - 20k^* + 11|A \cap B| - 40.
\end{aligned}$$

Since $\mathcal{N}_{\text{cliques}}(G) \geq 56(k-1) + 2^{s+3}$, we deduce that $-20k^* + 11|A \cap B| - 40 \geq 4$, and so $|A \cap B| = 3$ and $k^* = -1$. This implies that $s_A + s_B = s$, and either $s_A = s_B = 1$, or one of s_A or s_B is equal to 0. We may further assume that $\mathcal{N}_{\text{cliques}}(G[A \cap B]) = 8$, or else (ii) holds. Now

$$\begin{aligned}
\mathcal{N}_{\text{cliques}}(G[A]) + \mathcal{N}_{\text{cliques}}(G[B]) - 8 &\leq 56(k_A + k_B - 2) + 2^{s_A+3} + 2^{s_B+3} - 8 \\
&\leq 56(k-1) + 2^{s+3} \leq \mathcal{N}_{\text{cliques}}(G)
\end{aligned}$$

and since equality holds, so does (i).   □

The following theorem immediately implies Theorem 1.3.

Theorem 6.11. *Let n, k and s be non-negative integers with $s \in \{0, 1, 2\}$ and $n = 3k + s$. An n-vertex 1-planar graph G contains the extremal number $f(n)$ of cliques if and only if $G \in \mathcal{E}$. In particular, if $n \geq 3$ then*

$$f(n) = 56(k-1) + 2^{s+3}.$$

Proof. We proceed by induction on n. The claim is trivial when $n \leq 6$, so assume $n \geq 7$.

Consider an n-vertex graph G in $\mathcal{E}$. If $G \cong K_3 + C_4$, then $\mathcal{N}_{\text{cliques}}(G) = 72$, as required. Otherwise, for each $i \in [2]$ there exist integers $k_i \geq 1$ and $s_i \in \{0, 1, 2\}$ and a graph $H_i \in \mathcal{E}$ with $|V(H_i)| = 3k_i + s_i$

such that G is a stitch of H_1 and H_2, with $s_1 = 0$. Since $|V(G)| = |V(H_1)| + |V(H_2)| - 3$ and $s_1 = 0$, we have that $k_1 + k_2 - 1 = k$ and $s_2 = s$. Now $\mathcal{N}_{\text{cliques}}(G) = \mathcal{N}_{\text{cliques}}(H_1) + \mathcal{N}_{\text{cliques}}(H_2) - \mathcal{N}_{\text{cliques}}(K_3)$. By induction, we have

$$\begin{aligned}\mathcal{N}_{\text{cliques}}(G) &= 56(k_1 - 1) + 2^3 + 56(k_2 - 1) + 2^{s_2 + 3} - 8 \\ &= 56(k - 1) + 2^{s+3}.\end{aligned}$$

Now suppose that G is an n-vertex graph containing $f(n)$ cliques. To complete the proof, we will show that $G \in \mathcal{E}$. Note that $\mathcal{N}_{\text{cliques}}(G) \geq 56(k - 1) + 2^{s+3}$ by the previous argument, and that G is an edge maximal 1-planar graph since adding edges to a graph increases the number of cliques. Also G is 3-connected by Lemma 6.10 and the induction hypothesis.

First, assume that G contains no subgraph isomorphic to K_6. By Corollary 5.3, we deduce that $n \in \{7, 8\}$. If $n = 7$, then by Corollary 2.9, G is either $K_3 + \overline{K_{1,3}}$ or $K_3 + C_4$, as desired. Now suppose for a contradiction that $n = 8$. First, assume that G has a vertex v of degree at most 4. Now $\mathcal{N}_{\text{cliques}}(G) \leq \mathcal{N}_{\text{cliques}}(G - v) + 2^4 \leq 88$ by induction, and since equality holds we have that $G - v \cong K_3 + C_4$ and that the degree of v is exactly 4. Since G is edge maximal, there is a rich 1-drawing φ of G. By Corollary 2.3(i) no edge of $G - v$ is crossed with respect to φ, so at least four vertices are contained in the boundary of the region of $\mathbb{S}^2 \setminus \varphi(G - v)$ containing $\varphi(v)$. This contradicts Corollary 2.3(ii). We therefore can assume that $\delta(G) \geq 5$ and hence $\Delta(\overline{G}) \leq 2$. It was shown by Bodendiek, Schuhmacher and Wagner [45] that the graph $K_{2,2,2,2}$ is up to isomorphism the only 8-vertex 1-planar graph with 24 edges, and since it contains exactly 81 cliques, we may assume that $\overline{G}$ has at least 5 edges. It is straightforward to verify that of the nine 8-vertex graphs with five edges and maximum degree at most 2, only $\overline{C_5 + K_3}$ contains 88 independent sets of vertices. However, $C_5 + K_3$ contains 31 triangles, and hence is not 1-planar by Lemma 6.3. We therefore deduce that $n \neq 8$, as required.

So we may assume that G contains a subgraph H isomorphic to K_6. By Lemma 6.1, Lemma 3.3 and the fact that G is 3-connected, there are graphs H_1 and H_2 such that G is a stitch of H_1 and H_2. By Lemma 6.10 and the induction hypothesis, we deduce that $G \in \mathcal{E}$, as required. $\qquad\square$

7. Generating extremal graphs efficiently with tree-decompositions

In Section 3, we gave a recursive definition of the classes of graphs which are extremal in the sense of Theorems 6.6, 6.9 and 6.11. While this recursive definition was useful for the proofs in Section 6, it does not by itself provide a way to generate all of the graphs in these classes. The reason for this is that in order to determine which graphs are stitches of two extremal graphs H_1 and H_2, we need to determine which triangles of H_1 and H_2 are facial with respect to some 1-drawing. We solve this problem in Section 7.1, which allows us to provide a structural characterisation of the classes of extremal graphs in terms of tree-decompositions. In Section 7.2, we discuss how to use these characterisations to generate the n-vertex graphs in each of these classes in polynomial time.

7.1. A structural characterisation of the extremal graphs in terms of tree-decompositions

A pair $(T, \mathcal{V})$ is a *tree-decomposition* of G if T is a tree and $\mathcal{V}$ is a family $(V_t : t \in V(T))$ of sets of vertices of G such that

(T1) $V(G) = \bigcup \mathcal{V}$;
(T2) for every edge $e \in E(G)$ there is a $t \in V(T)$ such that $e \in E(G[V_t])$; and
(T3) $V_s \cap V_t \subseteq V_u$ whenever u is a node on the unique path from s to t in T.

We call the sets in $\mathcal{V}$ the *parts* of $(T, \mathcal{V})$. We call the sets $V_s \cap V_t$ for each edge $st \in E(T)$ the *adhesion sets* of $(T, \mathcal{V})$. We call $V(T)$ the set of *nodes* of the tree.

Definition 7.1. We define sets $\mathcal{A}(\mathcal{E}_3)$, $\mathcal{A}(\mathcal{E}_4)$, $\mathcal{A}(\mathcal{E}_5)$, $\mathcal{A}(\mathcal{E}_6)$ and $\mathcal{A}(\mathcal{E})$ as follows:
$\mathcal{A}(\mathcal{E}_3) := \{K_6, K_3 + C_4, K_2 + \overline{P_6}\}$;
$\mathcal{A}(\mathcal{E}_4) := \{K_4, K_5, K_6, K_3 + C_4\}$;

$\mathcal{A}(\mathcal{E}_5) := \{K_4, K_5, K_6\}$;

$\mathcal{A}(\mathcal{E}_6) := \{K_3, K_6\} \cup \{H \subseteq K_5 : H \text{ is 3-connected}\}$;

$\mathcal{A}(\mathcal{E}) := \{K_4, K_5, K_6, K_3 + C_4\}$.

Lemma 7.2. *Let $\mathcal{C}$ be one of $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$ or $\mathcal{E}$. Every 3-connected graph $G \in \mathcal{C}$ with at least 9 vertices has a tree-decomposition $(T, \mathcal{V})$ with the following properties.*

(a) *For each node $t \in V(T)$ the graph $G[V_t]$ is isomorphic to a graph in $\mathcal{A}(\mathcal{C}) \setminus \{K_3\}$;*
(b) *each graph in $\mathcal{H} := \{G[V_s \cap V_t] : st \in E(T)\}$ is a triangle;*
(c) *one of the following holds:*

- *at most one part induces a graph not isomorphic to K_6;*
- *$\mathcal{C}$ is $\mathcal{E}_3$ and two parts each induce a subgraph isomorphic to $K_3 + C_4$ and all other parts induce a subgraph isomorphic to K_6;*
- *$\mathcal{C}$ is $\mathcal{E}_6$ and two parts each induce a subgraph isomorphic to K_4 that share no triangle and all other parts induce a subgraph isomorphic to K_6.*

Proof. Since the stitch of two graphs is 3-connected if and only if both of them are, by the definition of $\mathcal{C}$ we inductively obtain a tree-decomposition $\mathcal{T} = (T, \mathcal{V})$ of G satisfying properties (a), (b) and (c), except that if $\mathcal{C}$ is $\mathcal{E}_6$, then two distinct parts V_x and V_y with $G[V_x] \cong G[V_y] \cong K_4$ may share a triangle D. In this case, for any distinct nodes s and t on the unique path P from x to y in T, we have $D = G[V_s \cap V_t]$ by property (b) and (T3). We now define a tree T' from T by first deleting the edge in $E(P)$ incident to x, adding a new edge between x and y and contracting that edge to a new node z. Moreover, we define $V'_t := V_t$ for all $t \in V(T') \setminus \{z\}$ and $V'_z := V_x \cup V_y$. Now $G[V'_z]$ is a stitch of two copies of K_4, and hence is isomorphic to a graph in $\mathcal{A}(\mathcal{E}_6)$. Hence, the tree-decomposition $(T', (V'_t : t \in V(T')))$ satisfies all desired properties. $\square$

The following lemma now completes the picture, allowing us to determine when a graph with a tree-decomposition as in Lemma 7.2 is in the given class $\mathcal{C}$.

Lemma 7.3. *Let G be a 3-connected graph with tree-decomposition $(T, \mathcal{V})$ as in Lemma 7.2. Let $\mathcal{D}$ be a set of triangles in G and let $F \subseteq E(G)$. The following statements are equivalent.*

(1) *There is a 1-drawing φ of G with respect to which all triangles in $\mathcal{D}$ are facial and no edge in F is crossed.*
(2) - *$\mathcal{D}$ and $\mathcal{H}$ are disjoint;*
   - *for each $D \in \mathcal{H}$, there is a unique edge $st \in E(T)$ such that $G[V_s \cap V_t] = D$;*
   - *for every node $t \in V(T)$ there is a 1-drawing of $G[V_t]$ with respect to which all triangles $D \subseteq G[V_t]$ in $\mathcal{D} \cup \mathcal{H}$ are facial and all edges in $E(G[V_t]) \cap F$ are uncrossed.*

Proof. Suppose that (1) holds. We may assume that φ is quasi-rich, since redrawing a crossed edge so that it is no longer crossed can only increase the set of facial triangles. In particular, we may assume that $\varphi \upharpoonright G[V_t]$ is quasi-rich for every node t of T. Consider a triangle D in $\mathcal{H}$. Note that by property (c) in Lemma 7.2, at most one part of $(T, \mathcal{V})$ induces a graph which contains D and is not isomorphic to a graph in $\{K_6, K_3 + C_4\}$. Let k be the number of distinct parts of $(T, \mathcal{V})$ containing $V(D)$, and let $\{H_i : i \in [k]\}$ be the graphs induced on these parts, so that every graph in $\{H_i : i \in [k-1]\}$ is isomorphic to a graph in $\{K_6, K_3 + C_4\}$. Note that $k \geq 2$ by the definition of $\mathcal{H}$.

Let $x \in [k-1]$ and $y \in [k] \setminus \{x\}$ and let $H := H_x \cup H_y$. By Corollary 2.3(i) applied to $\varphi \upharpoonright H_x$ no edge of H_y crosses any edge of H_x, since $\varphi \upharpoonright H$ is quasi-rich. It follows that for every component C of $H_y - V(D)$ there is a unique region R of $\mathbb{S}^2 \setminus \varphi(H_x)$ containing $\varphi(C)$. Since H_y is 3-connected, the boundary of R contains $V(D)$. The intersection of $\varphi(H_x - E(D))$ with the boundary of R is $V(D)$. Since H_x is 3-connected, the boundary of R is a simple closed curve. It follows that we can redraw the edges of D inside R, so that they do not cross any edge of H, forming a new quasi-rich drawing φ' of H. Again by Corollary 2.3(i), no edge which is crossed with respect to $\varphi \upharpoonright H_x$ can be redrawn so that it is uncrossed. In particular, the edges of D are uncrossed with respect to $\varphi \upharpoonright (H)$. Note that for

each edge $vw \in E(D)$, at most one of the regions of $\mathbb{S}^2 \setminus (\varphi(vw) \cup \varphi'(vw))$ contains a point of $\varphi(H)$, since H is 3-connected. It follows that φ' and $\varphi \upharpoonright H$ are equivalent. Hence $\varphi(D)$ is the boundary of the region of $\mathbb{S}^2 \setminus \varphi(H_x)$ containing $\varphi(C)$, and by symmetry is the boundary of every region of $\mathbb{S}^2 \setminus \varphi(H_x)$ containing $\varphi(C')$ for any component C' of $H_y - V(D)$. Since H_x is 3-connected, it follows that one region of $\mathbb{S}^2 \setminus \varphi(D)$ contains $\varphi(H_y - V(D))$, the other region of $\mathbb{S}^2 \setminus \varphi(D)$ contains $\varphi(H_x - V(D))$, and D is not facial with respect to φ. In particular, $D \notin \mathcal{D}$. Since no edge of H_y crosses any edge of H_x with respect to φ, it follows that D is a facial triangle with respect to both $\varphi \upharpoonright H_x$ and $\varphi \upharpoonright H_y$, and hence D is facial with respect to the restriction of φ to any subgraph of G induced on a part of $(T, \mathcal{V})$ which contains $V(D)$. If $k \geq 3$, then by the pigeonhole principle we can find distinct i and j in $[k]$ such that $\varphi(H_i - V(D))$ and $\varphi(H_j - V(D))$ are contained in the same region of $\mathbb{S}^2 \setminus \varphi(D)$, a contradiction. Hence there is a unique edge $st \in E(T)$ with $D = G[V_s \cap V_t]$. Uncrossed edges with respect to φ are still uncrossed in all restrictions φ to subgraphs of G, so (1) implies (2).

It is an easy consequence of Lemma 3.1 that (2) implies (1). □

Corollary 7.4. *Each tree-decomposition as in Lemma 7.2 also satisfies*

(d) for each $D \in \mathcal{H}$, there is a unique edge $st \in E(T)$ such that $G[V_s \cap V_t] = D$;
(e) for each node $t \in V(T)$, the graph $G[V_t]$ has a 1-drawing with respect to which all triangles $D \subseteq G[V_t]$ in $\mathcal{H}$ are facial.

Proof. By property (c) of Lemma 7.2, for each edge $st \in E(T)$ either $G[V_s]$ or $G[V_t]$ is isomorphic to a graph in $\{K_6, K_3 + C_4\}$. The result now follows from Lemma 7.3. □

Now for the classes $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}$, these results immediately yield a characterisation in terms of these tree-decompositions.

Proposition 7.5. *Let $\mathcal{C}$ be one of $\mathcal{E}_3$, $\mathcal{E}_4$ or $\mathcal{E}$. A graph G with at least 9 vertices is in $\mathcal{C}$ if and only if it has a tree-decomposition $(T, \mathcal{V})$ with the following properties.*

(a) For each $t \in V(T)$ the graph $G[V_t]$ is isomorphic to a graph in $\mathcal{A}(\mathcal{C})$;
(b) each graph in $\mathcal{H} := \{G[V_s \cap V_t] : st \in E(T)\}$ is a triangle;
(c) the number of nodes $t \in V(T)$ such that $G[V_t]$ is not isomorphic to K_6 is at most two, and this inequality is strict unless $\mathcal{C}$ is $\mathcal{E}_3$ and two distinct parts each induce a subgraph isomorphic to $K_3 + C_4$;
(d) for each $D \in \mathcal{H}$, there is a unique edge $st \in E(T)$ such that $G[V_s \cap V_t] = D$;
(e) for each node $t \in V(T)$, the graph $G[V_t]$ has a 1-drawing with respect to which all triangles $D \subseteq G[V_t]$ in $\mathcal{H}$ are facial.

Proof. First note that each $G \in \mathcal{C}$ with at least 9 vertices is 3-connected. Hence the existence of such a tree-decomposition is given from Lemma 7.2 and Corollary 7.4.

For the other direction, if G has a tree-decomposition $(T, \mathcal{V})$ with these properties, then it follows easily from Lemma 3.1 that G is 1-planar. Using property (c), a simple induction on $|V(T)|$ lets us conclude that G is in $\mathcal{C}$. □

Note that not all sufficiently large graphs in $\mathcal{E}_5$ and $\mathcal{E}_6$ are 3-connected, so a structural characterisation in terms of tree-decompositions needs to take separations of order less than 3 into account. Generalising the well-known notion of the block-cutvertex tree, Tutte [46] proved a decomposition theorem about "3-blocks" in graphs, for which we will use the following special case.

Theorem 7.6 *([46]).* *Let G be a 2-connected graph in which every non-trivial 2-separator induces a clique. Then G has a tree-decomposition of adhesion 2 in which every part induces a 3-connected graph or a triangle.*

Corollary 7.7. *Let G be an edge-maximal 1-planar graph, and let φ be a rich 1-drawing of G. Then G has a tree-decomposition in which each part induces a triangle or a 3-connected subgraph, and each adhesion set induces a subgraph isomorphic to K_2, whose unique edge is uncrossed with respect to φ.*

Proof. Let (A, B) be a non-trivial separation of order at most 2 and let R be a region of $\mathbb{S}^2 \setminus \varphi(G[A])$ containing a vertex in $B \setminus A$. We may assume that no vertex of $A \setminus B$ is on the boundary of R, or else we could draw a new uncrossed edge from $A \setminus B$ to $B \setminus A$ inside R, contradicting the edge-maximality of G. Now the boundary of R contains a simple closed curve but contains at most two vertices of $G[A]$, so in particular it contains a crossing of $\varphi{\restriction}G[A]$. It follows that $A \cap B$ consists of exactly two vertices v and w, and that some edge incident to v crosses some edge incident to w. Since φ is rich, $G[A \cap B] \cong K_2$, and the edge in $G[A \cap B]$ is uncrossed with respect to φ. The result now follows from Theorem 7.6. □

This allows us to complete our characterisation for the classes $\mathcal{E}_5$ and $\mathcal{E}_6$. Note that for the sake of simplicity the following characterisation only characterises the graphs in the respective class up to possible deletion of a small constant number of edges that are not contained in any copy of one of the respective cliques.

Proposition 7.8. *A graph G with $n \geq 3$ vertices is in $\mathcal{E}_6$ if and only if it is a spanning subgraph of some graph G' with $\mathcal{N}(G', K_6) = \mathcal{N}(G, K_6)$ which has a tree-decomposition $(T, \mathcal{V})$ satisfying the following properties.*

(a) For each node $t \in V(T)$, the graph $G'[V_t]$ is isomorphic to a graph in $\mathcal{A}(\mathcal{E}_6)$;
(b) each graph in $\mathcal{H} := \{G[V_s \cap V_t] : st \in E(T)\}$ is a clique of order 2 or 3;
(c) there are exactly $\lfloor (n-3)/3 \rfloor$ nodes $t \in V(T)$ such that $G'[V_t]$ is isomorphic to K_6;
(d) for each triangle $D \in \mathcal{H}$, there is a unique edge $st \in E(T)$ such that $G'[V_s \cap V_t] = D$;
(e) for each node $t \in V(T)$, the graph $G'[V_t]$ has a 1-drawing with respect to which all triangles $D \subseteq G[V_t]$ in $\mathcal{H}$ are facial and all edges of $G'[V_t] \cap \bigcup \mathcal{H}$ are uncrossed.

Proof. If there is a supergraph G' with such a tree-decomposition, then G contains $\lfloor (n-3)/3 \rfloor$ subgraphs isomorphic to K_6 by (c). Hence, by Theorem 6.9 it is sufficient to show that G' is 1-planar. This follows easily from Lemma 3.1.

For the other direction, suppose G is in $\mathcal{E}_6$, and let G' be an n-vertex edge-maximal 1-planar graph containing G. By Theorem 6.9, G' is also in $\mathcal{E}_6$. Let $\mathcal{T}' := (T', (V'_t : t \in V(T)))$ be a tree-decomposition of G' as in Corollary 7.7. It is easy to construct a 1-planar graph G'' with a tree-decomposition $(T', (V''_t : V(T)))$ such that each adhesion set has size two, for each $t \in V(T)$ the graph $G[V''_t]$ is in $\mathcal{E}_6$ and $|V''_t| = |V'_t|$. Hence, by Theorem 6.9, each part of $\mathcal{T}'$ induces a graph in $\mathcal{E}_6$.

By Lemmas 7.2 and 7.3, for each $t \in V(T')$, the graph $G[V_t]$ has a tree-decomposition $\mathcal{T}^t := (T^t, (V^t_s))$ satisfying properties (a), (b), (d) as well as

(e') for each node $s \in V(T^t)$ the graph $G'[V^t_s]$ has a 1-drawing with respect to which all triangles $D \subseteq G'[V^t_s]$ in $\mathcal{H}$ are facial and all edges of $G'[V^t_s] \cap \bigcup \mathcal{H}'$ are uncrossed, where $\mathcal{H}' := \{G'[V'_s \cap V'_t] : st \in E(T')\}$.

We now construct a tree T as the disjoint union of the trees T^t for all $t \in V(T')$ by adding for each edge st of T' an edge between some arbitrary nodes $u \in V(T^s)$ and $v \in V(T^t)$ for which $V^s_u \cap V^t_v = V'_s \cap V'_t$. Setting for each $s \in V(T)$ the part V_s to be equal to the part V^t_s for the unique $t \in V(T')$ with $s \in V(T^t)$ yields a tree-decomposition that is easily seen to satisfy properties (a), (b), (d) and (e). By Theorem 6.9, it also satisfies (c). □

For the class $\mathcal{E}_5$ we observe the following.

Proposition 7.9. *An n-vertex graph G is in $\mathcal{E}_5$ if and only if $G \in \mathcal{E}_6$ and either n is congruent to 1 modulo 3 or $G \in \mathcal{E}$.*

7.2. Efficiently generating the extremal graphs

We begin by observing that the $3k$-vertex extremal graphs for any integer $k \geq 2$ are isomorphic to the strong product of a triangle and a path of length k, as mentioned in the introduction.
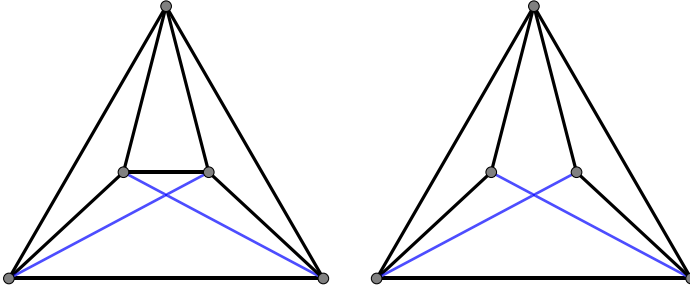
Fig. 7.1. On the left, the unique simple 1-drawing of K_5 (up to weak equivalence), and on the right, the only simple 1-drawing of $K_3 + \overline{K_2}$ (up to weak equivalence) such that the separating triangle is facial.

Corollary 7.10. *Let $\mathcal{C}$ be one of $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\mathcal{E}_6$ or $\mathcal{E}$ and let $k \geq 2$ be an integer, then a 3k-vertex graph G is in $\mathcal{C}$ if and only if G is isomorphic to the strong product of a triangle and a path of length $k - 1$.*

Proof. By the characterisations given in Propositions 7.5, 7.8 and 7.9, a 3k-vertex graph G in $\mathcal{C}$ has a tree-decomposition $(T, \mathcal{V})$ in which each part is isomorphic to K_6 with a drawing in which each adhesion set is a distinct facial triangle. By Corollary 2.3(iii), each part has at most two such triangles, and these triangles are vertex disjoint. Hence T is a path of length $k - 2$, and the result now follows from the definition of the strong product. □

If n is not divisible by 3, Propositions 7.5, 7.8 and 7.9 still allow us to generate all extremal graphs of size n in polynomial time.

If $\mathcal{C}$ is $\mathcal{E}_3$, we observe that $K_3 + C_4$ has a unique 1-drawing which has exactly two facial triangles (see Lemma 2.2 and Corollary 2.3(iii)). Moreover, by Lemma 4.2, each 1-drawing of $K_2 + \overline{P_6}$ is rich, and there is a unique rich 1-drawing of $K_2 + \overline{P_6}$ up to weak equivalence by the computer search in Lemma 6.3. Hence, the tree T in the tree-decomposition from Proposition 7.5 is again a path of length $\lfloor n/3 \rfloor$. With the restrictions on the parts given in the characterisation in Proposition 7.5, we conclude that we can generate all n-vertex graphs in $\mathcal{E}_3$ in polynomial time.

If $\mathcal{C}$ is either $\mathcal{E}_4$ or $\mathcal{E}$, then as there is at most one part inducing a graph not isomorphic to K_6, we observe that T is the subdivision of a star. Moreover, K_4 contains only four triangles, and since every simple 1-drawing of K_5 is rich by Lemma 4.2, and it is easy to observe that the drawing depicted in Fig. 7.1 is the unique 1-drawing of K_5 up to weak equivalence, we conclude that T has maximum degree at most 4. As before, the restrictions on the parts given in the characterisation in Proposition 7.5 allow us to generate all n-vertex graphs in $\mathcal{E}_4$ or $\mathcal{E}$ in polynomial time.

In the case of $\mathcal{E}_6$, note that the only non-planar graphs in $\mathcal{A}(\mathcal{E}_6)$ are K_6 and K_5. In a planar drawing of a graph, every non-separating triangle is facial. Thus in order to apply Proposition 7.8, the only remaining graph to consider is $K_3 + \overline{K_2}$, since this is the only graph in $\mathcal{A}(\mathcal{E}_6)$ with a separating triangle. It is easy to show that there is only one 1-drawing up to weak equivalence such that the separating triangle is facial, see Fig. 7.1. Thus we can again generate all n-vertex graphs in $\mathcal{E}_6$ in polynomial time.

By Proposition 7.9, we can extract all n-vertex graphs in $\mathcal{E}_5$ from the previously generated lists.

8. Concluding remarks

For 1-planar graphs, our results show that for any t and t' in $\{3, 4, 5, 6\}$, there is not much difference between maximising cliques of size t and maximising cliques of size t'. In fact, when n is divisible by 3, the n-vertex graphs with $f_t(n)$ copies of K_t also have $f_{t'}(n)$ copies of $K_{t'}$. This behaviour is similar to what we observe for planar graphs, where Apollonian networks are the graphs maximising both triangles and cliques of size 4. Interestingly however, the 1-planar graphs which maximise cliques of any size greater than 2 do not maximise the number of edges. The n-vertex 1-planar graphs with $4n - 8$ edges, called optimal 1-planar graphs, form an interesting class
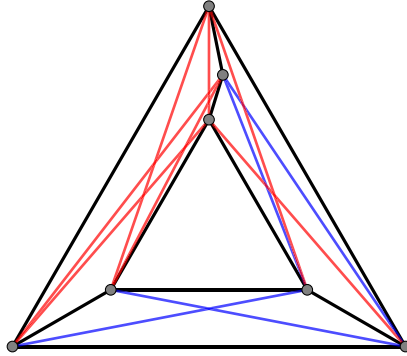
Fig. 8.1. A 2-planar drawing of K_7. The black edges are the edges of the true-planar skeleton, the blue edges are crossed exactly once and the red edges are crossed exactly twice. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

in their own right. Indeed, much of the research into 1-planar graphs has focused explicitly on this subclass. We therefore pose the following question.

Question 8.1. *Let n and t be positive integers with $t \in \{3, 4, 5\}$ and $n \geq 10$. What is the maximum number of subgraphs isomorphic to K_t in an n-vertex 1-planar graph with $4n - 8$ edges?*

For $t = 3$, Lemma 4.4 implies that this question is equivalent to maximising the number of non-trivial 3-separators in a 3-connected planar quadrangulation. In this case, we suspect the answer is given by following construction. Let H_0 be the 4-cycle $v_0 v_1 v_2 v_3 v_0$, let φ_0 be a planar drawing of H_0, and let F_1^0 and F_2^0 be the faces of φ_0. For each positive integer i, let H_i be obtained from H_{i-1} by adding a new vertex v_{i+3} adjacent to v_{i+2} and the non-neighbour of v_{i+2} in the boundary of F_2^{i-1} and let φ_i extend φ_{i-1} to a drawing of H_i by drawing the new vertices and edges inside F_2^{i-1}. Let $F_1^i := F_1^{i-1}$ and let F_2^i be a face of φ_i whose boundary contains v_{i+3}, (chosen arbitrarily). Finally let G_i be obtained from H_i by taking two disjoint copies of the cube Q_3, and identifying a 4-cycle in the first copy with the boundary cycle of F_1^i and a 4-cycle in the second copy with the boundary cycle of F_2^i. We will call the class of graphs G_i constructed this way $\mathcal{G}^*$. Every graph in $\mathcal{G}^*$ is a 3-connected planar quadrangulation, and it can be shown that n-vertex graphs in $\mathcal{G}^*$ have exactly $4n - 14$ non-trivial 3-separators. Lemma 4.4 and Lemma 5.2 together imply that this is within ten of the maximum possible value. We conjecture the following.

Conjecture 8.2. *For $n \geq 14$, the n-vertex 3-connected planar quadrangulations with the maximum number of 3-separators are exactly the n-vertex graphs in $\mathcal{G}^*$.*

In another direction, it is natural to consider extending our results to k-planar graphs for higher values of k. For $k \geq 3$, not even the maximum number of edges is known for n-vertex k-planar graphs, so counting larger cliques is likely to be an extremely difficult problem. However, counting triangles in 2-planar graphs is a logical next step toward broadening our understanding. Here, we conjecture the following.

Conjecture 8.3. *For $n \geq 7$, the maximum number of triangles in an n-vertex 2-planar graph is at most* $(17n - 49)/2$.

The bound in Conjecture 8.3 is achieved by the 2-planar graphs formed by stitching copies of K_7 together. This stitching is possible since K_7 has a 2-drawing with two facial triangles, see Fig. 8.1. It may even be the case that for all positive integers k, maximising copies of the largest k-planar cliques also maximises triangles, up to an additive constant, and vice versa.

References

[1] G. Ringel, Ein Sechsfarbenproblem auf der Kugel, German, Abh. Math. Sem. Univ. Hamburg 29 (1965) 107–117, http://dx.doi.org/10.1007/BF02996313, MR187232.

[2] O.V. Borodin, Solution of the ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs, Russian, Metody Diskret. Analiz. 41 (108) (1984) 12–26, MR832128.

[3] W. Didimo, G. Liotta, F. Montecchiani, A survey on graph drawing beyond planarity, ACM Comput. Surv. 52 (1) (2019) 1–37, http://dx.doi.org/10.1145/3301281.

[4] S.-H. Hong, T. Tokuyama (Eds.), Beyond Planar Graphs, Communications of NII Shonan Meetings, Springer, 2020, http://dx.doi.org/10.1007/978-981-15-6533-5.

[5] S.G. Kobourov, G. Liotta, F. Montecchiani, An annotated bibliography on 1-planarity, Comput. Sci. Rev. 25 (2017) 49–67, http://dx.doi.org/10.1016/j.cosrev.2017.06.002, MR3697129.

[6] K. Kuratowski, Sur le problème des courbes gauches en topologie, Fund. Math. (in French) 15 (1930) 271–283.

[7] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, German, Math. Ann. 114 (1) (1937) 570–590, http://dx.doi.org/10.1007/BF01594196, MR1513158.

[8] J. Hopcroft, R. Tarjan, Efficient planarity testing, J. Assoc. Comput. Mach. 21 (1974) 549–568, http://dx.doi.org/10.1145/321850.321852, MR359387.

[9] A. Grigoriev, H.L. Bodlaender, Algorithms for graphs embeddable with few crossings per edge, Algorithmica 49 (1) (2007) 1–11, http://dx.doi.org/10.1007/s00453-007-0010-x, MR2344391.

[10] V.P. Korzhik, M. Bojan, Minimal obstructions for 1-immersions and hardness of 1-planarity testing, J. Graph Theory 72 (1) (2013) 30–71, http://dx.doi.org/10.1002/jgt.21630, MR2993075.

[11] X. Zhang, G. Liu, J.-L. Wu, $(1, \lambda)$-Embedded graphs and the acyclic edge choosability, Bull. Korean Math. Soc. 49 (3) (2012) 573–580, http://dx.doi.org/10.4134/BKMS.2012.49.3.573, MR2963420.

[12] P. Turán, On an external problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452.

[13] R. Bodendiek, H. Schumacher, K. Wagner, Bemerkungen zu einem Sechsfarbenproblem von G. Ringel, German, Abh. Math. Semin. Univ. Hambg. 53 (1983) 41–52, http://dx.doi.org/10.1007/BF02941309, MR732806.

[14] A.A. Zykov, On some properties of linear complexes, Amer. Math. Soc. Translation, 1952 79 (1952) 33, MR0051516.

[15] P. Erdős, On the number of complete subgraphs contained in certain graphs, English, with Russian summary, Magyar Tud. Akad. Mat. KutatÓ Int. Közl. 7 (1962) 459–464, MR151956.

[16] S.L. Hakimi, E.F. Schmeichel, On the number of cycles of length k in a maximal planar graph, J. Graph Theory 3 (1) (1979) 69–86, http://dx.doi.org/10.1002/jgt.3190030108, MR519175.

[17] N. Alon, Y. Caro, On the number of subgraphs of prescribed type of planar graphs with a given number of vertices, in: Convexity and Graph Theory, Jerusalem, 1981, in: North-Holland Math. Stud., vol. 87, North-Holland, Amsterdam, 1984, pp. 25–36, http://dx.doi.org/10.1016/S0304-0208(08)72803-2, MR791009.

[18] D.R. Wood, On the maximum number of cliques in a graph, Graphs Combin. 23 (3) (2007) 337–352, http://dx.doi.org/10.1007/s00373-007-0738-8, MR2320588.

[19] H. Hatami, J. Hladký, D. KráĬ, S. Norine, A. Razborov, On the number of pentagons in triangle-free graphs, J. Combin. Theory Ser. A 120 (3) (2013) 722–732, http://dx.doi.org/10.1016/j.jcta.2012.12.008, MR3007147.

[20] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, J. Combin. Theory Ser. B 102 (5) (2012) 1061–1066, http://dx.doi.org/10.1016/j.jctb.2012.04.001, MR2959390.

[21] B. Bollobás, E. Győri, Pentagons vs. triangles, Discrete Math. 308 (19) (2008) 4332–4336, http://dx.doi.org/10.1016/j.disc.2007.08.016, MR2433860.

[22] N. Alon, C. Shikhelman, Many T copies in H-free graphs, J. Combin. Theory Ser. B 121 (2016) 146–172, http://dx.doi.org/10.1016/j.jctb.2016.03.004, MR3548290.

[23] E. Győri, H. Li, The maximum number of triangles in C_{2k+1}-free graphs, Combin. Probab. Comput. 21 (1–2) (2012) 187–191, http://dx.doi.org/10.1017/S0963548311000629, MR2900057.

[24] J. Kahn, An entropy approach to the hard-core model on bipartite graphs, Combin. Probab. Comput. 10 (3) (2001) 219–237, http://dx.doi.org/10.1017/S0963548301004631, MR1841642.

[25] Y. Zhao, The number of independent sets in a regular graph, Combin. Probab. Comput. 19 (2) (2010) 315–320, http://dx.doi.org/10.1017/S0963548309990538, MR2593625.

[26] A.J. Cutler, Extremal problems for independent set enumeration, Electron. J. Combin. 18 (1) (2011) 17, Paper 169, post=MR2831105.

[27] T. Huynh, G. Joret, D. Wood, Subgraph densities in a surface, 2020, available at arXiv:2003.13777, Preprint.

[28] V. Dujmović, G. Fijavž, G. Joret, T. Sulanke, D.R. Wood, On the maximum number of cliques in a graph embedded in a surface, European J. Combin. 32 (8) (2011) 1244–1252, http://dx.doi.org/10.1016/j.ejc.2011.04.001, MR2838013.

[29] D.R. Wood, Cliques in graphs excluding a complete graph minor, Electron. J. Combin. 23 (3) (2016) 16, MR3558055, Paper 3.18.

[30] S. Norine, P. Seymour, R. Thomas, P. Wollan, Proper minor-closed families are small, J. Combin. Theory Ser. B 96 (5) (2006) 754–757, http://dx.doi.org/10.1016/j.jctb.2006.01.006, MR2236510.

[31] B. Reed, D.R. Wood, A linear-time algorithm to find a separator in a graph excluding a minor, ACM Trans. Algorithms 5 (4) (2009) 39, http://dx.doi.org/10.1145/1597036.1597043, MR2571902.

[32] F.V. Fomin, S. Oum, D.M. Thilikos, Rank-width and tree-width of H-minor-free graphs, European J. Combin. 31 (7) (2010) 1617–1628, http://dx.doi.org/10.1016/j.ejc.2010.05.003, MR2673004.

[33] C. Lee, S. Oum, Number of cliques in graphs with a forbidden subdivision, SIAM J. Discrete Math. 29 (4) (2015) 1999–2005, http://dx.doi.org/10.1137/140979988, MR3414466.

[34] J. Fox, F. Wei, On the number of cliques in graphs with a forbidden minor, J. Combin. Theory Ser. B 126 (2017) 175–197, http://dx.doi.org/10.1016/j.jctb.2017.04.004, MR3667668.

[35] J. Fox, F. Wei, On the number of cliques in graphs with a forbidden subdivision or immersion, SIAM J. Discrete Math. 34 (4) (2020) 2556–2582, http://dx.doi.org/10.1137/18M1206126, MR4187140.

[36] K. Kawarabayashi, D. Wood, Cliques in odd-minor-free graphs, 2011, available at arXiv:1108.2949, Preprint.

[37] N. Chiba, T. Nishizeki, Arboricity and subgraph listing algorithms, SIAM J. Comput. 14 (1) (1985) 210–223, http://dx.doi.org/10.1137/0214017, MR774940.

[38] H. Schumacher, Zur Struktur 1-planarer graphen, German, Math. Nachr. 125 (1986) 291–300, http://dx.doi.org/10.1002/mana.19861250122, MR847368.

[39] Y. Suzuki, Re-embeddings of maximum 1-planar graphs, SIAM J. Discrete Math. 24 (4) (2010) 1527–1540, http://dx.doi.org/10.1137/090746835, MR2746706.

[40] V.P. Korzhik, Minimal non-1-planar graphs, Discrete Math. 308 (7) (2008) 1319–1327, http://dx.doi.org/10.1016/j.disc.2007.04.009, MR2382367.

[41] J. Barát, G. Tóth, Improvements on the density of maximal 1-planar graphs, J. Graph Theory 88 (1) (2018) 101–109, http://dx.doi.org/10.1002/jgt.22187, MR3781606.

[42] F.J. Brandenburg, Characterizing and recognizing 4-map graphs, Algorithmica 81 (5) (2019) 1818–1843, http://dx.doi.org/10.1007/s00453-018-0510-x, MR3968961.

[43] M.A. Bekos, G. Da Lozzo, S.M. Griesbach, M. Gronemann, F. Montecchiani, C. Raftopoulou, Book embeddings of nonplanar graphs with small faces in few pages, in: 36th International Symposium on Computational Geometry, LIPIcs, in: Leibniz Int. Proc. Inform., vol. 164, (16) Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020, p. 17, MR4117729.

[44] G. Brinkmann, B.D. McKay, Fast generation of planar graphs, MATCH Commun. Math. Comput. Chem. 58 (2) (2007) 323–357, MR2357364.

[45] R. Bodendiek, H. Schumacher, K. Wagner, Über 1-optimale Graphen, German, Math. Nachr. 117 (1984) 323–339, http://dx.doi.org/10.1002/mana.3211170125, MR755314.

[46] W.T. Tutte, Graph Theory, in: Encyclopedia of Mathematics and Its Applications, vol. 21, Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1984, p. xxi+333, With a foreword by C. St. J. A. Nash-Williams MR746795.