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Fast algorithm of equitably partitioning degenerate graphs into graphs with lower degeneracy ☆

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A B S T R A C T

A *q-degenerate k-partition* of a graph G is a collection $(V_1, V_2, \dots, V_k)$ of k pairwise disjoint subsets of $V(G)$ such that $V(G) = \bigcup_{i=1}^k V_i$ and each V_i induces a q-degenerate subgraph. Such a partition is called *equitable* if $\big||V_i| - |V_j|\big| \le 1$ for every $1 \le i < j \le k$. Equitable partition of graphs can model the problem of partitioning a large network into smaller sub-modules based on some cardinal principles, and has many other applications in network science and information science. In this work, we establish theoretical and algorithmic results on equitably partitioning degenerate graphs into graphs with lower degeneracy. Specifically, we show that every n-vertex d-degenerate graph with maximum degree at most n/β has a q-degenerate k-partition $(V_1, V_2, \dots, V_k)$ for every $k \ge \alpha d$ so that each V_i has size at most $\lceil n/k \rceil$ whenever q, α, and β satisfy a well-defined inequality, and such a partition can be computed in cubic time.

1. Introduction

Given a network with a huge number of nodes, we are interested in partitioning this network into smaller sub-modules based on some cardinal principles. For the sake of safety, we may require that each smaller sub-module is sparse enough so as to identify possible node failures or attacks quickly. For economic reasons, we also require that the sizes of every two smaller sub-modules are almost the same and thus we may establish a common standard to manage all sub-modules. This can be modeled by *equitable partition* or *equitable coloring* of graphs, which has many applications in network science and information science.

From now on, we do not distinguish between network and graph and in most cases we use the language of graph theory and assume every considering graph is finite and simple.

The degeneracy is a powerful parameter to measure how sparse a graph is. The *degeneracy* of a graph G is defined to be the minimum integer d such that G is *d-degenerate*, i.e., every subgraph of G contains a vertex of degree at most d. A linear time algorithm to compute the degeneracy of a graph is due to Matula and Beck [11].

An *equitable k-partition* of a graph G is a collection $(V_1, V_2, \dots, V_k)$ of k pairwise disjoint subsets of $V(G)$ such that $V(G) = \bigcup_{i=1}^k V_i$ and $\big||V_i| - |V_j|\big| \le 1$ for every $1 \le i < j \le k$. An equitable k-partition $(V_1, V_2, \dots, V_k)$ of a graph G is *q-degenerate* if each V_i induces a q-degenerate subgraph of G. In some occasions we may say equitable q-degenerate k-

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Table 1
Values of q, α, β satisfying $\Omega(q, \alpha, \beta) > 0$.

q	α	β	α	β	α	β	q	α	β	α	β	α	β
1	8	56	12	13	17	9	2	7	65	11	12	17	8
	9	26	13	12	20	8		8	25	12	11	23	7
	10	18	14	11	27	7		9	17	13	10	37	6
	11	15	15	10	52	6		10	14	14	9		
3	7	36	11	11	20	7	4	6	204	10	12	19	7
	8	20	12	10	31	6		7	29	11	10	29	6
	9	15	13	9	229	5		8	18	13	9	140	5
	10	12	15	8				9	14	15	8		

coloring instead of saying equitable q-degenerate k-partition, or in other words we do not distinguish between partition and coloring. So the subset V_i in an equitable k-partition $(V_1, V_2, \dots, V_k)$ is sometimes called *color class*.

Equitable 0-degenerate coloring and equitable 1-degenerate coloring are also known as *equitable proper coloring* and *equitable tree coloring* in the literature, which were introduced by Meyer [12] in 1973 and by Wu, Zhang, and Li [14] in 2013, respectively. A survey on equitable proper coloring is due to Lih [10] and the list version of equitable q-degenerate coloring was investigated by many research groups including [2–4,9,15,17].

The class of planar graphs is well-established subclasses of 5-degenerate graphs. Nakprasit [13] showed that every planar graph with maximum degree Δ at least 9 admits an equitable proper Δ-coloring, which improved the lower bound for Δ of Zhang and Yap [19] from 13. Indeed, improving the lower bound 9 (best known until now) for Δ in this problem is still interesting. For the equitable tree coloring of planar graphs, Esperet, Lemoine and Maffray [5] proved that every planar graph has an equitable tree k-coloring for every $k \geq 4$, answering a conjecture of Wu, Zhang and Li [14]. It is interesting to determine whether every planar graph has an equitable tree 3-coloring (see [6,16] for some partial results). Recently, Kim, Oum, and Zhang [6] showed that every planar graph has an equitable 2-degenerate k-coloring for every $k \geq 3$, and has an equitable 3-degenerate k-coloring for every $k \geq 2$.

For degenerate graphs in general, Zhang et al. [18] proved that every d-degenerate graph with maximum degree Δ has an equitable tree-k-coloring for every $k \geq (\Delta + 1)/2$ whenever $\Delta \geq 9.818d$. Kostochka and Nakprasit [7] proved that every d-degenerate graph with maximum degree at most k has an equitable proper k-coloring if $d \geq 2$ and $k \geq 14d + 1$. Kostochka, Nakprasit, and Pemmaraju [8] showed that every n-vertex d-degenerate graph G with maximum degree at most $n/15$ admits an equitable proper k-coloring for every $k \geq 16d$, and also proved that every d-degenerate graph with $d \geq 2$ has a $(d-1)$-degenerate k-partition for every $k \geq 3$.

The aim of this paper is to establish new theoretical and algorithmic results on equitably partitioning degenerate graphs into graphs with lower degeneracy by proving the following main theorem, where

$$\Omega(q, \alpha, \beta) = \Big(2\alpha(\mu - 4)(\mu - 6) - 10(\mu - 3)^2\Big)(\mu - 1)\beta - 3\Big(\alpha(\mu - 6) - 3(\mu - 1)(\mu - 3)\Big)(\mu - 4)\alpha$$

and

$$\mu = (q + 1)\alpha. \tag{1.1}$$

Theorem 1.1. *Let α, β, k, q be positive integers and let G be an n-vertex d-degenerate graph with maximum degree at most n/β. If $k \geq \alpha d$ and $\Omega(q, \alpha, \beta) > 0$, then G has a q-degenerate k-coloring with the size of each color classes being at most $\lceil n/k \rceil$, and this coloring can be constructed in cubic time.*

Before proving Theorem 1.1, we look back into the function $\Omega(q, \alpha, \beta)$. Let

$$\Phi(q) = \min\{\alpha \in \mathbb{N} \mid \exists \beta \in \mathbb{N},\ s.t.\ \Omega(q, \alpha, \beta) > 0\},$$

$$\Psi(q) = \min\{\beta \in \mathbb{N} \mid \exists \alpha \in \mathbb{N},\ s.t.\ \Omega(q, \alpha, \beta) > 0\}.$$

Using MATLAB, one can check that

- $\Phi(1) = 8$, $\Phi(2) = \Phi(3) = 7$, and $\Phi(q) = 6$ for every $q \geq 4$;
- $\Psi(1) = \Psi(2) = 6$ and $\Psi(q) = 5$ for every $q \geq 3$.

Hence we assume $\alpha \geq 6$ (thus $\mu \geq 2\alpha \geq 12$) and $\beta \geq 5$ throughout this paper. Table 1 displays some values of q, α, and β satisfying $\Omega(q, \alpha, \beta) > 0$.

Applying Theorem 1.1, we have the following.

Theorem 1.2. *Let α, β, k, t, q be positive integers and let G be a kt-vertex d-degenerate graph with maximum degree at most kt/β. If $k \geq \alpha d$ and $\Omega(q, \alpha, \beta) > 0$, then G has an equitable q-degenerate k-coloring and this coloring can be constructed in cubic time.*

According to Theorem 1.2 and the above table, the following result on equitable tree coloring of graphs is immediate.

Theorem 1.3. *If G is an n-vertex d-degenerate graph with maximum degree at most $n/56$ (resp. $n/6$), then there is a cubic time algorithm to construct an equitable tree k-coloring of G for every $k \geq 8d$ (resp. $k \geq 52d$) that divides n.*

Notations: The set of vertices and the maximum degree of G is denoted by $V(G)$ and $\Delta(G)$, respectively. If G is a graph and U is a subset of $V(G)$, then $\deg(v, G)$ and $\deg(v, U)$ respectively denote the degree of v in G and in the subgraph of G induced by U, i.e., the number of neighbors of v contained in $V(G)$ and U. For two disjoint subsets U_1 and U_2 of $V(G)$, $e(U, V)$ denotes the number of edges that have one end-vertex in U and the other in V. Other undefined notations follow those from [1].
Organizations: We divide the proof of Theorem 1.1 into two separate parts according to the value of $\lceil n/k \rceil$. In Section 2, we prove it for the case that $\lceil n/k \rceil \leq \beta(q + 1 - 1/\alpha)$ by Proposition 2.2. In Sections 3 and 4, we assume $\lceil n/k \rceil \geq \beta(q + 1 - 1/\alpha)$ and prove the theorem for this case by Proposition 4.3.

2. The case when the size of each color class is small

Throughout this section, $\alpha, \beta, \Delta, k, t, q$ are positive integers and G is an n-vertex d-degenerate graph with $\Delta(G) \leq \Delta \leq n/\beta$, $(k-1)t < n \leq kt$, and $k \geq \alpha d$. Note that $(k-1)t < n \leq kt$ is equivalent to $t = \lceil n/k \rceil$.

A *degenerate ordering* $v_1, v_2, \cdots, v_n$ of $V(G)$ is a vertex sequence so that v_i has at most d neighbors among $\{v_1, \cdots, v_{i-1}\}$ for each $2 \leq i \leq n$. One can see from Algorithm 1 that such a degenerate ordering can be found in $O(n^2)$ time. In each algorithm of this paper, the sentence "break" means break out of the current for-iteration.

Algorithm 1: ORDERING(G).

Input: A d-degenerate graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$;

```
G_1 ← G;
for i = 1 to n do
    for j = 1 to n − i + 1 do
        if deg(v_j, G_i) ≤ d then
            G_{i+1} ← G_i \ {v_j};
            w ← v_{n−i+1};
            v_{n−i+1} ← v_j;
            v_j ← w;
            break;
```

Output: A degenerate ordering $v_1, v_2, \cdots, v_n$ of $V(G)$.

Proposition 2.1. *The running time of Algorithm 1 is $O(n^2)$.* $\square$

The main result of this section is as follows.

Proposition 2.2. *If $t \leq \beta(q + 1 - 1/\alpha)$, then Algorithm 2 is a cubic-time algorithm to color the vertices of G in a degenerate ordering so that in each step, each color class induces a q-degenerate graph and contains at most t vertices.*

Proof. Algorithm 2 starts with sorting the vertices of G into a degenerate ordering $v_1, v_2, \ldots, v_n$ and then colors them in this ordering. When the algorithm enters the i-th iteration of the second "for" of line 4, it is going to color v_i in the situation that $(V_1, V_2, \ldots, V_k)$ is already a partition of the vertex set $\{v_1, \ldots, v_{i-1}\}$ such that each V_i has size at most t and induces a q-degenerate subgraph.

Indeed, lines 5 to 13 reorder the partition $(V_1, V_2, \ldots, V_k)$ into $(V_1^1, \ldots V_{a-1}^1, V_1^2, \ldots, V_{b-1}^2)$ so that v_i has most q neighbors in each of the first $a-1$ subsets and has least $q+1$ neighbors in each of the last $b-1$ subsets.

If there is a subset V_j^1 for some $1 \leq j \leq a-1$ satisfying $|V_j^1| \leq t-1$, then $V_j^1 \cup \{v_i\}$ is a subset of size at most t and induces a q-degenerate subgraph. In this case, the for-iteration of line 20 changes nothing because the pointer η becomes 1 before it starts and thus the condition of line 21 is not satisfied. Hence line 28 finally outputs a partition of the vertex set $\{v_1, \ldots, v_{i-1}, v_i\}$ into k subsets such that each subset has size at most t and induces a q-degenerate subgraph.

On the other hand, if $|V_j^1| = t$ for every $1 \leq j \leq a-1$, then the "for" interaction of line 15 changes nothing and the algorithm enters line 20 with $\eta = 0$. In this case, the condition of line 21 is satisfied by some $1 \leq j \leq b-1$, because otherwise $|V_j^2| \geq t$ for every $1 \leq j \leq b-1$ and thus $|\bigcup_{j=1}^{a-1} V_j^1 \cup \bigcup_{j=1}^{b-1} V_j^2| \geq (a+b-2)t = kt \geq n$, a contradiction.

If the condition of line 23 is satisfied by some $1 \leq \ell \leq a-1$, then v_i will be moved into V_ℓ^1 by line 24, and the vertex $w \in V_\ell^1$ with $\deg(w, V_j^2) \leq q$ will be moved into V_j^2 by line 25. Now the pointer η resumes 1, so the algorithm changes nothing before coming to the last line. Since both $V_\ell^1 \cup \{v_i\} \setminus \{w\}$ and $V_j^2 \cup \{w\}$ induce q-degenerate graphs and have sizes

Algorithm 2: MAIN-PROCEDURE-1$(G, \alpha, \beta, \Delta, k, t, q)$.

Input: Positive integers $\alpha, \beta, \Delta, k, t, q$, and an n-vertex d-degenerate graph G such that $\Delta(G) \le \Delta \le n/\beta$, $(k-1)t < n \le kt$, $k \ge \alpha d$, and $t \le \beta(q + 1 - 1/\alpha)$;

1 **ORD**(G);
2 **for** $i = 1$ *to* k **do**
3   $V_i \leftarrow \emptyset$;
4 **for** $i = 1$ *to* n **do**
5   $a \leftarrow 1$;
6   $b \leftarrow 1$;
7   **for** $j = 1$ *to* k **do**
8   **if** $\deg(v_i, V_j) \le q$ **then**
9       $V_a^1 \leftarrow V_j$;
10       $a \leftarrow a + 1$;
11   **if** $\deg(v_i, V_j) \ge q + 1$ **then**
12       $V_b^2 \leftarrow V_j$;
13       $b \leftarrow b + 1$;
14   $\eta \leftarrow 0$;
15   **for** $j = 1$ *to* $a - 1$ **do**
16   **if** $|V_j^1| \le t - 1$ **then**
17       $V_j^1 \leftarrow V_j^1 \cup \{v_i\}$;
18       $\eta \leftarrow 1$;
19   break;
20   **for** $j = 1$ *to* $b - 1$ **do**
21   **if** $|V_j^2| \le t - 1$ *and* $\eta = 0$ **then**
22   **for** $\ell = 1$ *to* $a - 1$ **do**
23   **if** *there is a vertex* $w \in V_\ell^1$ *such that* $\deg(w, V_j^2) \le q$ **then**
24           $V_\ell^1 \leftarrow V_\ell^1 \cup \{v_i\} \setminus \{w\}$;
25           $V_j^2 \leftarrow V_j^2 \cup \{w\}$;
26           $\eta \leftarrow 1$;
27   break;
28   $(V_1, V_2, \dots, V_k) \leftarrow (V_1^1, \dots V_{a-1}^1, V_1^2, \dots, V_{b-1}^2)$;

Output: An equitable q-degenerate partition $(V_1, V_2, \dots, V_k)$ of $V(G)$.

at most t, line 28 finally partitions the vertex set $\{v_1, \dots, v_{i-1}, v_i\}$ into k subsets so that each subset has size at most t and induces a q-degenerate subgraph.

Hence the remaining argument is dedicated to showing that there is a vertex $w \in V_\ell^1$ for some $1 \le \ell \le a - 1$ such that $\deg(w, V_j^2) \le q$. Suppose, for a contradiction, that $\deg(w, V_j^2) \ge q + 1$ for every $w \in V_\ell^1$ and every $1 \le \ell \le a - 1$. It follows

$$\Delta(t-1) \ge \Delta|V_j^2| \ge e\Big(\bigcup_{\ell=1}^{a-1} V_\ell^1, V_j^2\Big) \ge (a-1)t(q+1). \tag{2.1}$$

Since

$$d \ge \deg\Big(v_i, \bigcup_{j=1}^{b-1} V_j^2\Big) \ge (b-1)(q+1) = k(q+1) - (a-1)(q+1),$$

we have

$$(a-1)t(q+1) \ge kt(q+1) - dt \ge kt(q+1) - kt/\alpha \ge n(q+1-1/\alpha) \ge \beta\Delta(q+1-1/\alpha). \tag{2.2}$$

Combining (2.1) with (2.2), we deduce that $t - 1 \ge \beta(q + 1 - 1/\alpha)$, a contradiction.

For the complexity, it costs $O(n)$, $O(a-1)$, and $O((a-1)(b-1)t^2)$ time to complete the for-iterations of lines 7, 15, and 20, respectively. This implies that the main for-iteration of line 4 runs in $n(O(n) + O(a-1) + O((a-1)(b-1)t^2) + O(1)) = O(n^3)$ time. Note that a and b are bounded by the constant k and $t = \lceil n/k \rceil$. Since line 1 can be done in $O(n^2)$ time by Proposition 2.1 and the next two lines runs in $O(k)$ time, the complexity of Algorithm 2 is at most $O(n^3)$. $\square$

3. Partition the vertex set into disjoint subsets

Let $\alpha, \beta, \Delta, k, t, q$ be positive integers. In this section we assume that G is an n-vertex d-degenerate graph with $\Delta(G) \le \Delta \le n/\beta$, $(k-1)t < n \le kt$, $k \ge \alpha d$, and

$$t \geq \beta(q+1-1/\alpha). \tag{3.1}$$

Let

$$t = 3^m\omega_1 + 3^{m-1}\omega_2 + \cdots + \omega_{m+1} \tag{3.2}$$

and

$$\ell_0 = 0,\ \ \ell_i = 3^{i-1}\omega_1 + 3^{i-2}\omega_2 + \cdots + \omega_i,\ i = 1, 2, \cdots, m+1 \tag{3.3}$$

where m is an integer and $\omega_1, \omega_2, \cdots, \omega_{m+1}$ are integers chosen from $\{0, 1, 2\}$ such that $\omega_1 \neq 0$. Note that m and ω_i's come from the 3-ary representations of t, and can be computed in $O(t) \leq O(n)$ time. It is easy to see that

$$\ell_{m+1} = t, \quad \ell_i = 3\ell_{i-1} + \omega_i,\ i = 1, 2, \cdots, m+1. \tag{3.4}$$

Algorithm 3: PARTITION$(G, \alpha, \beta, \Delta, k, t, q)$.

Input: Positive integers $\alpha, \beta, \Delta, k, t, q$, and an n-vertex d-degenerate graph G such that $\Delta(G) \leq \Delta \leq n\beta$, $(k-1)t < n \leq kt$, $k \geq \alpha d$, and $t \geq \beta(q+1-1/\alpha)$;

1 Compute m and ℓ_i with $1 \leq i \leq m$ according to (3.2) and (3.3).
2 $\mathcal{A}_0, \mathcal{H}_0 \leftarrow \emptyset$;
3 **for** $i = 1$ *to* m **do**
4   **for** $j = 1$ *to* $(\ell_i - \ell_{i-1})k$ **do**
5   Find a vertex v that has maximum degree in $G\big[V(G) - \mathcal{H}_{i-1} - \bigcup_{a=1}^{j-1}\{v_a^i\}\big]$;
6         $v_j^i \leftarrow v$;
7         $\mathcal{A}_i \leftarrow \mathcal{A}_i \cup \{v_j^i\}$;
8     $\mathcal{B}_i \leftarrow \emptyset$;
9     $\mathcal{D}_i \leftarrow \mathcal{H}_{i-1} \cup \mathcal{A}_i$;
10   **while** *there is a vertex* $v \in V(G) - \mathcal{D}_i$ *so that* $\deg(v, \mathcal{D}_i) \geq (\mu-4)d$ **do**
11         $\mathcal{B}_i \leftarrow \mathcal{B}_i \cup \{v\}$;
12         $\mathcal{D}_i \leftarrow \mathcal{D}_i \cup \{v\}$;
13     $\mathcal{C}_i \leftarrow \mathcal{A}_i \cup \mathcal{B}_i$;
14     $\mathcal{H}_i \leftarrow \mathcal{H}_{i-1} \cup \mathcal{C}_i$;
15 $\mathcal{C}_{m+1} \leftarrow V(G) - \bigcup_{a=1}^{m}\mathcal{C}_a$;

Output: Vertex sets $\mathcal{H}_1, \cdots, \mathcal{H}_m$ and a partition of $V(G)$ into $m+1$ disjoint subsets $\mathcal{C}_1, \cdots, \mathcal{C}_m, \mathcal{C}_{m+1}$.

Proposition 3.1. *Algorithm 3 outputs vertex sets $\mathcal{H}_1, \cdots, \mathcal{H}_m$ and a partition of $V(G)$ into $m+1$ disjoint subsets $\mathcal{C}_1, \cdots, \mathcal{C}_m, \mathcal{C}_{m+1}$ in $O(n^3)$ time such that*

$$\left|\bigcup_{j=1}^{i}\mathcal{C}_j\right| < \frac{\mu-4}{\mu-5}\ell_i k \tag{3.5}$$

for each $1 \leq i \leq m$.

Proof. One can easily check that Algorithm 3 works. Since lines 5–7 run in $O(n^2)$ time and they would be carried out at most $\sum_{i=1}^{m}(\ell_i - \ell_{i-1})k = \ell_m k < n$ times throughout the algorithm, and the while-iteration of line 10 takes $O(n^2)$ time, the for-iteration of line 3 runs in at most $n \cdot O(n^2) + m \cdot O(n^2) \leq O(n^3)$ time. Note that $m < \log_3 t < t \leq n$ by (3.2), and the first two lines and the last line of Algorithm 3 only run in $O(n)$ time. Hence the complexity of Algorithm 3 is at most $O(n^3)$.

We now prove (3.5). Let $H_i = G\big[\bigcup_{j=1}^{i}\mathcal{C}_j\big]$ with $1 \leq i \leq m$. According to the constructions of $\mathcal{A}_i$'s, $\mathcal{B}_i$'s, and $\mathcal{C}_i$'s, we have

$$|H_i| = \left|\bigcup_{j=1}^{i}\mathcal{A}_j\right| + \left|\bigcup_{j=1}^{i}\mathcal{B}_j\right| = \sum_{j=1}^{i}(\ell_j - \ell_{j-1})k + \left|\bigcup_{j=1}^{i}\mathcal{B}_j\right| = \ell_i k + \left|\bigcup_{j=1}^{i}\mathcal{B}_j\right| \tag{3.6}$$

and

$$|E(H_i)| \geq (\mu-4)d\left|\bigcup_{j=1}^{i}\mathcal{B}_j\right|. \tag{3.7}$$

Since H_i is d-degenerate, $|E(H_i)| < d|H_i|$ and thus

$$\left|\bigcup_{j=1}^{i}\mathcal{B}_j\right| < \frac{1}{\mu-5}\ell_i k$$

by (3.6) and (3.7). It follows

$$\left|\bigcup_{j=1}^{i}\mathcal{C}_j\right| = |H_i| < \frac{\mu-4}{\mu-5}\ell_i k.$$

by (3.6). □

In the following arguments, we use the notations $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{H}_i$, and v_j^i based on the output of Algorithm 3. The next two propositions will be useful for the proofs of the next section.

Proposition 3.2. *For $1 \le i \le m$, if Δ_i is the maximum degree of the graph $G\big[V(G) \setminus \cup_{j=1}^{i}\mathcal{C}_j\big]$, then*

$$\ell_1\Delta + (\ell_2-\ell_1)\Delta_1 + (\ell_3-\ell_2)\Delta_2 + \cdots + (\ell_{m+1}-\ell_m)\Delta_m \le 2\Delta + \frac{10}{3}dt.$$

Proof. By the definitions of $\mathcal{A}_i$ and Δ_i, we conclude

$$\begin{aligned}|E(G)| &\ge \sum_{i=1}^{m}\sum_{j=1}^{(\ell_i-\ell_{i-1})k}\deg(v_j^i, V(G)-(\mathcal{H}_{i-1}\cup\{v_1^i,\cdots,v_{j-1}^i\})\\ &\ge \ell_1 k\Delta_1 + (\ell_2-\ell_1)k\Delta_2 + (\ell_3-\ell_2)k\Delta_3 + \cdots + (\ell_m-\ell_{m-1})k\Delta_m.\end{aligned}$$

Since G is d-degenerate, $|E(G)| < dn \le dkt$. Hence

$$\ell_1\Delta_1 + (\ell_2-\ell_1)\Delta_2 + (\ell_3-\ell_2)\Delta_3 + \cdots + (\ell_m-\ell_{m-1})\Delta_m < dt. \tag{3.8}$$

If $i \ge 3$, then by (3.4) (note that $\omega_{i+1} \le 2$, $\omega_i \ge 0$ and $\ell_{i-1} \ge 3$)

$$\frac{\ell_{i+1}-\ell_i}{\ell_i-\ell_{i-1}} = \frac{3\ell_i+\omega_{i+1}-\ell_i}{3\ell_{i-1}+\omega_i-\ell_{i-1}} \le \frac{2(3\ell_{i-1}+\omega_i)+2}{2\ell_{i-1}+\omega_i} = 3 + \frac{2-\omega_i}{2\ell_{i-1}+\omega_i} \le 3 + \frac{1}{\ell_{i-1}} \le \frac{10}{3}.$$

Hence by (3.8), we have

$$\begin{aligned}\frac{10}{3}dt &> \frac{10}{3}\ell_1\Delta_1 + \frac{10}{3}(\ell_2-\ell_1)\Delta_2 + \frac{10}{3}\Big((\ell_3-\ell_2)\Delta_3 + \cdots + (\ell_m-\ell_{m-1})\Delta_m\Big)\\ &\ge \frac{10}{3}\ell_1\Delta_1 + \frac{10}{3}(\ell_2-\ell_1)\Delta_2 + (\ell_4-\ell_3)\Delta_3 + \cdots + (\ell_{m+1}-\ell_m)\Delta_m\\ &= \ell_1\Delta + (\ell_2-\ell_1)\Delta_1 + (\ell_3-\ell_2)\Delta_2 + (\ell_4-\ell_3)\Delta_3 + \cdots + (\ell_{m+1}-\ell_m)\Delta_m\\ &\qquad - \Big(\ell_1\Delta + (\ell_2 - \frac{13}{3}\ell_1)\Delta_1 + (\ell_3 - \frac{13}{3}\ell_2 + \frac{10}{3}\ell_1)\Delta_2\Big)\end{aligned}$$

Now, it is sufficient to prove that

$$\xi := \ell_1\Delta + (\ell_2 - \frac{13}{3}\ell_1)\Delta_1 + (\ell_3 - \frac{13}{3}\ell_2 + \frac{10}{3}\ell_1)\Delta_2 \le 2\Delta. \tag{3.9}$$

Since $\ell_1 = \omega_1$, $\ell_2 = 3\ell_1 + \omega_2 = 3\omega_1 + \omega_2$ and $\ell_3 = 3\ell_2 + \omega_3 = 9\omega_1 + 3\omega_2 + \omega_3$ by (3.4),

$$3\xi = 3\omega_1\Delta + (3\omega_2 - 4\omega_1)\Delta_1 + (3\omega_3 - 4\omega_2 - 2\omega_1)\Delta_2.$$

Recall that $\Delta \ge \Delta_1 \ge \Delta_2$ and $\omega_i \in \{0,1,2\}$.

Suppose first that $3\omega_2 - 4\omega_1 \ge 0$. If $3\omega_3 - 4\omega_2 - 2\omega_1 \ge 0$, then $3\xi \le 3\omega_1\Delta + (3\omega_2 - 4\omega_1)\Delta + (3\omega_3 - 4\omega_2 - 2\omega_1)\Delta = (3\omega_3 - 3\omega_1 - \omega_2)\Delta \le 6\Delta$. If $3\omega_3 - 4\omega_2 - 2\omega_1 < 0$, then $3\xi \le 3\omega_1\Delta + (3\omega_2 - 4\omega_1)\Delta = (3\omega_2 - \omega_1)\Delta \le 6\Delta$.

Suppose, on the other hand, that $3\omega_2 - 4\omega_1 < 0$. If $3\omega_3 - 4\omega_2 - 2\omega_1 \le 0$, then $3\xi \le 3\omega_1\Delta \le 6\Delta$. If $3\omega_3 - 4\omega_2 - 2\omega_1 > 0$, then $3\xi \le 3\omega_1\Delta + (3\omega_2 - 4\omega_1)\Delta_1 + (3\omega_3 - 4\omega_2 - 2\omega_1)\Delta_1 = 3\omega_1\Delta + (3\omega_3 - \omega_2 - 6\omega_1)\Delta_1$. If $3\omega_3 - \omega_2 - 6\omega_1 \le 0$, we then have $3\xi \le 3\omega_1\Delta \le 6\Delta$. If $3\omega_3 - \omega_2 - 6\omega_1 \ge 0$, then $3\xi \le 3\omega_1\Delta + (3\omega_3 - \omega_2 - 6\omega_1)\Delta = (3\omega_3 - 3\omega_1 - \omega_2)\Delta \le 6\Delta$.

Therefore, in each case we conclude that $3\xi \le 6\Delta$, and (3.9) holds. □

Let

$$L_\theta = \begin{cases}\left\lceil \frac{\mu-3}{\mu-5}\ell_\theta \right\rceil, & 1 \le \theta \le m;\\ t, & \theta = m+1.\end{cases}$$

Proposition 3.3. *For $2 \le \theta \le m+1$, if $\alpha \ge 6$, then*

$$\frac{L_{\theta-1}}{L_\theta} \le \frac{1}{2}.$$

Proof. Recall (3.4) that $\ell_\theta = 3\ell_{\theta-1} + \omega_\theta \ge 3\ell_{\theta-1}$ and $\ell_{m+1} = t$.

If $2 \le \theta \le m$, then we consider two subcases. If $\ell_{\theta-1} \ge 2$, then $\ell_\theta \ge 6$ and thus

$$\begin{aligned}\frac{L_{\theta-1}}{L_\theta} = \frac{\left\lceil \frac{\mu-3}{\mu-5}\ell_{\theta-1}\right\rceil}{\left\lceil \frac{\mu-3}{\mu-5}\ell_\theta\right\rceil} &\le \frac{\frac{\mu-3}{\mu-5}\ell_{\theta-1} + \frac{\mu-6}{\mu-5}}{\frac{\mu-3}{\mu-5}\ell_\theta} \\ &\le \frac{1}{3} + \frac{\mu-6}{6(\mu-3)} \\ &= \frac{\mu-4}{2\mu-6} < \frac{1}{2}.\end{aligned}$$

If $\ell_{\theta-1} = 1$, then $\ell_\theta \ge 3$ and

$$\frac{L_{\theta-1}}{L_\theta} = \frac{\left\lceil \frac{\mu-3}{\mu-5}\ell_{\theta-1}\right\rceil}{\left\lceil \frac{\mu-3}{\mu-5}\ell_\theta\right\rceil} \le \frac{\left\lceil \frac{\mu-3}{\mu-5}\right\rceil}{\left\lceil 3\cdot\frac{\mu-3}{\mu-5}\right\rceil} \le \frac{2}{4} = \frac{1}{2}.$$

If $\theta = m+1$, then $\ell_\theta = t$, $\ell_m/t \le 1/3$ and

$$\begin{aligned}\frac{L_{\theta-1}}{L_\theta} &= \frac{\left\lceil \frac{\mu-3}{\mu-5}\ell_{\theta-1}\right\rceil}{t} \le \frac{\frac{\mu-3}{\mu-5}\ell_m + 1}{t} \le \frac{\mu-3}{3(\mu-5)} + \frac{\alpha}{\beta(\mu-1)} \\ &< \frac{\mu-3}{3(\mu-5)} + \frac{\alpha\left(2\alpha(\mu-4)(\mu-6) - 10(\mu-3)^2\right)}{3(\mu-4)(\mu-6)\alpha^2 + 9\alpha(\mu-4)(\mu-3)(\mu-1)} \\ &< \frac{\mu-3}{3(\mu-5)} + \frac{\alpha\left(2\alpha(\mu-4)(\mu-6) - 10(\mu-4)(\mu-6)\right)}{3(\mu-4)(\mu-6)\alpha^2 + 9\alpha(\mu-4)(\mu-6)(\mu-1)} \\ &= \frac{1}{3} + \frac{2}{3(\mu-5)} + \frac{2\alpha-10}{3\alpha+9(\mu-1)} < \frac{1}{3} + \frac{2}{3(2\alpha-5)} + \frac{2\alpha-10}{3\alpha+9(2\alpha-1)} \\ &= \frac{1}{3} + \frac{4}{3}\cdot\frac{\alpha^2-4\alpha+11}{14\alpha^2-41\alpha+15} < \frac{1}{3} + \frac{4}{3}\cdot\frac{1}{8} = \frac{1}{2}.\end{aligned}$$

Note that $t \ge \beta(\mu-1)/\alpha$ by (3.1) and

$$\beta(\mu-1) > \frac{3(\mu-4)(\mu-6)\alpha^2 + 9\alpha(\mu-4)(\mu-3)(\mu-1)}{2\alpha(\mu-4)(\mu-6) - 10(\mu-3)^2},$$

derived from $\Omega(q,\alpha,\beta) > 0$. $\square$

4. The case when the size of each color class is large

In this section, we continue to prove Theorem 1.1 algorithmically for the case $t \ge \beta(q+1-1/\alpha)$. We follow all assumptions and notations of Section 3.

Proposition 4.1. *Algorithm 4 outputs a q-degenerate coloring of $G[\mathcal{C}_1]$ with k color classes $V_1, V_2, \ldots, V_k$ in $O(n^3)$ time such that*

$$|V_i| \le \left\lceil \frac{\mu-3}{\mu-5}\ell_1 \right\rceil \tag{4.1}$$

for each $1 \le i \le k$.

Proof. To show the correctness of Algorithm 4, it is sufficient to show that for each $1 \le i \le c_1$, there is an integer $1 \le j \le k$ such that $\deg(v_i^1, V_j) \le q$ and $|V_j| < \lceil \frac{\mu-3}{\mu-5}\ell_1 \rceil$. This is equivalent to say that in the i-th stage of the for-iteration of line 4, lines 7 and 8 are executed exactly once, and thus v_i^1 would be added to some color class V_j such that $V_j \cup \{v_i^1\}$ induces a q-degenerate graph and has size at most $\lceil \frac{\mu-3}{\mu-5}\ell_1 \rceil$, as desired.

Algorithm 4: COLORING-C1$(G, \alpha, \beta, \Delta, k, t, q)$.

Input: Positive integers $\alpha, \beta, \Delta, k, t, q$, and an n-vertex d-degenerate graph G such that $\Delta(G) \le \Delta \le n/\beta$, $(k-1)t < n \le kt$, $k \ge \alpha d$, and $t \ge \beta(q+1-1/\alpha)$;

```
1 PARTITION(G, α, β, Δ, k, t, q);
2 ORDERING(G[C_1]);
  /* Let v^1_1, v^1_2, ..., v^1_{c_1} be the ordering of C_1 outputted by line 2.   */
3 (V_1, V_2, ..., V_k) ← (∅, ∅, ..., ∅);
4 for i = 1 to c_1 do
5     for j = 1 to k do
6         if deg(v^1_i, V_j) ≤ q and |V_j| < ⌈(μ-3)/(μ-5) ℓ_1⌉ then
7             V_j ← V_j ∪ {v^1_i};
8             break ;
```

Output: A q-degenerate coloring $(V_1, V_2, \dots, V_k)$ of $G[\mathcal{C}_1]$ such that $|V_i| \le \lceil \frac{\mu-3}{\mu-5}\ell_1 \rceil$ for each $i \in [k]$.

Suppose for a contradiction that for each vertex v_i^1 with $1 \le i \le c_1$ and each integer $1 \le j \le k$, either $\deg(v_i^1, V_j) \ge q+1$ or $|V_j| \ge \lceil \frac{\mu-3}{\mu-5}\ell_1 \rceil$. Since v_i^1 has at most d neighbors among $\bigcup_{k=1}^{i-1}\{v_i^1\}$ by line 2 of Algorithm 4,

$$\left|\left\{V_j \;\middle|\; \deg(v_i^1, V_j) \ge q+1\right\}\right| \le \frac{d}{q+1}.$$

It follows that

$$\left|\left\{V_j \;\middle|\; |V_j| \ge \left\lceil \frac{\mu-3}{\mu-5}\ell_1 \right\rceil\right\}\right| \ge k - \frac{d}{q+1},$$

and thus

$$\begin{aligned}
\left|\bigcup_{j=1}^{k} V_j\right| &\ge \left(\frac{\mu-3}{\mu-5}\ell_1\right)\left(k - \frac{d}{q+1}\right) \\
&> \left(\frac{\mu-4}{\mu-5}\ell_1\right)\left(\frac{\mu}{\mu-1}(k - \frac{d}{q+1})\right) \\
&= \left(\frac{\mu-4}{\mu-5}\ell_1\right)\left(\frac{\mu k - \alpha d}{\mu-1}\right) \\
&\ge \left(\frac{\mu-4}{\mu-5}\ell_1\right)\left(\frac{\mu k - k}{\mu-1}\right) \\
&= \frac{\mu-4}{\mu-5}\ell_1 k.
\end{aligned} \tag{4.2}$$

On the other hand, by Proposition 3.1,

$$\left|\bigcup_{j=1}^{k} V_j\right| \le |\mathcal{C}_1| < \frac{\mu-4}{\mu-5}\ell_1 k,$$

contradicting (4.2).

For the complexity, the running time of the for-iteration of line 4 is at most

$$c_1\left|\bigcup_{j=1}^{k} V_j\right| \le c_1\frac{\mu-4}{\mu-5}\ell_1 k \le n\frac{\mu-4}{\mu-5}kt < \frac{\mu-4}{\mu-5}n(n+t) \le \frac{2(\mu-4)}{\mu-5}n^2 = O(n^2)$$

Furthermore, line 1 runs in $O(n^3)$ time by Proposition 3.1, line 2 can be done in $O(n^2)$ time by Proposition 2.1, and line 3 uses constant time. Hence the complexity of Algorithm 4 is $O(n^3)$. $\square$

The idea of proving Theorem 1.1 for the case $t \ge \beta(q+1-1/\alpha)$ is to color $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$ and $\mathcal{C}_{m+1}$ in this ordering based on certain principles. Specifically, let Q_i be the subgraph of G induced by $\bigcup_{j=1}^{i}\mathcal{C}_j$. Algorithm 4 already constructs a q-degenerate k-coloring of Q_1 so that each color class has size at most L_1. The next step is extending this coloring to a q-degenerate k-coloring of Q_2 so that each color class has size at most L_2, and then we repeat this idea recursively. Rephrased, once we have q-degenerate k-colored Q_{i-1} for some $2 \le i \le m+1$ so that each color class has size at most L_{i-1}, our next goal is to k-color the vertices of $\mathcal{C}_i$ in a degenerate ordering so that each color class of Q_i induces a q-degenerate subgraph and has size at most L_i, and no vertex in Q_{i-1} is recolored. We win if Q_{m+1} is colored based on this principle.

A *partial q-degenerate coloring* $(V_1, V_2, \ldots, V_k)$ of G is a q-degenerate coloring of a subgraph G' of G such that $\bigcup_{i=1}^{k} V_i = V(G')$. Algorithm 5 constructs a digraph $\mathcal{D}$ on the vertex set $\{V_1, V_2, \ldots, V_k\}$, where V_i's are color classes of a partial q-degenerate coloring of G. In particular, we conclude the following.

Proposition 4.2. *Algorithm 5 constructs in $O(n^2)$ time the adjacency matrix of a digraph $\mathcal{D}$ on the class classes $V_1, V_2, \ldots, V_k$ of a partial q-degenerate k-coloring of G, and meanwhile outputs a set of vertices x_{ij} such that $V_j \cup \{x_{ij}\}$ induces a q-degenerate subgraph of G.* $\square$

Let $\mathcal{D}$ be the digraph constructed by Algorithm 5. For two disjoint vertex sets $\mathcal{S}, \mathcal{T} \subseteq V(\mathcal{D})$, an $(\mathcal{S}, \mathcal{T})$*-path* of $\mathcal{D}$ is a directed path from some vertex $X \in \mathcal{S}$ to some vertex $Y \in \mathcal{T}$. If $\mathcal{S}$ owns one single vertex, i.e., $\mathcal{S} = \{X\}$, then we write $(X, \mathcal{T})$-path instead of $(\{X\}, \mathcal{T})$-path. Similarly, if both $\mathcal{S}$ and $\mathcal{T}$ are single, i.e., $\mathcal{S} = \{X\}$ and $\mathcal{T} = \{Y\}$, then we use (X, Y)-path instead of $(\{X\}, \{Y\})$-path. If u is a vertex of G outside of $\bigcup_{i=1}^{k} V_i$, then a $(u, \mathcal{S})$*-path* is a $(U, \mathcal{S})$-path of $\mathcal{D}$ such that $U \in V(\mathcal{D})$ and $\deg(u, U) \le q$.

Algorithm 5: CONSTRUCT-D$(V_1, V_2, \ldots, V_k, W)$.

Input: A partial q-degenerate coloring $(V_1, V_2, \ldots, V_k)$ of G and a subset $W \subseteq \bigcup_{i=1}^{k} V_i$;

```
for i = 1 to k do
    M[i, i] ← 0;
    for j = 1 to k and j ≠ i do
        M[i, j] ← 0;
        while there is a vertex z ∈ V_i \ W such that deg(z, V_j) ≤ q do
            M[i, j] ← 1;
            x_ij ← z;
            break;
```

Output: A digraph $\mathcal{D}$ with adjacent matrix $\mathbb{M}$, vertex set $\{V_1, V_2, \ldots, V_k\}$, and $V_i V_j$ being a directed edge if and only if $\mathbb{M}[i, j] = 1$, and a set of vertices x_{ij} such that $x_{ij} \in V_i \setminus W$ and $\deg(x_{ij}, V_j) \le q$.

Algorithm 6: SWITCHING-WITNESS$(V_1, V_2, \ldots, V_k, W, u, \theta)$.

Input: A q-degenerate coloring $(V_1, V_2, \ldots, V_k)$ of a subgraph G' of G, a subset $W \subseteq V(G')$, a vertex $u \in V(G \setminus G')$, and an integer θ such that $|V_i| \le L_\theta$ for each $i \in [k]$;

```
CONSTRUCT-D(V_1, V_2, ..., V_k, W);
Y_0 ← ∅;
for i = 1 to k do
    if |V_i| < L_θ then
        Y_0 ← Y_0 ∪ {V_i};
Find a (u, Y_0)-path P in D using breadth-first search;
/* Assume P = V_{s_1} V_{s_2} ... V_{s_r} .                          */
V_{s_1} ← V_{s_1} ∪ {u} \ {x_{s_1 s_2}};
for i = 2 to r − 1 do
    V_{s_i} ← V_{s_i} ∪ {x_{s_{i−1} s_i}} \ {x_{s_i s_{i+1}}};
V_{s_r} ← V_{s_r} ∪ {x_{s_{r−1} s_r}};
```

Output: A q-degenerate coloring $(V_1, V_2, \ldots, V_k)$ of $G' + u$ such that $|V_i| \le L_\theta$ for each $i \in [k]$.

The following is the main result of this section.

Proposition 4.3. *Algorithm 7 outputs in $O(n^3)$ time a q-degenerate partition $(V_1, V_2, \ldots, V_k)$ of G such that $|V_i| \le t$ for each $i \in [k]$.*

Proof. Its first line partitions $V(G)$ into $m + 1$ disjoint subsets $\mathcal{C}_1, \cdots, \mathcal{C}_m, \mathcal{C}_{m+1}$, where $\mathcal{C}_i = \mathcal{A}_i \cup \mathcal{B}_i$ for each $1 \le i \le m$ and $\mathcal{A}_i, \mathcal{B}_i$ are defined by Algorithm 3. Our goal is to show that for each $2 \le i \le m + 1$, if the vertices of $\bigcup_{\ell=1}^{i-1} \mathcal{C}_\ell$ have been q-degenerate k-colored so that the size of each color class is at most L_{i-1}, then lines 4 to 7 of Algorithm 7 extend this q-degenerate k-coloring to a q-degenerate k-coloring of $G[\bigcup_{\ell=1}^{i} \mathcal{C}_\ell]$ so that the size of each color class is at most L_i by coloring the vertices $v_1^i, v_2^i, \ldots, v_{c_i}^i$ of $\mathcal{C}_i$ in a degenerate ordering. If so, then $G[\bigcup_{\ell=1}^{m+1} \mathcal{C}_\ell]$, which is G itself, admits a q-degenerate k-coloring so that the size of each color class is at most $L_{m+1} = t$, as desired. Note that the second line of Algorithm 7 guarantees that $G[\mathcal{C}_1]$ has a q-degenerate k-coloring so that the size of each color class is at most L_1 by Algorithm 4, and thus the condition of the base case $i = 2$ of the above recursion satisfies.

Algorithm 7: MAIN-PROCEDURE-2$(G, \alpha, \beta, \Delta, k, t, q)$.

Input: Positive integers $\alpha, \beta, \Delta, k, t, q$, and a d-degenerate n-vertex graph G such that $\Delta(G) \le \Delta \le n/\beta$, $(k-1)t < n \le kt$, $k \ge \alpha d$, and $t \ge \beta(q + 1 - 1/\alpha)$;

1 **PARTITION**$(G, \alpha, \beta, \Delta, k, t, q)$;
2 **COLORING-C1**$(G, \alpha, \beta, \Delta, k, t, q)$;
3 **for** $i = 2$ *to* $m + 1$ **do**
4   **ORDERING**$(G[\mathcal{C}_i])$;
  /* The output ordering of $\mathcal{C}_i$ is assumed to be $v^i_1, v^i_2, \ldots, v^i_{c_i}$. */
5   $W \leftarrow \bigcup_{j=1}^{i-1} \mathcal{C}_j$;
6   **for** $j = 1$ *to* c_i **do**
7   **SWITCHING-WITNESS**$(V_1, V_2, \ldots, V_k, W, v^i_j, i)$;

Output: A q-degenerate partition $(V_1, V_2, \ldots, V_k)$ of G such that $|V_i| \le t$ for each $i \in [k]$.

It is sufficient to prove the correctness of line 7 of Algorithm 7 and thus we look back into Algorithm 6, inputting v^i_j and i into u and θ there, respectively. For convenience, in the following arguments, we still use u to represent the vertex v^i_j being colored, where $2 \le i \le m$ and $1 \le j \le c_i$ are fixed, and use G' to represent the colored subgraph of G before coloring u. At this stage, the coloring $(V_1, V_2, \ldots, V_k)$ inputted into Algorithm 6 is actually a q-degenerate k-coloring of G'. We use c to denote this coloring below.

The first line of Algorithm 6 constructs a digraph on the color classes of c in $O(n^2)$ time by Proposition 4.2. Lines 2 to 5 take $O(k) = O(1)$ time to define Y_0 to be a set of color classes of c with sizes less than L_i. Since $|\bigcup_{\ell=1}^{i} \mathcal{C}_\ell| < \frac{\mu-4}{\mu-5}\ell_i k < L_i k$ for each $1 \le i \le m$ by Proposition 3.1, and $|\bigcup_{\ell=1}^{m+1} \mathcal{C}_\ell| = n \le kt = L_{m+1}k$, there is always a color class containing less than L_i vertices when the vertex u is being colored, and thus $Y_0 \neq \emptyset$. The fundamental part of Algorithm 6 is its line 6. We first assume in advance that this line works (i.e., the (u, Y_0)-path in $\mathcal{D}$ exists) and prove it later. We now analysis its complexity. Indeed, searching for the (u, Y_0)-path can be divided into two stages. The first stage is finding a set $\mathcal{S}$ of vertices $X \in V(\mathcal{D})$ such that $\deg(u, X) \le q$, which costs at most $O(n)$ time. The second stage is searching for an $(\mathcal{S}, Y_0)$-path using breadth-first search. Since $\mathcal{D}$ owns k vertices and k is actually a constant independent of n, this stage costs $O(1)$ time. Hence line 6 of Algorithm 6 can be done in $O(n)$ time. Additionally, One can see that another lines of Algorithm 6 run in $O(1)$ time.

Therefore, we conclude that if line 6 of Algorithm 6 works, then line 7 of Algorithm 7 works too, and it takes $O(n^2)$ time. In this case, the complexity of Algorithm 7 is $O(n^3)$, since line 4 of Algorithm 7 takes $O(n^2)$ time by Proposition 2.1, the for-iteration of line 6 costs $\sum_{i=2}^{m+1} O(c_i n^2) \le O(n^3)$ time, and the first two lines of Algorithm 7 take $O(n^3)$ time by Propositions 3.1 and 4.1.

Now we pay attention to showing that the (u, Y_0)-path in $\mathcal{D}$ exists, and thus line 6 of Algorithm 6 works, as desired.

By Y_i $(i \ge 1)$, we denote the set of color classes of G' such that

(i) $Y_i \cap \bigcup_{j=0}^{i-1} Y_j = \emptyset$, and

(ii) for any color class $M_i \in Y_i$ there exists a color class $M_{i-1} \in Y_{i-1}$ so that $M_i M_{i-1} \in E(\mathfrak{D})$.

Let $\mathfrak{Y} = \bigcup Y_j$ and let $y = |\mathfrak{Y}|$. It is sufficient to prove that there exists one color class $M_j \in Y_j \in \mathfrak{Y}$ containing at most q neighbor of u, which implies the existence of an (u, Y_0)-path by the definition of Y_i. We facilitate a contradictory argument as follows.

Suppose, for a contradiction, that every color class of $\mathfrak{Y}$ contains at least $q+1$ neighbors of u. Note that every vertex $v \in V(G') \setminus W$ not in any color class of $\mathfrak{Y}$ has at least $q+1$ neighbors in every color class of $\mathfrak{Y}$, because otherwise the color class containing v would be included in $\mathfrak{Y}$ by line 1 of Algorithm 6 and the definition of $\mathfrak{Y}$.

We first claim that y is upper-bounded. Actually, there are less than $(\mu - 4)d$ neighbors of u in W (otherwise u would had already been selected for $\mathcal{B}_{i-1}$ and thus $u \in \mathcal{C}_{i-1}$, see Lines 8 to 14 of Algorithm 3), and in $\mathcal{C}_i$ there are at most d neighbors of u that are already colored (recall that the vertices of $\mathcal{C}_i$ are being colored in a degenerate ordering). Therefore, among the neighbors of u, less than $(\mu - 3)d$ are colored under c. This implies that there are less than $(\mu - 3)d/(q+1)$ color classes that contain at least $q+1$ neighbors of u. Hence

$$y < \frac{\mu - 3}{q+1} d, \qquad \frac{y}{d} < \frac{\mu - 3}{q+1}. \tag{4.3}$$

Let $\mathcal{S}$ be the set of vertices w such that there exists a color class of $\mathfrak{Y}$ containing w, and let $\mathcal{T}$ be the set of colored vertices in $\mathcal{C}_i$ that do not belong to $\mathcal{S}$. By the d-degeneracy of G and by the above analysis, we have

$$d(|\mathcal{S}| + |\mathcal{T}|) > e(\mathcal{T}, \mathcal{S}) \ge (q+1)y|\mathcal{T}|,$$

which implies

$$\big((q+1)y - d\big)|\mathcal{T}| < d|\mathcal{S}|. \tag{4.4}$$

By the definition of Y_0, every color class of Y_0 contains less than L_i vertices and every other color class of c contains exactly L_i vertices (note that every color class of c has at most L_i vertices by Algorithms 6 and 7). Since no vertex in W

would be recolored when coloring vertices of $\mathcal{C}_i$ (see Algorithms 5 and 6), every color class of c has at most L_{i-1} vertices in W. So

$$|\mathcal{S}| \le yL_i, \quad |\mathcal{T}| \ge (k-y)L_i - (k-y)L_{i-1} = (k-y)(L_i - L_{i-1}) \ge (\alpha d - y)(L_i - L_{i-1}),$$

which imply by (4.4) that

$$\big((q+1)y - d\big)(\alpha d - y)(L_i - L_{i-1}) < dyL_i.$$

Write $\gamma = y/d$. We deduce from Proposition 3.3 and the above inequality that

$$f(\gamma) := (q+1)\gamma^2 - (\mu-1)\gamma + \alpha > 0.$$

Since

$$f\Big(\frac{\mu-3}{q+1}\Big) = \frac{6}{q+1} - \alpha < 0, \qquad f\Big(\frac{\alpha}{\mu-3}\Big) = \frac{\mu\big(\frac{6}{q+1} - \alpha\big)}{(\mu-3)^2} < 0,$$

we conclude by (4.3) that

$$\frac{y}{d} = \gamma < \frac{\alpha}{\mu-3}. \tag{4.5}$$

We count the number ζ_c of vertices that have already been colored under c. Actually, among the k color classes, there are only $|Y_0| \le y$ color classes containing less than L_i vertices. Therefore, $\zeta_c \ge (k-y)L_i$. Since $k \ge \alpha d$ and $y < \alpha d/(\mu-3)$ by (4.5),

$$\zeta_c \ge \frac{\mu-4}{\mu-3}kL_i = \frac{\mu-4}{\mu-3}k\Big\lceil \frac{\mu-3}{\mu-5}\ell_i \Big\rceil \ge \frac{\mu-4}{\mu-5}\ell_i k.$$

On the other hand, it is trivial that

$$\zeta_c \le \Big|\bigcup_{j=1}^{i} \mathcal{C}_j\Big| < \frac{\mu-4}{\mu-5}\ell_i k$$

by (3.5). They contradict each other by Proposition 3.1 when $2 \le i \le m$.

Hence there remains only one case: $i = m+1$.

Recall that we are now coloring a vertex $u \in \mathcal{C}_{m+1}$ and c is the partial coloring of G already constructed with the property that every color class of c contains at most $L_{m+1} = t$ vertices. Since $|V(G) - \{u\}| < n \le kt$, there is at least one color class in c that contains less than t vertices. This implies that $Y_0 \ne \emptyset$.

Let $\mathcal{M}$ be a color class of Y_0. For $1 \le j \le m$, let $\mathcal{Z}_j = \mathcal{M} \cap \mathcal{C}_j$ and $z_j = |\mathcal{Z}_j|$. Since no vertex in $\bigcup_{\ell=1}^{j} \mathcal{C}_\ell$ would be recolored while coloring vertices of $\mathcal{C}_{j+1}$ and every color class of the subgraph induced by $\bigcup_{\ell=1}^{j} \mathcal{C}_\ell$ contains at most L_j vertices,

$$\sum_{s=1}^{j} z_s \le L_j, \quad 1 \le j \le m. \tag{4.6}$$

.

Let $\mathcal{U}$ be the set of colored vertices in $\mathcal{C}_{m+1}$ that are adjacent to some vertex in $\mathcal{M}$.

For $1 \le j \le m$, recall that Δ_j is the maximum degree of the graph $G\big[V(G) \setminus \cup_{\ell=1}^{j} \mathcal{C}_\ell\big]$. It is easy to see that

$$|\mathcal{U}| \le z_1\Delta + z_2\Delta_1 + \cdots + z_{m+1}\Delta_m \tag{4.7}$$

Since $kt/\beta \ge n/\beta \ge \Delta \ge \Delta_1 \ge \cdots \ge \Delta_m$, by (4.6), (4.7) and Proposition 3.2, we have

$$\begin{aligned}
|\mathcal{U}| &\le L_1\Delta + (L_2 - L_1)\Delta_1 + \cdots + (L_{m+1} - L_m)\Delta_m \\
&= \Big\lceil \frac{\mu-3}{\mu-5}\ell_1 \Big\rceil \Delta + \Big(\Big\lceil \frac{\mu-3}{\mu-5}\ell_2 \Big\rceil - \Big\lceil \frac{\mu-3}{\mu-5}\ell_1 \Big\rceil\Big)\Delta_1 + \cdots + \Big(t - \Big\lceil \frac{\mu-3}{\mu-5}\ell_m \Big\rceil\Big)\Delta_m \\
&= \Big\lceil \frac{\mu-3}{\mu-5}\ell_1 \Big\rceil (\Delta - \Delta_1) + \Big\lceil \frac{\mu-3}{\mu-5}\ell_2 \Big\rceil (\Delta_1 - \Delta_2) + \cdots + \Big\lceil \frac{\mu-3}{\mu-5}\ell_m \Big\rceil (\Delta_{m-1} - \Delta_m) + t\Delta_m \\
&\le \Big(\frac{\mu-3}{\mu-5}\ell_1 + \frac{\mu-6}{\mu-5}\Big)(\Delta - \Delta_1) + \Big(\frac{\mu-3}{\mu-5}\ell_2 + \frac{\mu-6}{\mu-5}\Big)(\Delta_1 - \Delta_2) \\
&\qquad + \cdots + \Big(\frac{\mu-3}{\mu-5}\ell_m + \frac{\mu-6}{\mu-5}\Big)(\Delta_{m-1} - \Delta_m) + t\Delta_m
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\mu-6}{\mu-5}\Delta + \frac{\mu-3}{\mu-5}\Big(\ell_1\Delta + (\ell_2-\ell_1)\Delta_1 + (\ell_3-\ell_2)\Delta_2 + \cdots + (\ell_{m+1}-\ell_m)\Delta_m\Big) \\
&\le \frac{\mu-6}{\mu-5}\Delta + \frac{\mu-3}{\mu-5}\Big(2\Delta + \frac{10}{3}dt\Big) = \frac{3\mu-12}{\mu-5}\Delta + \frac{10\mu-30}{3\mu-15}dt \\
&\le \frac{1}{\mu-5}\Big(\frac{3\mu-12}{\beta}k + \frac{10\mu-30}{3}d\Big)t. \qquad (4.8)
\end{aligned}
$$

On the other hand, recall that any vertex $v \in \mathcal{C}_{m+1}$ in some color class outside of $\mathfrak{Y}$ has at least $q+1$ neighbors in every color class (e.g., $\mathcal{M}$) of $\mathfrak{Y}$. Therefore, all vertices containing in the $(k-y)$ color classes outside of $\mathfrak{Y}$ are neighbors of $\mathcal{M}$. By the definition of Y_0, we also know that each of those $(k-y)$ color classes of c contains exactly L_{m+1} vertices. Since no vertex in $W = \bigcup_{\ell=1}^{m}\mathcal{C}_\ell$ would be recolored while coloring $\mathcal{C}_{m+1}$, every color class of c has at most L_m vertices in W. Therefore,

$$
\begin{aligned}
|\mathcal{U}| &\ge (k-y)(L_{m+1}-L_m) = (k-y)\Big(t - \Big\lceil \frac{\mu-3}{\mu-5}\ell_m \Big\rceil\Big) \\
&\ge (k-y)\Big(1 - \frac{\mu-3}{\mu-5}\frac{\ell_m}{t} - \frac{\mu-6}{\mu-5}\frac{1}{t}\Big)t \\
&\ge (k-y)\Big(1 - \frac{\mu-3}{3\mu-15} - \frac{\mu-6}{\mu-5}\frac{\alpha}{(\mu-1)\beta}\Big)t \\
&\ge \frac{\mu-6}{\mu-5}\Big(k - \frac{\alpha}{\mu-3}d\Big)\Big(\frac{2}{3} - \frac{\alpha}{(\mu-1)\beta}\Big)t \qquad (4.9)
\end{aligned}
$$

by (3.1), (3.4), and (4.5).

Combining (4.9) with (4.8), we immediately conclude

$$(\mu-6)\Big(k - \frac{\alpha}{\mu-3}d\Big)\Big(\frac{2}{3} - \frac{\alpha}{(\mu-1)\beta}\Big) \le \frac{3\mu-12}{\beta}k + \frac{10\mu-30}{3}d$$

which implies

$$
\begin{aligned}
\frac{k}{d} &\le \frac{\frac{10\mu-30}{3} + \frac{2\alpha(\mu-6)}{3(\mu-3)} - \frac{\alpha^2(\mu-6)}{(\mu-1)(\mu-3)\beta}}{\frac{2\mu-12}{3} - \frac{\alpha(\mu-6)}{(\mu-1)\beta} - \frac{3\mu-12}{\beta}} \qquad (4.10) \\
&= \frac{\Big(10(\mu-3)^2 + 2\alpha(\mu-6)\Big)(\mu-1)\beta - 3\alpha^2(\mu-6)}{(\mu-3)\Big(2(\mu-6)(\mu-1)\beta - 3\alpha(\mu-6) - 9(\mu-4)(\mu-1)\Big)}. \qquad (4.11)
\end{aligned}
$$

Note that the denominators in (4.10) and (4.11) are positive if $\Omega(q,\alpha,\beta) > 0$.

Since $k \ge \alpha d$, we deduce from (4.11) that

$$\Big(2\alpha(\mu-4)(\mu-6) - 10(\mu-3)^2\Big)(\mu-1)\beta \le 3(\mu-4)(\mu-6)\alpha^2 + 9\alpha(\mu-4)(\mu-3)(\mu-1).$$

This contradicts the assumption that $\Omega(q,\alpha,\beta) > 0$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] R. Diestel, Graph Theory, Springer Berlin Heidelberg, Berlin, Heidelberg, 2017.

[2] E. Drgas-Burchardt, J. Dybizbański, H. Furmańczyk, E. Sidorowicz, Equitable list vertex colourability and arboricity of grids, Filomat 32 (18) (2018) 6353–6374.

[3] E. Drgas-Burchardt, H. Furmańczyk, E. Sidorowicz, Equitable d-degenerate choosability of graphs, in: Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), in: LNCS, vol. 12126, 2020, pp. 251–263.

[4] E. Drgas-Burchardt, H. Furmańczyk, E. Sidorowicz, Equitable improper choosability of graphs, Theor. Comput. Sci. 844 (2020) 34–45.

[5] L. Esperet, L. Lemoine, F. Maffray, Equitable partition of graphs into induced forests, Discrete Math. 338 (8) (2015) 1481–1483.

[6] R. Kim, S.-I. Oum, X. Zhang, Equitable partition of planar graphs, Discrete Math. 344 (6) (2021) #112351.

[7] A. Kostochka, K. Nakprasit, Equitable colourings of d-degenerate graphs, Comb. Probab. Comput. 12 (1) (2003) 53–60.

[8] A. Kostochka, K. Nakprasit, S. Pemmaraju, On equitable coloring of d-degenerate graphs, SIAM J. Discrete Math. 19 (1) (2005) 83–95.

[9] Y. Li, X. Zhang, Equitable list tree-coloring of bounded treewidth graphs, Theor. Comput. Sci. 855 (2021) 61–67.

[10] K.-W. Lih, Equitable Coloring of Graphs, Springer New York, New York, NY, 2013, pp. 1199–1248.
[11] D.W. Matula, L.L. Beck, Smallest-last ordering and clustering and graph coloring algorithms, J. ACM 30 (3) (July 1983) 417–427.
[12] W. Meyer, Equitable coloring, Am. Math. Mon. 80 (1973) 920–922.
[13] K. Nakprasit, Equitable colorings of planar graphs with maximum degree at least nine, Discrete Math. 312 (5) (2012) 1019–1024.
[14] J.-L. Wu, X. Zhang, H. Li, Equitable vertex arboricity of graphs, Discrete Math. 313 (23) (2013) 2696–2701.
[15] H. Zhang, X. Zhang, Theoretical aspects of equitable partition of networks into sparse modules, Theor. Comput. Sci. 871 (2021) 51–61.
[16] X. Zhang, Equitable vertex arboricity of planar graphs, Taiwan. J. Math. 19 (1) (2015) 123–131.
[17] X. Zhang, Equitable list point arboricity of graphs, Filomat 30 (2) (2016) 373–378.
[18] X. Zhang, B. Niu, Y. Li, B. Li, Equitable vertex arboricity conjecture holds for graphs with low degeneracy, Acta Math. Sin. Engl. Ser. 37 (8) (2021) 1293–1302.
[19] Y. Zhang, H.-P. Yap, Equitable colourings of planar graphs, J. Comb. Math. Comb. Comput. 27 (1998) 97–105.