An Improvement of the Bound on the Odd Chromatic Number of 1-Planar Graphs

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Abstract. An odd coloring of a graph is a proper coloring in such a way that every non-isolated vertex has some color that appears an odd number of times on its neighborhood. A graph is 1-planar if it has a drawing in the plane so that each edge is crossed at most once. Cranston, Lafferty, and Song showed that every 1-planar graph admits an odd 23-coloring [arXiv:2202.02586v4]. In this paper, we improve their bound to 16.

Keywords: 1-planar graph · Odd coloring · Discharging

1 Introduction

Throughout the paper, all graphs are finite, simple and undirected. By $V(G)$, $E(G)$, and $\delta(G)$, we denote the set of vertices, the set of edges, and the minimum degree of a graph G, respectively. If G is a plane graph, then $F(G)$ denotes the set of faces of G, The *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the size of $N_G(v)$, and the *degree* of a face f in a plane graph G, denoted by $d_G(f)$, is the the number of edges that are incident with f in G, where cut-edges are counted twice. A k-, k^+-, and k^--*vertex* (resp. face) is a vertex (resp. face) of degree k, at least k and at most k, respectively. For other undefined notation, we refer the readers to the book [1].

A coloring of vertices of a hypergraph is *conflict-free* if at least one vertex in each (hyper-)edge has a unique color, see [6]. Its research was initially motivated by a frequency assignment problem in cellular networks. Such networks consist of fixed-position base stations and roaming clients, each base station is assigned a certain frequency and transmits data in this frequency within some given region. Roaming clients have a range of communication and come under the influence of different subsets of base stations. This situation can be modeled by means of a hypergraph whose vertices correspond to the base stations. The range of communication of a mobile agent, that is, the set of base stations it can

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communicate with, is represented by a hyperedge $e \in E$. A conflict-free coloring of such a hypergraph implies an assignment of frequencies, to the base stations, which enables clients to connect to a base station holding a unique frequency in the client's range, thus avoiding interferences.

Recently, Petruševski and Škrekovski [9] introduced the notion of odd coloring, which is a relaxation of conflict-free coloring. Formally, an *odd c-coloring* of a graph is a proper c-coloring with the additional constraint that each vertex of positive degree has a color appearing an odd number of times among its neighborhood. A graph G is *odd c-colorable* if it has an odd c-coloring. The *odd chromatic number* of a graph G, denoted $\chi_o(G)$, is the minimum c such that G has an odd c-coloring. Petruševski and Škrekovski [9] put forward the following conjecture:

Conjecture 1. ([9]). *Every planar graph admits an odd 5-coloring.*

The best progress towards this conjecture is due to Petr and Portier [8], who proved that the odd chromatic number of every planar graph is at most 8, improving the preceding bound 9 of Petruševski and Škrekovski [9]. Supporting this conjecture, Cranston [4] showed that every planar graph of girth at least 7 is odd 5-colorable, Caro, Petruševski and Škrekovski [2] also proved that every outerplanar graph admits an odd 5-coloring. Qi and Zhang [10] later verified Conjecture 1 for another two subclasses of planar graphs, saying outer-1-planar graphs and 2-boundary planar graphs, generalizing the result of Petruševski and Škrekovski [9]. Note that the bound 5 in Conjecture 1 would be sharp as $\chi_o(C_5) = 5$.

There are normally two ways to generalize the planarity. One way is to allow a drawing without crossings in a surface, such as a torus, rather than a plane. In view of this, Metrebian [7] showed that every torodial graph admits an odd 9-coloring. Another generalization can be established in the way of allowing bounded number of crossings per edge. Ringel [11] introduced the notion of 1-planarity in 1965. Precisely, a graph is *1-planar* if it can be drawn in the plane so that each edge is crossed by at most one other edge. Recently, Cranston, Lafferty, and Song [5] showed that every 1-planar graph admits an odd 23-coloring (the first bound for the odd chromatic number of 1-planar graphs was 47, due to the first version of [5]).

The aim of this paper is to find a better upper bound for the odd chromatic number of 1-planar graphs by showing the following.

Theorem 2. *Every 1-planar graph admits an odd 16-coloring.*

The proof of Theorem 2 is relied on the proof of the odd 23-colorability of Cranston, Lafferty, and Song [5]. Readers will see that we borrow all structural lemmas there. However, with our new discharging rules, the counting of final charges becomes easier and surprisingly the bound descends.

2 The Proof of Theorem 2

Suppose for a contradiction that G is a minimal counterexample (in terms of $|V(G)| + |E(G)|$) to this theorem. The *associated plane graph* $G^\times$ of G is the plane graph obtained from G by turning all crossings of G into new vertices of degree four. Those new 4-vertices are *false* vertices of $G^\times$, and the original vertices of G are *true* vertices of $G^\times$. A face of $G^\times$ is *false* if it is incident with at least one false vertex, and *true* otherwise. For each vertex $v \in V(G)$, let $d_2(v)$ denote the number of 2-vertices adjacent to v in G. An *odd vertex* of G is a vertex having odd degree. If $v \in V(G)$ has even degree at most 6 then we call it *small vertex*, and if $v \in V(G)$ has degree at least 8 then we call it *big vertex*.

Claim 1. *[5, Claim 1]* $\delta(G) \geq 2$.

Claim 2. *[5, Claim 2] Every odd vertex in G has degree at least 9.*

Claim 3. *[5, Claim 3] No two small vertices are adjacent in G.*

Claim 4. *[5, Claim 4] Every edge incident to a small vertex in G has a crossing.*

Claim 5. *[3, Lemma 2.1] If v is a vertex with $d_2(v) \geq 1$, then $2d(v) \geq d_2(v)+16$.*

Claim 6. *[5, Claim 6] The graph $G^\times$ has no loop or 2-face, and every 3-face in $G^\times$ is incident to either three big vertices or two big vertices and one false 4-vertex.*

Claim 7. *[5, Claim 7] Every 2-vertex in $G^\times$ is incident to a 5^+-face and to another 4^+-face.*

Claim 8. *[5, in the proof of Claim 9] For a 4-face zz_1vz_2z and a 6-face $vz_1uz_3wz_2v$ of $G^\times$, if z_1, z_2, z_3 are false vertices, then at most two vertices among u, v, w are 2-vertices.*

Note that Claims 2 and 5 are different from their original forms. However, we can prove them using the same arguments only with certain numbers changed. Moreover, although we change the definitions of small and big vertices, comparing to the ones in [5], the proofs of Claims 3, 4, and 7 work in the same logic as in [5].

We apply the discharging method to $G^\times$. Formally, for each vertex $v \in V(G^\times)$, let $\text{ch}(v) := d_{G^\times}(v)-4$ be its initial charge, and for each face $f \in F(G^\times)$, let $\text{ch}(f) := d_{G^\times}(f) - 4$ be its initial charge. Clearly,

$$\sum_{x \in V(G^\times) \cup F(G^\times)} \text{ch}(x) = -8 < 0$$

by the well-known Euler's formula.

For convenience we use $d(v)$ and $d(f)$ instead of $d_{G^\times}(v)$ and $d_{G^\times}(f)$ if v is a true vertex and f is a face in $G^\times$, respectively. The discharging rules are defined as follows.

R1 Every big vertex sends 1/4 to each of its incidence 2-vertices;
R2 Every big vertex sends 1/3 to each of its incidence true 3-faces;
R3 Every big vertex sends 1/2 to each of its incidence false faces;

R4 If f is a 4^+-face with positive charge after applying **R1**–**R3** and f is incident with at least one 2-vertex, then f redistribute its positive charge to each of its incidence 2-vertices equally.

Let $\mathrm{ch}^*(x)$ be the charge of $x \in V(G^\times) \cup F(G^\times)$ after applying the above rules. Since our rules only move charge around, and do not affect the sum, we have

$$\sum_{x \in V(G^\times) \cup F(G^\times)} \mathrm{ch}^*(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} \mathrm{ch}(x) < 0.$$

Next, we prove that $\mathrm{ch}^*(x) \geq 0$ for each $x \in V(G^\times) \cup F(G^\times)$ by Propositions 1, 2 and 3. This gives

$$\sum_{x \in V(G^\times) \cup F(G^\times)} \mathrm{ch}^*(x) \geq 0,$$

a contradiction. Note that every true vertex of $G^\times$ is either small or big by Claim 2 and the final charge of any false vertex of $G^\times$ is trivially 0.

Proposition 1. *The final charge of every face of $G^\times$ is non-negative.*

Proof. By Claim 6, every face of $G^\times$ has degree at least 3. If f is a true 3-face, then f is incident only with big vertices by Claim 6 and thus $\mathrm{ch}^*(f) = 3-4+3\times\frac{1}{3} = 0$ by **R2**. If f is a false 3-face, then f is incident with two big vertices by Claim 6 and thus $\mathrm{ch}^*(f) = 3 - 4 + 2 \times \frac{1}{2} = 0$ by **R3**. If f is 4^+-face incident with 2-vertices, then $\mathrm{ch}^*(f) = 0$ by **R4**. If f is 4^+-face incident with no 2-vertex, then no rules will be applied to f and thus $\mathrm{ch}^*(f) = \mathrm{ch}(f) = d(f) - 4 \geq 0$. □

Proposition 2. *The final charge of every small vertex of $G^\times$ is non-negative.*

Proof. It is sufficient to prove this result for an arbitrary arbitrarily 2-vertex v, as $\mathrm{ch}^*(v) = \mathrm{ch}(v)$ for $d(v) = 4, 6$ by the discharging rules. Assume $N_G(v) = \{x, y\}$. By Claim 4, vx and vy are crossed. Assume that vx is crossed by u_1u_2 at a false vertex z_1, and vy is crossed by w_1w_2 at a false vertex z_2, such that u_1, z_1, v, z_2, w_1 are on one face, say f_1, and u_2, z_1, v, z_2, w_2 are on another face, say f_2. It may be possible that $u_1 = w_1$ or $u_2 = w_2$. Assume, without loss of generality, that $d(f_1) \leq d(f_2)$. Since v is incident to a 5^+-face and to another 4^+-face by Claim 7, we consider two cases.

Case 1. $d(f_1) = 4$.

This situation implies $u_1 = w_1$ and $u_2 \neq w_2$. For convenience we let $z = u_1 = w_1$. Note that $d(f_2) \geq 5$.

Subcase 1.1. z is a big vertex, see Fig. 1(a).

Now f_1 sends 1/2 to v by **R3** and **R4**. Next we look at f_2. Besides z_1, v, z_2, there are at most $\lceil \frac{d(f_2)-3}{2} \rceil$ 2-vertices on f_2 by Claim 3. Hence at most $\lceil \frac{d(f_2)-1}{2} \rceil$ 2-vertices exist on f_2. By **R4**, f_2 sends to v at least

$$\alpha := \frac{d(f_2) - 4}{\left\lceil \frac{d(f_2)-1}{2} \right\rceil}.$$

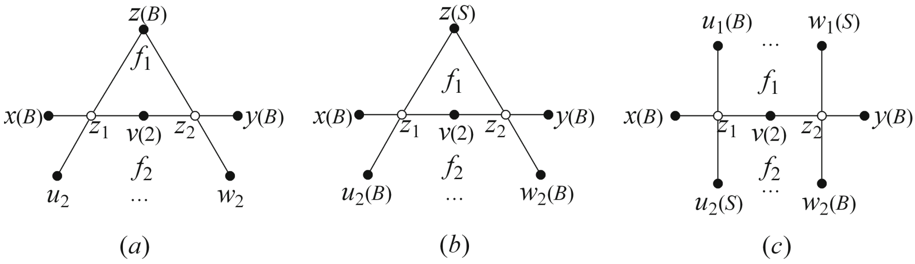

Fig. 1. Illustration for the proof of Proposition 2.

If $d(f_2) \geq 7$, then $\alpha \geq 1$.

If $d(f_2) = 6$, then f_2 is incidence with at most two 2-vertices by Claim 8. Hence f_2 sends to v at least $\frac{6-4}{2} = 1$, too.

If $d(f_2) = 5$, then $u_2w_2 \in E(G)$. This implies that u_2 and w_2 are both big by Claim 4. Hence f_2 has charge $5 - 4 + 2 \times \frac{1}{2} = 2$ after applying **R3**, which will be sent to v by **R4**.

In each case, v receives at least 1 from f_2, 1/2 from f_1, and 1/4 from each of x and y by **R1**. This gives $\mathrm{ch}^*(v) \geq 2 - 4 + 1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$.

Subcase 1.2. z is a small vertex, see Fig. 1(b).

This situation implies u_2 and w_2 are big by Claim 3.

Now we mainly look at f_2. Besides u_2, z_1, v, z_2, w_2, there are at most $\lfloor \frac{d(f_2)-5}{2} \rfloor$ 2-vertices on f_2 by Claims 3 and 4. Hence at most $\lfloor \frac{d(f_2)-3}{2} \rfloor$ 2-vertices exist on f_2. By **R3** and **R4**, f_2 sends to v at least

$$\frac{d(f_2) - 4 + 2 \times \frac{1}{2}}{\left\lfloor \frac{d(f_2)-3}{2} \right\rfloor} \geq 2$$

and thus $\mathrm{ch}^*(v) \geq 2 - 4 + 2 = 0$.

Case 2. $d(f_1) \geq 5$.

If u_i and w_i are big for some $i \in \{1, 2\}$, then by a similar argument as in Subcase 1.2, we conclude that f_i sends at least 2 to v and thus $\mathrm{ch}^*(v) \geq 2 - 4 + 2 = 0$. Hence we assume u_1 is big and w_1 is small. By Claim 3, w_2 is big. So we further assume u_2 is small, see Fig. 1(c).

Now besides u_1, z_1, v, z_2, there are at most $\lceil \frac{d(f_1)-4}{2} \rceil$ 2-vertices on f_1 by Claim 3. Hence at most $\lceil \frac{d(f_1)-2}{2} \rceil$ 2-vertices exist on f_1. By **R3** and **R4**, f_1 sends to v at least

$$\frac{d(f_1) - 4 + \frac{1}{2}}{\left\lceil \frac{d(f_1)-2}{2} \right\rceil} \geq \frac{3}{4}$$

as $d(f_1) \geq 5$. By symmetry, f_2 sends to v at least 3/4.

Since both x and y sends 1/4 to v by **R1**, $\mathrm{ch}^*(v) \geq 2 - -4 + 2 \times \frac{3}{4} + 2 \times \frac{1}{4} = 0$. □

Proposition 3. *The final charge of every big vertex of $G^{\times}$ is non-negative.*

Proof. Let v be a big vertex. If $d_2(v) = 0$, then $\text{ch}^*(v) \geq d(v) - 4 - \frac{1}{2}d(v) \geq 0$ by **R2** and **R3** as $d(v) \geq 8$. If $d_2(v) \geq 1$, then $d_2(v) \leq 2d(v) - 16$ by Claim 5. Hence

$$\begin{aligned}\text{ch}^*(v) &\geq d(v) - 4 - \frac{1}{4}d_2(v) - \frac{1}{2}d(v)\\ &\geq d(v) - 4 - \frac{1}{4}(2d(v) - 16) - \frac{1}{2}d(v)\\ &= 0\end{aligned}$$

by **R1**–**R3**. □

References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory. GTM 244, Springer, New York (2008)
2. Caro, Y., Petruševski, M., Škrekovski, R.: Remarks on odd colorings of graphs. (2022). arXiv: 2201.03608v1
3. Cho, E.-K., Choi, I., Kwon, H., Park, B.: Odd coloring of sparse graphs and planar graphs. (2022). arXiv: 2202.11267v1
4. Cranston, D.W.: Odd Colorings of Sparse Graphs (2022).arXiv:2201.01455v1
5. Cranston, D.W., Lafferty, M., Song, Z.-X.: A note on odd coloring of 1-planar graphs (2022). arXiv: 2202.02586v4
6. Even, G., Lotker, Z., Ron, D., Smorodinsky, S.: Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. SIAM J. Comput. **33**, 94–136 (2003)
7. Metrebian, H.: Odd coloring on the torus (2022). arXiv: 2205.05398v1
8. Petr, J., Portier, J.: The odd chromatic number of a planar graph is at most 8 (2022). arXiv: 2201.12381v2
9. Petruševski, M., Škrekovski, R.: Coloring with neighborhood parity condition (2022). arXiv: 2112.13710v2
10. Qi, M.K., Zhang, X.: Odd coloring of two subclasses of planar graphs (2022). arXiv: 2205.09317v1
11. Ringel, G.: Ein Sechsfarbenproblem auf der Kugel, Abhandlungen aus dem Mathematischen Seminar der Universität Hambur, **29** 107–117 (1965)