Theoretical Computer Science 895 (2021) 68–74

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

h-extra r-component connectivity of interconnection networks with application to hypercubes ☆

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ARTICLE INFO

Article history:
Received 24 March 2021
Accepted 18 September 2021
Available online 1 October 2021
Communicated by C. Kaklamanis

Keywords:
Interconnection networks
Extra connectivity
Component connectivity
h-extra r-component connectivity
Hypercube

ABSTRACT

The connectivity and its generalizations have been well studied due to their impact on the fault tolerance and diagnosability of the interconnection networks. In this paper, we introduce a novel generalized connectivity, which combines the h-extra connectivity and r-component connectivity. Given a connected graph $G = (V, E)$, for any $h \geq 0$ and $r \geq 2$, an h-extra r-component cut of G is a subset $S \subseteq V$ such that there are at least r components in $G \setminus S$ and each component has at least $h + 1$ vertices; h-extra r-component connectivity of G, denoted as $c\kappa_r^h(G)$, is the minimum size of any h-extra r-component cut of G. We determine the h-extra r-component connectivity of n-dimensional hypercube Q_n, $c\kappa_r^1(Q_n) = 2(r-1)(n-r+1)$ for $r \in \{2, 3, 4\}$.

1. Introduction

Fault tolerance is one of the major problems in the research of interconnection networks. Usually, an interconnection network can be modeled by a connected graph. Each processor in the network is represented by a vertex in the graph; and the communication link between any two processors is represented by an edge between two corresponding vertices. The reliability and fault tolerance of an interconnection network are closely related with the *connectivity* of a graph. Given a connected graph G, a subset of vertices is called a *cut* of G if its removal disconnects G. The minimum size of a cut of G is the *connectivity* of G, denoted as $\kappa(G)$. In a large interconnection network, faults occur in the processors frequently, which may induce disconnection of the network. The connectivity is a lower bound of the number of faulty processors to disconnect the network. While the minimum vertex-degree of the graph is an upper bound of the connectivity. This turns out that the connectivity is restricted by the disconnections with some isolated vertex and a large connected component. In this case, the "core" of the network is still connected. So it is more reasonable to estimate the connectivity or faulty tolerance of the network more precisely.

There are several generalizations of the classical connectivity of graphs. Chartrand et al. in [1] and Sampathkumar in [2] introduced independently the *r-component connectivity* of a connected graph G, which is the minimum size of a subset of vertices, called an *r-component cut* of G if exists, whose removing results in at least r components, denoted as $c\kappa_r(G)$. The network is more disconnected in the sense that there are more components remained. Many researches have studied the r-component connectivity in kinds of networks for small r. For n-dimensional hypercube Q_n, Hsu et al. [3] and Zhao

☆ Supported by the National Natural Science Foundation of China (No. 11701440, 11626181, 11871055, 61672025).

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https://doi.org/10.1016/j.tcs.2021.09.030
0304-3975/

et al. [4] studied the cases for $2 \le r \le n+1$ and $n+2 \le r \le 2n-4$ respectively. In [5,6], Cheng et al. studied the r-component connectivity of the n-dimensional hierarchical cubic and complete cubic networks for $2 \le r \le n+1$ respectively. Later, they gave the results of the generalized exchanged hypercubes in some special cases [7]. Chang et al. in [8,9] worked on the alternating group networks for $r \in \{3,4\}$. There are also some works in the twisted cubes and dual cubes, please see [10,11]. Recently, the Cayley graph generated by trees were explored in [12,13]

Another popular generalization of connectivity of a connected graph G is proposed by Fàbrega and Fiol in [14], called the *h-extra connectivity*. It is the minimum size of a subset of vertices, called an *h-extra cut* of G if exists, whose removal disconnects G and each remaining component has at least $h+1$ vertices, denoted as $\kappa_h(G)$. There are many papers working on the h-extra connectivity in special networks, such as k-ary n-cube networks [15], BC networks [16], folded cubes [17], alternating group graphs [18], Split-Star networks [19], bubble-sort star graphs [20], balanced hypercubes [21,22], data center network DCell [23], DQcubes [24], enhanced hypercubes [25] and so on. From the intuition of h-extra connectivity, h-extra diagnosability was proposed in [26]. Many researchers work on the h-extra diagnosability of special networks, such as hypercubes [26,27], folded hypercubes [28,29], bubble-sort star graph networks [30], locally twisted cubes [31], crossed cubes [32] and (n,k)-star networks [33] etc. Moreover, the relations between the h-extra connectivity and h-extra diagnosability attracted much attention, please see [34–38].

In this paper, we combine the r-component connectivity and h-extra connectivity of a graph and propose a more general connectivity as follows:

Definition 1. Given a connected graph $G=(V,E)$, for two integers $r \ge 2$ and $h \ge 0$, a subset $S \subseteq V$ is called an h-extra r-component cut of G if there are at least r connected components in $G \setminus S$ and each component has at least $h+1$ vertices. The minimum size of any h-extra r-component cut of G, if exists, is the h-extra r-component connectivity of G, denoted as $c\kappa_r^h(G)$.

From the definition, we get the following results immediately:

Lemma 1. *Given a connected graph $G=(V,E)$, for any integers $h \ge 0$ and $r \ge 2$, we have that:*

- $c\kappa_2^0(G) = \kappa(G)$;
- $c\kappa_r^0(G) = c\kappa_r(G)$;
- $c\kappa_2^h(G) = \kappa_h(G)$.

The rest part of the paper is organized as follows: in Section 2, some definitions and notations are introduced. Also, some previous results are displayed, which is useful for proving the main theorem later. In Section 3, the 1-extra r-component connectivity of Q_n for $r \in \{2,3,4\}$ is determined. An upper bound is displayed for $2 \le r \le n$. Finally, we conclude the paper in Section 4.

2. Preliminary

Hypercubes are fundamental models for interconnection networks. The *n-dimensional hypercube* $Q_n=(V,E)$ is an undirected graph, in which each vertex can be represented by an n-bit binary ($\{0,1\}$) string and every two vertices are adjacent if and only if their corresponding strings differ in exactly one dimension. Note that Q_n is a bipartite graph with one part containing exactly all vertices whose corresponding strings have odd number of 1s.

Notations: For any graph $G=(V,E)$ and a vertex $v \in V$, the set of all neighbors of v is denoted as $N_G(v)$; and for any $S \subseteq V$, $N_G(S) = \cup_{v\in S} N_G(v) \setminus S$ is called the set of all neighbors of S. The subgraph induced by S is denoted as $G[S]$. Following Latifi in [39], we always express Q_n as $Q_{n-1}^0 \odot Q_{n-1}^1$ where Q_{n-1}^0 and Q_{n-1}^1 are two $(n-1)$-dimensional hypercubes of Q_n induced by the vertices with the ith coordinates 0 and 1 respectively. Then there is a perfect matching between Q_{n-1}^0 and Q_{n-1}^1. The two ends of each matching edge differ exactly in the ith dimension.

First, let us justify the range of r, i.e. the maximum number of components, in $Q_n \setminus S$ for any cut S.

Lemma 2. *For $n \ge 2$, let S be a cut of Q_n such that there are exactly $r \ge 2$ connected components in $Q_n \setminus S$. Then it satisfies that*

1. $r \le 2^{n-1}$;
2. $r = 2^{n-1}$ *can be attained when S is an independent set containing exactly 2^{n-1} vertices and all the r components in $Q_n \setminus S$ are trivial, i.e. they are all isolated vertices.*

Proof. This is true for $n=2$. We prove the lemma by induction on n. Suppose that the lemma is true for Q_{n-1}.

1. Denote $S^0 = S \cap Q_{n-1}^0$ and $S^1 = S \cap Q_{n-1}^1$. Assume that there are exactly r_0 (resp. r_1) connected components in $Q_{n-1}^0 \setminus S^0$ (resp. $Q_{n-1}^1 \setminus S^1$). Then $r_0 \le 2^{n-2}$ (resp. $r_1 \le 2^{n-2}$). So $r \le r_0 + r_1 \le 2^{n-1}$.

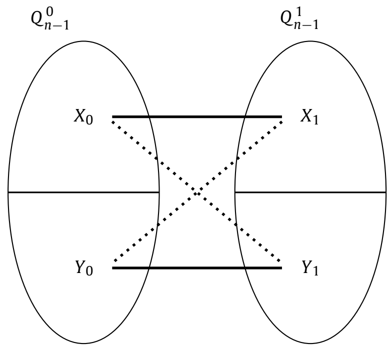

Fig. 1. The '·········' denotes that there is no edge between them; The '————' denotes that there is a perfect matching between them.

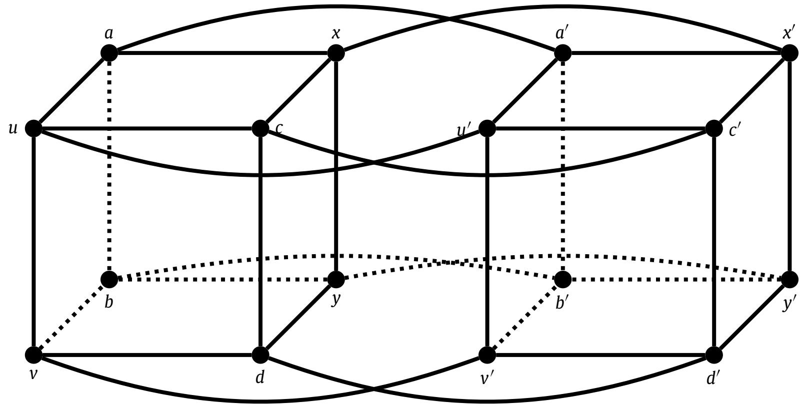

Fig. 2. For $n=4$, $T_4=\{u,v,x,y,a',b',c',d'\}$ or $T_4=\{a,b,c,d,u',v',x',y'\}$.

2. Since Q^0_{n-1} is a bipartite graph, let X and Y be its two parts. Then $|X|=|Y|=2^{n-2}$. Let $X_i=\{ix|x\in X, i\in\{0,1\}\}$ and $Y_i=\{iy|y\in Y, i\in\{0,1\}\}$, where $x\in X$ (resp. $y\in Y$) is a $(n-1)$-bit string. Then Q_n can be decomposed as in Fig. 1. Let $S=X_0\cup Y_1$. Then S is an independent set containing exactly 2^{n-1} vertices and all the r components in $Q_n\setminus S$ are trivial. □

Given an h-extra cut S of Q_n for $h\geq 1$, from this lemma one sees that there are less than 2^{n-1} components in $Q_n\setminus S$. But for $n\geq 3$ and $h=1$, there exists an 1-extra 2^{n-2}-component cut T_n of Q_n. In $Q_n\setminus T_n$, there are exactly 2^{n-2} components and each component contains exactly two adjacent vertices. The construction of T_n can be done recursively: for $n=3$, let $C=\{u,v,x,y\}$ where $u=(0,0,0)$, $v=(0,0,1)$, $x=(1,1,0)$ and $y=(1,1,1)$. Then $\bar{C}=V(Q_n)\setminus C$ is an 1-extra 2-component cut of Q_3 and the two components in $Q_3\setminus\bar{C}$ are the two subgraphs induced by $\{u,v\}$ and $\{x,y\}$ respectively. Symmetrically, one sees that C is also an 1-extra 2-component cut of Q_3 satisfying the required properties. It means that there are at least two choices, C or $\bar{C}$, of T_3 and one sees that $V(Q_3)=T_3\cup\bar{T}_3$. For $n\geq 4$, let T_{n-1} be an 1-extra 2^{n-3}-component cut of Q^0_{n-1} and $\bar{T}'_{n-1}$ be an 1-extra 2^{n-3}-component cut of Q^1_{n-1}. Note that there is no edge between T_{n-1} and $\bar{T}'_{n-1}$ in Q_n. Then $T_n=T_{n-1}\cup\bar{T}'_{n-1}$ and $\bar{T}_n=V(Q_n)\setminus T_n$ are two 1-extra 2^{n-2}-component cuts of Q_n satisfying the required properties respectively. Please see an example for $n=4$ in Fig. 2. This shows that 1-extra r-component cut is well defined for any $r\leq 2^{n-2}$.

At the end of this section, we display some lemmas, which play an important role in the proof of our main results.

Lemma 3. *[40] If $|S|\leq 2n-3$ for $n\geq 2$, then $Q_n\setminus S$ is either connected or disconnected with exactly two connected components, one of which is trivial, i.e., an isolated vertex.*

Lemma 4. *[41,42] For any integer $1\leq t\leq n+1$, the minimum number of vertices adjacent to t vertices in Q_n is $p_n(t)=-\frac{t^2}{2}+(n-\frac{1}{2})t+1$; For any integer $n+2\leq t\leq 2n$, the minimum number of vertices adjacent to t vertices in Q_n is $q_n(t)=-\frac{t^2}{2}+(2n-\frac{3}{2})t-n^2+2$.*

Lemma 5. *[16] If $|S|<3n-5$ for $n\geq 4$, then $Q_n\setminus S$ is either connected or disconnected with a large connected components, and remaining small components containing at most two vertices in total.*

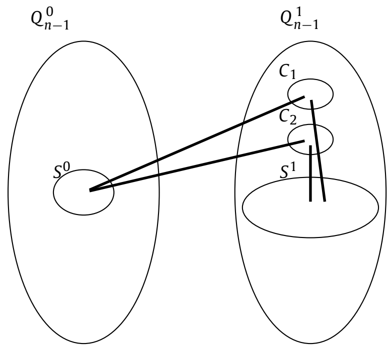

Fig. 3. $Q^0_{n-1}\setminus S^0$ is connected. All neighbors of C_1 and C_2 are contained in $S=S^0\cup S_1$. So all their neighbors in Q^0_{n-1} are contained in S^0. So $|V(C_1)|+|V(C_2)|\le|S^0|$.

Lemma 6. *[43] For $n\ge5$ and any $T\subseteq V(Q_n)$, if $|T|\ge 2n$ and $|V(Q_n)\setminus(T\cup N_{Q_n}(T))|\ge|T|$, then $|N_{Q_n}(T)|\ge q_n(2n)$, where $q_n(2n)$ is as defined in Lemma 4.*

3. $c\kappa^1_r(Q_n)$ for $r\in\{2,3,4\}$

In this section, we focus on computing $c\kappa^1_r(Q_n)$ for $r\in\{2,3,4\}$. For $r=2$, from the result of Yang and Meng in [44,45], we get the following result:

Lemma 7. *For $n\ge4$, $c\kappa^h_2(Q_n)=\kappa_h(Q_n)=(h+1)n-\frac{h^2+3h}{2}$ for $0\le h\le n-4$ and $c\kappa^h_2(Q_n)=\kappa_h(Q_n)=\frac{n(n-1)}{2}$ for $n-3\le h\le n$. In particular, $c\kappa^1_2(Q_n)=\kappa_1(Q_n)=2n-2$ for $n\ge3$.*

For $r=3$, before showing the main result, we prove that any two independent edges in Q_n have at least $4n-8$ neighbors.

Lemma 8. *For $n\ge3$, let $u_1v_1,u_2v_2\in E(Q_n)$ and $N_{Q_n}(\{u_1,v_1\})\cap\{u_2,v_2\})=\emptyset$. Then $|N_{Q_n}(\{u_1,v_1\})\cup N_{Q_n}(\{u_2,v_2\})|=|N_{Q_n}(\{u_1,v_1,u_2,v_2\})|\ge 4n-8$.*

Proof. There are no odd cycles in Q_n since Q_n is a bipartite graph. So u_1,v_1 (resp. u_2,v_2) have no common neighbors in Q_n; And u_1 (resp. v_1) cannot have common neighbors with both u_2 and v_2 otherwise there will exist a cycle of length 5 in Q_n. So at most 2 pairs of the 4 vertices can have common neighbors and the number is at most 4 since any pair of vertices in Q_n can have at most 2 common neighbors. $|N_{Q_n}(\{u_1,v_1,u_2,v_2\})|\ge 4n-2*2-4\ge 4n-8$ since in the induced subgraph of $\{u_1,v_1,u_2,v_2\}$, there are exactly 2 edges. $\square$

Theorem 1. *$c\kappa^1_3(Q_n)=4n-8$ for $n\ge7$.*

Proof. Let $u=(0,0,0,...,0,0)$, $v=(0,0,0,...,0,1)$ and $x=(1,1,0,...,0,0)$, $y=(1,1,0,...,0,1)$. Let $F=N_{Q_n}(\{u,v\})\cup N_{Q_n}(\{x,y\})$. Then $|F|=4n-8$. There are at least three components in $Q_n\setminus F$, two of which are $Q_n[\{u,v\}]$ and $Q_n[\{x,y\}]$, because $2^n-(4n-8)-4>0$ for $n\ge7$. One sees that F does not contain the neighborhood of any vertex w in $Q_n\setminus F\setminus\{u,v,x,y\}$ since $|N_{Q_n}(w)\cap F|\le4$ by its bipartite property and that $Q_n\setminus F$ contains at least three components. There are exactly one edge in these two components respectively. Moreover, there is no isolated vertex in $Q_n\setminus F$ since the neighborhood of any vertex is not contained in F. So $c\kappa^1_3(Q_n)\le 4n-8$ for $n\ge7$.

Next, we prove that $c\kappa^1_3(Q_n)\ge4n-8$ for $n\ge7$. Suppose that S is an 1-extra 3-component cut of Q_n and $|S|\le4n-9$. Let $C_1,C_2,...,C_l$ be the connected components in $Q_n\setminus S$, where $l\ge3$ and $2\le|V(C_1)|\le|V(C_2)|\le...\le|V(C_l)|$.

Denote $S^0=S\cap Q^0_{n-1}$ and $S^1=S\cap Q^1_{n-1}$. Then $S=S^0\cup S^1$ and $|S|=|S^0|+|S^1|\le4n-9$. W. l. o. g, let $|S^0|\le|S^1|$. Then $|S^0|\le2n-5=2(n-1)-3$. From Lemma 3, we consider the following two cases:

- Case 1) $Q^0_{n-1}\setminus S^0$ is connected. Then $Q^1_{n-1}\setminus S^1$ is disconnected with at least l components, two of which are C_1 and C_2. See in Fig. 3. So $4\le|V(C_1)|+|V(C_2)|\le|S^0|\le2n-5$. Note that $N_{Q_n}(V(C_1))\cap V(C_2)=\emptyset$ and $N_{Q_n}(V(C_2))\cap V(C_1)=\emptyset$. So $S\ge|N_{Q_n}(V(C_1))\cup N_{Q_n}(V(C_2))|=|N_{Q_n}(V(C_1)\cup V(C_2))|$. Let $|V(C_1)\cup V(C_2)|=t$. Then $4\le t\le 2n-5$. From Lemma 4, $|N_{Q_n}(V(C_1)\cup V(C_2))|\ge p_n(t)$ for $4\le t\le n+1$ and $|N_{Q_n}(V(C_1)\cup V(C_2))|\ge q_n(t)$ for $n+2\le t\le2n-5$, where $p_n(t)$ and $q_n(t)$ are both quadratic functions with t. So the minimum value of $p_n(t)$ for $4\le t\le n+1$ and $q_n(t)$ for $n+2\le t\le 2n-5$ is $\min\{p_n(4),p_n(n+1),q_n(n+2),q_n(2n-5)\}=p_n(4)$ for $n\ge7$ by some simple computations; and the minimum value attains only when $t=4$. So $|N_{Q_n}(V(C_1)\cup V(C_2))|\ge p_n(4)=4n-9$ for $n\ge7$ and the equality holds only when $|V(C_1)|+|V(C_2)|=4$. In this case C_1 and C_2 are both components with exactly one edge respectively.

Let u_1v_1 (resp. u_2v_2) be the unique edge in C_1 (resp. C_2). Then u_1, v_1 have no common neighbor, because there is no triangle in Q_n. So $|N_{Q_n}(V(C_1)| = |N_{Q_n}(u_1, v_1)| = 2n-2$. Similarly, $|N_{Q_n}(V(C_2)| = 2n-2$. If u_1 and u_2 have a common neighbor, then u_1 and v_2 have no common neighbors since otherwise there is a cycle of length 5 in Q_n, contradiction with its bipartiteness. Let u be a common neighbor of u_1 and u_2 if exists. Then the corresponding strings of u and u_1 (resp. u_2) differ in exactly one dimension. So u_1 and u_2 differs in exactly two dimensions. So they have two common neighbors. In this case, one sees similarly that v_1 and u_2, v_2 have at most two common neighbors in total. So $|N_{Q_n}(V(C_1)) \cap N_{Q_n}(V(C_2))| \leq 4$. This implies that $|N_{Q_n}(V(C_1)) \cup N_{Q_n}(V(C_2))| \geq 4n-8$. So $|S| \geq 4n-8$. It is a contradiction.

- Case 2) $Q^0_{n-1} \setminus S^0$ is disconnected with exactly two connected components, one of which is an isolated vertex, denoted as w^0. Let C^0 be the other component not containing w^0. In this case, $|S^0| \geq n-1$. Let w^1 be the unique neighbor of w^0 in Q^1_{n-1}. Then $w^1 \notin S^1$ and w^0 and w^1 are in the same connected component of $Q_n \setminus S$.

  If $Q^1_{n-1} \setminus S^1$ is connected, then there are at most two connected components in $Q_n \setminus S$. It is a contradiction. So $Q^1_{n-1} \setminus S^1$ is disconnected. Suppose there are j (≥ 2) connected components in $Q^1_{n-1} \setminus S^1$.
  - If there exist two components in $Q^1_{n-1} \setminus S^1$ which are also components of $Q_n \setminus S$, then the theorem can be proved as in case 1.
  - At most one component in $Q^1_{n-1} \setminus S^1$ is also a component of $Q_n \setminus S$. Since $Q_n - S$ has at least 3 components and w^0 and C^0 can be in at most 2 components. So there are exactly 3 components in $Q_n - S$. One containing C^0, one containing w^0 and another one is also a component of $Q^1_{n-1} \setminus S^1$. We use C^1 to denote the component containing w^0 and C^2 to denote the component not containing w^0 and C^0. Then $N_{Q^0_{n-1}}(V(C^1) \cup V(C^2) - \{w_0, w_1\}) \subset S^0$. So $|V(C^1)| + |V(C^2)| \leq 2n-5+2 = 2n-3$. Since S is a 1-extra 3-component cut, C^1, C^2 both have at least 2 vertices. So $4 \leq |V(C^1)| + |V(C^2)|$. Similar calculation as in case 1) can get that $|N_{Q_n}(V(C^1) \cup V(C^2))| \geq \min\{p_n(4), p_n(n+1), q_n(n+2), q_n(2n-3)\} \geq p_n(4) = 4n-9$ for $n \geq 7$; and the minimum value attains only when $t=4$. So $|S| \geq 4n-9$ and the equality attains only when $S = N_{Q_n}(V(C^1) \cup V(C^2))$ and $|V(C^1)| = |V(C^2)| = 2$. But by Lemma 8 this implies $|S| \geq 4n-8$. It is a contradiction. □

Before we prove the result for $r=4$, let us prove an upper bound of $c\kappa^1_r$ for $2 \leq r \leq n$.

Theorem 2. *For $n \geq 5$ and $2 \leq r \leq \frac{n}{2}$, $c\kappa^1_r(Q_n) \leq 2(r-1)(n-r+1)$.*

Proof. To prove the theorem, it is sufficient to find an 1-extra r-component cut of Q_n with size at most $2(r-1)(n-r+1)$. Let u_i be the vertex with only the ith coordinate 1 for $i = 1, 2, \ldots, r-1$. Let v_i be the vertex with only two coordinates the ith and the nth 1s for $i = 1, 2, \ldots, r-1$. Then $u_iv_i \in E(Q_n)$. For any $i, j \in \{1, 2, \ldots, r-1\}$ and $i \neq j$, u_i and u_j have exactly two different coordinates, the ith and the jth, so they have exactly two common neighbors, $(0, 0, \ldots, 0, 0)$ and the vertex with the ith and the jth coordinates 1s, all others 0s; similarly, v_i and v_j have exactly two common neighbors, $(0, 0, \ldots, 0, 1)$ and the one with the $\{i, j, n\}$th coordinates 1s, all others 0s for any $i, j \in \{1, 2, \ldots, r-1\}$ and $i \neq j$. Let $S = \cup_{i \in \{1,2,\ldots r-1\}} N_{Q_n}(\{u_i, v_i\})$. Then $|S| = (r-1)(2n-2) - 4C^2_{r-1} = 2(r-1)(n-r+1)$ and S does not contain the neighborhood of any vertex in Q_n for $r \leq \frac{n}{2}$. One sees that S is an 1-extra r-component cut of Q_n, with components $C_i = Q_n[\{u_i, v_i\}]$ for $i = 1, 2, \ldots, r-1$ and components in $Q_n \setminus (S \cup_{i=1,2,\ldots,n-1} C_i)$, which contains no isolated vertex and is not empty from simple counts. This proves the theorem. □

Finally, in the following theorem, we prove that the upper bound in Theorem 2 is attained for $r=4$.

Theorem 3. $c\kappa^1_4(Q_n) = 6n-18$ *for $n \geq 9$.*

Proof. From Theorem 2, it is sufficient to prove that $c\kappa^1_4(Q_n) \geq 6n-18$. Suppose that S is an 1-extra 4-component cut of Q_n and $|S| \leq 6n-19$. Let $C_1, C_2, \ldots, C_l$ be the connected components in $Q_n \setminus S$, where $l \geq 4$ and $2 \leq |V(C_1)| \leq |V(C_2)| \leq \cdots \leq |V(C_l)|$. Denote $S^0 = S \cap Q^0_{n-1}$ and $S^1 = S \cap Q^1_{n-1}$. Then $S = S^0 \cup S^1$ and $|S| = |S^0| + |S^1| \leq 6n-19$. W. l. o. g, let $|S^0| \leq |S^1|$. Then $|S^0| \leq 3n-10 = 3(n-1)-7$. From Lemma 5, we consider the following four cases:

- $Q^0_{n-1} \setminus S^0$ is connected. Then $Q^1_{n-1} \setminus S^1$ is disconnected with at least l components, three of which are C_1, C_2 and C_3. So $6 \leq |V(C_1)| + |V(C_2)| + |V(C_3)| \leq |S^0| \leq 3n-10$. Note that the neighborhood of any one of $V(C_1), V(C_2)$ and $V(C_3)$ is disjoint with the other two components. So $S \geq |\cup^3_{j=1} N_{Q_n}(V(C_j))| = |N_{Q_n}(\cup^3_{j=1} V(C_j))|$. Let $T = V(C_1 \cup C_2 \cup C_3)$ and $|T| = t$. Then $6 \leq t \leq 3n-10$. For $n \geq 9$, from Lemma 4 and 6, $|N_{Q_n}(T)| \geq p_n(t)$ for $6 \leq t \leq n+1$ and $|N_{Q_n}(T)| \geq q_n(t)$ for $n+2 \leq t \leq 2n$; and for $3n-10 \geq t > 2n$, $N_{Q_n}(T) > q_n(2n)$ since $|V(Q_n) \setminus (T \cup N_{Q_n}(T))| \geq 2^n - (3n-10) - (6n-19) > 3n-10 \geq |T|$. The minimum value of $p_n(t)$ for $6 \leq t \leq n+1$ and $q_n(t)$ for $n+2 \leq t \leq 2n$ is $\min\{p_n(6), p_n(n+1), q_n(n+2), q_n(2n)\} = p_n(6) = 6n-20$ for $n \geq 9$ by some simple computations; and the minimum value attains only when $t=6$. Moreover, $|N_{Q_n}(T)| > 6n-19$ for $3n-10 \geq t > 6$ and $n \geq 9$ from similar calculations. Then $t=6$ since $6n-19 \geq |S| \geq |N_{Q_n}(T)|$. In this case C_1, C_2 and C_3 are all components with exactly one edge respectively. As proved in Lemma 8, one can prove similarly that any three independent edges have at least $6n-18$ neighbors. This implies that $|\cup_{j=1}^{3} N_{Q_n}(V(C_j))| \geq 6n-18$. It is a contradiction.

- $Q_{n-1}^0 \setminus S^0$ is disconnected with exactly two connected components, one of which contains exactly one vertex, denoted as w^0. Then there are at least three components C_1', C_2', C_3' in $Q_{n-1}^1 \setminus S^1$ such that $N_{Q_{n-1}^0}(V(C_1' \cup C_2' \cup C_3')) \subseteq S^0 \cup \{w^0\}$. So $|V(C_1' \cup C_2' \cup C_3')| \leq 3n-9$. Moreover, the three components including C_1', C_2', C_3' separately in the subgraph induced by vertices in $C_1' \cup C_2' \cup C_3' \cup \{w^0\}$ are also three components in $Q_n \setminus S$. Then from the same discussion as the first case one gets the contradiction.
- $Q_{n-1}^0 \setminus S^0$ is disconnected with exactly three connected components, two of which are isolated vertices, denoted as w_1^0 and w_2^0. Then there are at least three components D_1, D_2, D_3 in $Q_{n-1}^1 \setminus S^1$ such that $N_{Q_{n-1}^0}(V(D_1 \cup D_2 \cup D_3)) \subseteq S^0 \cup \{w_1^0, w_2^0\}$. So $|V(D_1 \cup D_2 \cup D_3)| \leq 3n-8$. Moreover, the three components including D_1, D_2, D_3 separately in the subgraph induced by vertices in $D_1 \cup D_2 \cup D_3 \cup \{w_1^0, w_2^0\}$ are also three components in $Q_n \setminus S$. Then from the same discussion as the first case one gets the contradiction.
- $Q_{n-1}^0 \setminus S^0$ is disconnected with two connected components, one of which contains exactly one edge, denoted as $w_3^0 w_4^0$.
  - If $w_3^0 w_4^0$ is also a component, denoted as C, in $Q_n \setminus S$, then there are at least two components F_1, F_2 in $Q_{n-1}^1 \setminus S^1$, which are also two components in $Q_n \setminus S$. So $|V(F_1 \cup F_2)| \leq |S^0| \leq 3n-10$. Then there are three components C, F_1, F_2 in $Q_n \setminus S$ with at most $3n-8$ vertices. From the same discussion as the first case one gets the contradiction.
  - Otherwise $w_3^0 w_4^0$ is not a component in $Q_n \setminus S$. Then there are at least three components D_1', D_2', D_3' in $Q_{n-1}^1 \setminus S^1$ such that $N_{Q_{n-1}^0}(V(D_1' \cup D_2' \cup D_3')) \subseteq S^0 \cup \{w_3^0, w_4^0\}$. We arrive the same situation as the third case above.

The theorem is proved. □

4. Perspectives

We propose the following conjecture for the h-extra r-component connectivity of hypercubes, which are proved to be true for $2 \leq r \leq 4$ in this paper:

Conjecture 4. *For $2 \leq r \leq n$, $c\kappa_r^1(Q_n) = 2(r-1)(n-r+1)$.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Bi Li was supported by the National Natural Science Foundation of China (Nos. 11701440, 11626181). Qiang Zhu was supported by the National Natural Science Foundation of China (No. 61672025). Xin Zhang was supported by the National Natural Science Foundation of China (No. 11871055).

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