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Incidence coloring of Mycielskians with fast algorithm ☆

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A B S T R A C T

An *incidence* of a graph G is a vertex-edge pair (v, e) such that the vertex v is incident with the edge e. A *proper incidence k-coloring* of a graph is a coloring of its incidences involving k colors so that two incidences (u, e) and (w, f) receive distinct colors if and only if $u = w$, or $e = f$, or $uw \in \{e, f\}$. In this paper, we present some idea of using the incidence coloring to model a kind of *multi-frequency assignment problem*, in which each transceiver can be simultaneously in both sending and receiving modes, and then establish some theoretical and algorithmic aspects of the incidence coloring.

Specifically, we conjecture that if G is the Mycielskian of some graph then it has a proper incidence $(\Delta(G)+2)$-coloring. Actually, our conjecture is motivated by the "$(\Delta+2)$ conjecture" of Brualdi and Quinn Massey in 1993, which states that every graph G has a proper incidence $(\Delta(G)+2)$-coloring, and was disproved in 1997 by Guiduli, who pointed out that the Paley graphs with large maximum degree are counterexamples (yet they are all known counterexamples to the "$(\Delta+2)$ conjecture", and are not Mycielskians of any graph).

To support our conjecture, we prove in this paper that if G is the Mycielskian of a graph H with $|H| \geq 3\Delta(H)+2$, then we can construct a proper incidence $(\Delta(G)+1)$-coloring of G in cubic time, and if G is the Mycielskian of an incidence $(\Delta(H)+1)$-colorable graph H with $|H| \leq 2\Delta(H)$, or the Mycielskian of an incidence $(\Delta(H)+2)$-colorable graph H with $|H| \geq 2\Delta(H)+1$, then G has a proper incidence $(\Delta(G)+2)$-coloring. The minimum positive integer k such that the Mycielskian of a cycle or a path has a proper incidence k-coloring is also determined.

1. Introduction

The frequency assignment problem is an important problem that arises in the design of the wireless radio network consisting of a group of transceivers in space communicating with each other via the link between them [2,6,7,10,19]. In the standard model, each transceiver can be in either sending or receiving mode but not both at the same time, and it is required that adjacent transceivers are assigned distinct frequencies so as to avoid collisions of simultaneous transmissions over the same frequency [5,24,25]. This can be modeled by the proper vertex coloring of graphs, which becomes, however, invalid if each transceiver can be simultaneously in both sending and receiving modes (for example, this phenomenon may appear in the two-way radio network). So it is necessary to revise the model to adapt to such a situation.

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For two adjacent transceivers T_1 and T_2 in the wireless radio network, image that there are two directed links (T_1, T_2) and (T_2, T_1) between them. If data transfers from T_1 to T_2 (imaging that T_1 is in sending mode and T_2 is in receiving mode), then a frequency $F_{1,2}$ would be assigned to the link (T_1, T_2) from T_1 to T_2, and in the other direction, if data transfers from T_2 to T_1 (imaging that T_1 is in receiving mode and T_2 is in sending mode), then we assign another frequency $F_{2,1}$ to the link (T_2, T_1) from T_2 and T_1. We call $F_{1,2}$ (resp. $F_{2,1}$) the *sending frequency* of T_1 through (T_1, T_2) (resp. T_2 through (T_2, T_1)), and thus the *receiving frequency* of T_2 through (T_1, T_2) (resp. T_1 through (T_2, T_1)). To avoid the communication interference, this wireless radio network shall naturally obey the following requirements:

(a) for each pair of adjacent transceivers T_1 and T_2, the sending frequency of T_1 through (T_1, T_2) is different from the sending frequency of T_2 through (T_2, T_1);
(b) for three distinct transceivers T_1, T_2, and T_3 such that T_1 is adjacent to both T_2 and T_3, the sending frequency of T_1 through (T_1, T_2) is different from the sending frequency of T_1 through (T_1, T_3);
(c) for three distinct transceivers T_1, T_2, and T_3 such that T_2 is adjacent to both T_1 and T_3, the receiving frequency of T_2 through (T_1, T_2) is different from the sending frequency of T_2 through (T_2, T_3).

Surprisingly, this can be modeled by the incidence coloring of graphs, which was initially introduced by Brualdi and Quinn Massey [4] in 1993.

In this paper, all graphs we consider are finite and simple. We denote the set of vertices and edges of G by $V(G)$ and $E(G)$, respectively, and the maximum degree of G by $\Delta(G)$. We use $|G|$ and $\|G\|$ to indicate $|V(G)|$ and $|E(G)|$, and use C_n, P_n, and K_n to denote a cycle, a path, and a complete graph on n vertices, respectively. Throughout this paper, $[k]$ denotes the set $\{1, 2, \dots, k\}$. For terminologies not given here, we refer the readers to [3].

An *incidence* of a graph G is a pair (v, e) such that $v \in V(G)$, $e \in E(G)$, and v is incident with e. Two incidences (u, e) and (w, f) are *adjacent* if $u = w$, or $e = f$, or $uw \in \{e, f\}$. For a vertex u and an edge $e = uw$, the incidence (u, e) is called a *strong incidence* of u, and the incidence (w, e) is called a *weak incidence* of u. The set of incidences of G is denoted by $I(G)$. A *proper incidence k-coloring* of a graph G is a mapping $\varphi : I(G) \longrightarrow \{1, 2, \dots, k\}$ such that $\varphi(u, e) \neq \varphi(w, f)$ if (u, e) and (w, f) are adjacent. The smallest integer k such that G admits a proper incidence k-coloring (or saying in other words that G is *incidence k-colorable*) is the *incidence chromatic number* of G, denoted by $\chi_i(G)$.

In the above mentioned multi-frequency assignment problem, we model each transceiver into a vertex of a graph G and then for each edge $uv \in E(G)$ we assign two colors α_u and α_v to the incidence (u, uv) and (v, uv), respectively. Here α_u (resp. α_v) stands for the sending frequency of the transceiver u (resp. v) through the link from u to v (resp. from v to u). It is easy to see that a proper incidence k-coloring of the graph G just corresponds to a multi-frequency assignment of the network with k frequencies satisfying the requirements (a), (b), and (c). Therefore, it is interesting to do some necessary investigations into the theoretical and algorithmic aspects of the incidence coloring.

Now let us come back to graph theory. It is easy to see that $\chi_i(G) \geq \Delta(G) + 1$ for every graph G (looking at the strong and weak incidences of a vertex with the maximum degree). The first result concerning the upper bound for $\chi_i(G)$ is due to Brualdi and Quinn Massey [4], who proved that $\chi_i(G) \leq 2\Delta(G)$ for every graph G. In 1997, Guiduli [9] improved this bound to $\Delta(G) + 20 \log \Delta(G) + 84$, and showed that Paley graphs have incidence chromatic number at least $\Delta(G) + \Omega(\log \Delta(G))$ (by applying a result of Algor and Alon [1] on the star arboricity of graphs). This disproved the following conjecture of Brualdi and Quinn Massey [4].

Conjecture 1.1. *[4] For every graph G, $\chi_i(G) \leq \Delta(G) + 2$.*

Although Conjecture 1.1 is false in general, finding graphs G with $\chi_i(G) \leq \Delta(G) + 2$ is still interesting and this topic has attracted much interest in recent years. More specifically, $\chi_i(G) \leq \Delta(G) + 2$ holds for the following classes of graphs:

(i) paths, cycles, trees, and complete graphs [4];
(ii) complete multipartite graphs [14];
(iii) graphs with maximum degree at most three [21];
(iv) square meshes, honeycomb meshes, and hexagonal meshes [13];
(v) toroidal grids [23];
(vi) pseudo-Halin graphs [16];
(vii) n-dimensional hypercubes [8,20];
(viii) squares of cycles [18];
(ix) partial 2-trees (and thus also outerplanar graphs) [12];
(x) graphs with maximum average degree less than 3 (and thus also planar graphs with girth at least 6) [11,15];
(xi) graphs with maximum average degree less than 10/3 and $\Delta(G) \geq 8$ (and thus also planar graphs with girth at lest 5 and $\Delta(G) \geq 8$) [15].

We refer the readers to a real-time online survey contributed by Éric Sopena [22] for more information about the recent progresses of the study of incidence colorings.

The *Mycielskian* $M(G)$ of a graph G with $V(G)=\{v_1,v_2,\dots,v_n\}$ is a supergraph of G with

$$V(M(G))=\{v_1,v_2,\dots,v_n,u_1,u_2,\dots,u_n,w\}, \text{ and}$$
$$E(M(G))=E(G)\cup\{u_iv_j \mid i \text{ and } j \text{ are integers such that } v_iv_j\in E(G)\}\cup\{wu_i \mid 1\le i\le n\}.$$

Sometimes we call $M(G)$ the *Mycielskian of G based on the ordering* $v_1,v_2,\dots,v_n$ of $V(G)$.

To our best knowledge, the only known counterexamples to Conjecture 1.1 are Paley graphs with large maximum degree [1,9], and they are not Mycielskians of any graph. This motivates us to conjecture the following.

Conjecture 1.2. *If G is the Mycielskian of some graph, then* $\chi_i(G)\le\Delta(G)+2$.

This paper is organized as follows. In Section 2, we prove that if G is the Mycielskian of a graph H with $\Delta(H)\le(|H|-2)/3$ then we can construct a proper incidence $(\Delta(G)+1)$-coloring of G in cubic time. In Section 3, we show that if G is the Mycielskian of a graph H with $\Delta(H)\ge(|H|-1)/3$ then we can give a reasonable upper bound for $\chi_i(G)$, the gap between which and the one in Conjecture 1.2 (i.e., $\Delta(G)+2$) is at most $\chi_i(H)-(\Delta(H)+1)$. This implies that Conjecture 1.2 satisfies if G is the Mycielskian of a graph H with $\chi_i(H)=\Delta(H)+1$. In Section 4, we confirm Conjecture 1.2 if G is the Mycielskian of a cycle or a path, by determining the incidence chromatic numbers of $M(C_n)$ and $M(P_n)$.

2. Mycielskians of sparse graphs

A proper incidence k-coloring φ of a graph G is *amicable* if there are k distinct vertices $y_1,y_2,\dots,y_k$ such that the color i does not appear in any strong incidence of y_i for each $1\le i\le k$. We call the vertex set $\{y_1,y_2,\dots,y_k\}$ an *amicable set* of the coloring φ.

Let Δ be a positive integer and let G be a graph such that $\Delta(G)\le\Delta$. By the well-known Vizing's theorem, we can give a coloring ϕ with color set $\{1,2,\dots,\Delta+1\}$ on the edges of G so that adjacent edges receive distinct colors. Based on ϕ, we construct a $(2\Delta+2)$-coloring φ on $I(G)$ as follows. If $\phi(uv)=i$, then color the incidences (u,uv) and (v,uv) so that $\{\varphi(u,uv),\varphi(v,uv)\}=\{i,i'\}$ (sets are unordered throughout this paper).

Proposition 2.1. *φ is a proper incidence $(2\Delta+2)$-coloring of G.*

Proof. If (u,e) and (w,f) are two adjacent incidences, then either $e=f$ or e and f are adjacent. If $e=f$, then

$$\{\varphi(u,e),\varphi(w,f)\}=\{\phi(e),\phi(e)'\}.$$

If e and f are adjacent, then $\phi(e)\ne\phi(f)$. This implies

$$\{\varphi(u,e)\}\cap\{\varphi(w,f)\}\subseteq\{\phi(e),\phi(e)'\}\cap\{\phi(f),\phi(f)'\}=\emptyset.$$

Hence $\varphi(u,e)\ne\varphi(w,f)$ in each case and thus φ is a proper incidence $(2\Delta+2)$-coloring. $\square$

In the following, a *Vizing-based incidence coloring* of G is a proper incidence $(2\Delta+2)$-coloring of G using colors from $\{1,2,\dots,\Delta+1,1',2',\dots,(\Delta+1)'\}$ such that for each edge $uv\in E(G)$, $\{\varphi(u,uv),\varphi(v,uv)\}=\{i,i'\}$ for some $i\in\{1,2,\dots,\Delta+1\}$. Proposition 2.1 guarantees that every graph G has a Vizing-based incidence coloring.

Lemma 2.2. *Let Δ be a positive integer and let G be a graph such that $|G|\ge2\Delta+2$ and $\Delta(G)\le\Delta$. If φ is a Vizing-based incidence coloring of G and S is a set of $2\Delta+2$ vertices of G, then there is a set E of edges (E may be empty) such that exchanging the colors of (u,uv) and (v,uv) for each $uv\in E$ results in an amicable Vizing-based incidence coloring of G with amicable set S.*

Proof. We prove it by applying induction on $\|G\|$. If $\|G\|=1$, then G has only two incidences and the desired conclusion follows. We may henceforth assume $\|G\|\ge2$.

Let uv be an edge of G and let $G'=G-uv$. Without loss of generality, assume $\varphi(u,uv)=j$ and $\varphi(v,uv)=j'$. Restricting the coloring φ to G', we obtain a Vizing-based incidence coloring ϕ of G'. Since $\|G'\|<\|G\|$, $|G'|=|G|\ge2\Delta+2$ and $\Delta(G')\le\Delta(G)\le\Delta$, by the induction hypothesis, there is a set $E'\subseteq E(G')$ such that ϕ can be modified, via exchanging the colors of (u,uv) and (v,uv) for each $uv\in E'$, into an amicable Vizing-based incidence coloring φ' of G' so that $S=\{y_1,y_2,\dots,y_{2\Delta+2}\}$ is an amicable set of φ', and the color i or i' does not appear in any strong incidence of y_i for each $1\le i\le\Delta+1$ or $\Delta+2\le i\le2\Delta+2$, respectively.

We can extend φ' to a Vizing-based incidence coloring of G in two possible ways. The first way is to color (u,uv) and (v,uv) with j and j', respectively, and we denote such an extended incidence coloring of G by φ_1. The second way is to color (u,uv) and (v,uv) with j' and j, respectively, and we denote this extended incidence coloring of G by φ_2. If $S\cap\{u,v\}=\emptyset$, or $S\cap\{u,v\}=\{u\}$ and $u\ne y_j$, or $S\cap\{u,v\}=\{v\}$ and $v\ne y_{\Delta+j+1}$, or $S\cap\{u,v\}=\{u,v\}$, $u\ne y_j$, and $v\ne y_{\Delta+j+1}$, then φ_1 is amicable Vizing-based incidence coloring of G so that S is an amicable set of φ_1. Since $\varphi_1(u,uv)=$

$\varphi(u, uv)$ and $\varphi_1(v, uv) = \varphi(v, vu)$, in this case the set E' acts as the desired set E satisfying the lemma. On the other hand, if $S \cap \{u, v\} = \{u\}$ and $u = y_j$, or $S \cap \{u, v\} = \{v\}$ and $v = y_{\Delta+j+1}$, or $S \cap \{u, v\} = \{u, v\}$ and $u = y_j$, or $S \cap \{u, v\} = \{u, v\}$ and $v = y_{\Delta+j+1}$, then φ_2 is an amicable Vizing-based incidence coloring of G so that S is an amicable set of φ_2, and in this case $E' \cup \{uv\}$ is the desired set E satisfying the lemma, since $\varphi_2(u, uv) = \varphi(v, uv)$ and $\varphi_2(v, uv) = \varphi(u, vu)$. □

Based on the proof of Lemma 2.2, for any given parameters G, S, and Δ satisfying the conditions of Lemma 2.2, we can release Algorithm 1 that outputs an amicable Vizing-based incidence coloring of G using colors from $\{1, 2, \ldots, \Delta + 1, 1', 2', \ldots, (\Delta + 1)'\}$ such that S is an amicable set of this coloring. Using the algorithm of Misra and Gries [17], one can construct in $O(mn)$ time a proper edge coloring of G using $\Delta + 1$ colors and thus line 1 in Algorithm 1 can be done in $O(mn)$ time, where $n = |G|$ and $m = \|G\|$. It is easy to see that lines 2–8 of Algorithm 1 take $O(m)$ time. Since $m \le O(n^2)$, Algorithm 1 is indeed a polynomial-time algorithm, with running time $O(mn) + O(m) = O(mn) \le O(n^3)$.

Algorithm 1: Construct Amicable Vizing-Based Incidence Coloring **CAVBIC**(G, S, Δ).

Input: An integer Δ, a graph G with $|G| \ge 2\Delta + 2$ and $\Delta(G) \le \Delta$, and a set $S = \{y_1, y_2, \ldots, y_{2\Delta+2}\}$ of $2\Delta + 2$ vertices of G;
Output: An amicable Vizing-based incidence coloring ϕ of G using colors from $\{1, 2, \ldots, \Delta + 1, 1', 2', \ldots, (\Delta + 1)'\}$ such that S is an amicable set of this coloring.

1 Construct a proper edge coloring φ of G using $\Delta + 1$ colors;
  /* edges of G are $e_1 = u_1v_1, e_2 = u_2v_2, \ldots, e_m = u_mv_m$, where $m = \|G\|$. */
2 **for** $i = 1$ *to* m **do**
3   **if** $S \cap \{u_i, v_i\} = \emptyset$ **or** $S \cap \{u_i, v_i\} = \{u_i\}$ *and* $u_i \neq y_{\varphi(e_i)}$ **or** $S \cap \{u_i, v_i\} = \{v_i\}$ *and* $v_i \neq y_{\Delta+\varphi(e_i)+1}$ **or** $S \cap \{u_i, v_i\} = \{u_i, v_i\}$, $u_i \neq y_\varphi(e_i)$, *and* $v_i \neq y_{\Delta+\varphi(e_i)+1}$, **then**
4     $\phi(u_i, e_i) \leftarrow \varphi(e_i)$;
5     $\phi(v_i, e_i) \leftarrow \varphi(e_i)'$;
6   **else**
7     $\phi(u_i, e_i) \leftarrow \varphi(e_i)'$;
8     $\phi(v_i, e_i) \leftarrow \varphi(e_i)$.

Algorithm 2: Incidence Color the Mycielskian **ICM**(G, Δ).

Input: An integer Δ, a graph G with $|G| \ge 3\Delta + 2$ and $\Delta(G) \le \Delta$;
Output: A proper incidence coloring of $M(G)$ using $|G| + 1$ colors.

  /* vertices of G are $v_1, v_2, \ldots, v_n$, where $n = |G|$. */
1 **CAVBIC**$(G, \{v_1, v_2, \ldots, v_{2\Delta+2}\}, \Delta)$;
  /* coloring outputted from Line 1 is denoted by φ. */
2 **for** $i = 1$ *to* n **do**
3   $\ell \leftarrow 1$;
4   **for** $j = 1$ *to* n **do**
5   **if** $v_iv_j \in E(G)$, **then**
6       $\varphi(v_i, u_jv_i) \leftarrow \ell''$;
7       $\varphi(u_j, u_jv_i) \leftarrow \varphi(v_j, v_iv_j)$;
8       $\ell \leftarrow \ell + 1$;
9   $\varphi(u_i, wu_i) \leftarrow (\Delta + 1)''$;
10   **if** $1 \le i \le \Delta + 1$, **then**
11     $\varphi(w, wu_i) \leftarrow i$;
12   **if** $\Delta + 2 \le i \le 2\Delta + 2$, **then**
13     $\varphi(w, wu_i) \leftarrow (i - \Delta - 1)'$;
14   **if** $2\Delta + 3 \le i \le 3\Delta + 2$, **then**
15     $\varphi(w, wu_i) \leftarrow (i - 2\Delta - 2)''$;
16   **if** $3\Delta + 3 \le i \le n$, **then**
17     $\varphi(w, wu_i) \leftarrow i - 2\Delta - 1$.

While constructing a proper incidence coloring of $M(G)$, we distinguish the incidences of $M(G)$ by five types as follows:

Type 1 incidences of G;
Type 2 incidences (v_i, u_jv_i) with $1 \le i \neq j \le n$;
Type 3 incidences (u_j, u_jv_i) with $1 \le i \neq j \le n$;
Type 4 incidences (u_i, wu_i) with $1 \le i \le n$;
Type 5 incidences (w, wu_i) with $1 \le i \le n$.

Theorem 2.3. *If G is a graph such that $|G| \geq 3\Delta + 2$ and $\Delta(G) \leq \Delta$, where Δ is a positive integer, then $\chi_i(M(G)) \leq |G| + 1$.*

Proof. It is sufficient to show that Algorithm 2 returns a proper incidence coloring φ of $M(G)$. Since line 1 outputs a proper incidence coloring of G, any two adjacent Type 1 incidences receive different colors.

If two Type 2 incidences $(v_{j_1}, u_{j_1}v_{i_1})$ and $(v_{j_2}, u_{j_2}v_{i_2})$ are adjacent, then $j_1 = j_2$, and thus

$$\varphi(v_{j_1}, u_{j_1}v_{i_1}) \neq \varphi(v_{j_2}, u_{j_2}v_{i_2})$$

by lines 4–8.

If two Type 3 incidences $(u_{j_1}, u_{j_1}v_{i_1})$ and $(u_{j_2}, u_{j_2}v_{i_2})$ are adjacent, then $j_1 = j_2$. It follows that

$$\varphi(u_{j_1}, u_{j_1}v_{i_1}) = \varphi(v_{j_1}, v_{i_1}v_{j_1}) \neq \varphi(v_{j_2}, v_{i_2}v_{j_2}) = \varphi(u_{j_2}, u_{j_2}v_{i_2})$$

by lines 1 and 7.

If $(u_{j_1}, u_{j_1}v_{i_1})$ is a Type 3 incidence adjacent to a Type 1 incidence $(v_{j_2}, v_{i_2}v_{j_2})$, then $i_1 = j_2$. This implies

$$\varphi(u_{j_1}, u_{j_1}v_{i_1}) = \varphi(v_{j_1}, v_{i_1}v_{j_1}) \neq \varphi(v_{i_1}, v_{i_2}v_{i_1}) = \varphi(v_{j_2}, v_{i_2}v_{j_2})$$

by lines 1 and 7.

Since the color set used by Type 2 incidences is a subset of $\{1'', 2'', \ldots, \Delta''\}$ by line 6, and the color set used by Type 1 or Type 3 incidences is chosen from $\{1, 2, \ldots, \Delta+1, 1', 2', \ldots, (\Delta+1)'\}$ by lines 1 and 7, every Type 2 incidence is colored with a color different from the color of any Type 1 or Type 3 incidence.

One can see from line 9 that all Type 4 incidences are colored with $(\Delta+1)''$. Lines 10–17 imply that any two Type 5 incidences receive different colors, and moreover, the color set used by Type 5 incidences is a subset of $\{1, 2, \ldots, \Delta + 1, 1', 2', \ldots, (\Delta+1)', 1'', 2'', \ldots, \Delta''\} \cup S$, where $S = \emptyset$ if $|G| \leq 3\Delta + 2$, or $S = \{\Delta + 2, \ldots, |G| - 2\Delta - 1\}$ if $|G| \geq 3\Delta + 3$. Therefore, the color of a Type 4 incidence is different from the color of its every adjacent incidence.

Let

$$J(v_i) = \{\varphi(v_i, v_iv_\ell) \mid v_iv_\ell \in E(G)\}$$

and

$$J(u_i) = \{\varphi(u_i, u_iv_\ell) \mid u_iv_\ell \in E(M(G))\}.$$

By the definition of $M(G)$, and by lines 1 and 7,

$$J(v_i) = J(u_i) \subseteq \{1, 2, \ldots, \Delta + 1, 1', 2', \ldots, (\Delta + 1)'\}$$

for every $1 \leq i \leq n$. Since line 1 outputs an amicable Vizing-based incidence coloring φ of G using colors from $\{1, 2, \ldots, \Delta + 1, 1', 2', \ldots, (\Delta+1)'\}$ such that $\{v_1, v_2, \ldots, v_{2\Delta+2}\}$ is an amicable set of φ, $i \notin J(v_i) = J(u_i)$ if $1 \leq i \leq \Delta + 1$ and $(i - \Delta - 1)' \notin J(v_i) = J(u_i)$ if $\Delta + 2 \leq i \leq 2\Delta + 2$. Moreover, it is clear that $(i - 2\Delta - 2)'' \notin J(u_i)$ if $2\Delta + 3 \leq i \leq 3\Delta + 2$, and $i - 2\Delta - 1 \notin J(u_i)$ if $3\Delta + 3 \leq i \leq n$.

Since any Type 1 or Type 2 incidence is not adjacent to any Type 5 incidence, the final task is to check that if a Type 3 incidence is adjacent to a Type 5 incidence then they receive different colors. If $(u_i, u_iv_{j_1})$ is a Type 3 incidence adjacent to a Type 5 incidence (w, wu_ℓ), then $\ell = i$. Lines 10 to 17 guarantee that

$$\varphi(w, wu_\ell) = \varphi(w, wu_i) \notin J(u_i) \supseteq \varphi(u_i, u_iv_{j_1}),$$

implying

$$\varphi(w, wu_\ell) \neq \varphi(u_i, u_iv_{j_1}),$$

as desired. □

We claim that Algorithm 2 is a cubic-time algorithm. Actually, for a graph G having n vertices and m edges, line 1 takes $O(mn)$ time by the complexity of Algorithm 1. Since lines 3 and 9–17 can be done in $O(1)$ time, and lines 4–8 take $O(n)$ time, lines 2–17 can be populated in $O(n^2)$ time. Hence the running time of Algorithm 2 is $O(mn) + O(n^2) \leq O(n^3)$.

Theorem 2.4. *If G is a graph such that $|G| \geq 3\Delta(G) + 2$, then $\chi_i(M(G)) = \Delta(M(G)) + 1$, and a proper incidence $(\Delta(M(G) + 1)$-coloring can be constructed in cubic time.*

Proof. Taking $\Delta = \Delta(G)$ into Theorem 2.3, we can construct a proper incidence $(|G|+1)$-coloring φ of $M(G)$ by Algorithm 2 in cubic time. It follows that

$$|G| + 1 \geq \chi_i(M(G)) \geq \Delta(M(G)) + 1 = \max\{2\Delta(G), |G|\} + 1 = |G| + 1.$$

This implies that $\chi_i(M(G)) = \Delta(M(G)) + 1$ and φ is a proper incidence $(\Delta(M(G) + 1)$-coloring of $M(G)$. □

3. Mycielskians of dense graphs

Lemma 3.1. *For every graph G, there is an ordering $v_1, v_2, \dots, v_n$ of $V(G)$ and a proper incidence coloring of G using $\chi_i(G)$ colors so that the color i does not appear in any strong incidence of v_i for each $1 \le i \le \Delta(G)+1$.*

Proof. Let $u \in V(G)$ such that $N(u) = \{y_1, y_2, \dots, y_{\Delta(G)}\}$ and let φ be a proper incidence coloring of G using $\chi_i(G)$ colors. Since φ is proper, $\varphi(u, y_i u) \ne \varphi(u, y_j u)$ for each $i \ne j$. Hence by redistributing the colors of φ, we can construct a new proper incidence coloring ϕ of G using $\chi_i(G)$ colors such that $\phi(u, y_i u) = i$ for each $1 \le i \le \Delta(G)$. Since $(u, y_i u)$ is a weak incidence of y_i, the color i would not appear in any strong incidence of y_i under ϕ for each $1 \le i \le \Delta(G)$. One can also see that the color $\Delta(G)+1$ does not appear in any strong incidence of u, therefore, $y_1, y_2, \dots, y_{\Delta(G)}, u, \dots$ is the desired ordering of $V(G)$ and ϕ is the desired proper incidence coloring of G. $\square$

Theorem 3.2. *If G is a graph such that $|G| \le 3\Delta(G)+1$, then*

$$\chi_i(M(G)) \le \begin{cases} \chi_i(G) + |G| - \Delta(G) & \text{if } |G| \ge 2\Delta(G)+2, \\ \chi_i(G) + \Delta(G) + 1 & \text{otherwise.} \end{cases}$$

Proof. According to Lemma 3.1, there is an ordering $v_1, v_2, \dots, v_n$ of $V(G)$ and a proper incidence $\chi_i(G)$-coloring φ of G such that the color i does not appear in any strong incidence of v_i for each $1 \le i \le \Delta(G)+1$. Let $M(G)$ be the Mycielskian of G based on the ordering $v_1, v_2, \dots, v_n$. Note that φ constructs a proper coloring of the Type 1 incidences of $M(G)$. We extend φ to an incidence coloring (still denoted by φ) of $M(G)$ as follows.

For each $i \in [n]$, if the neighbors of v_i among $\{u_1, \dots, u_n\}$ are $\{u_{j_1}, \dots, u_{j_s}\}$, then let $\varphi(v_i, u_{j_\ell} v_i) = \ell''$ for each $\ell \in [s]$. For each pair $i, j \in [n]$ such that u_j is adjacent to v_i in $M(G)$, let $\varphi(u_j, u_j v_i) = \varphi(v_j, v_i v_j)$, and for each $j \in [n]$, let $\varphi(u_j, w u_j) = (\Delta(G)+1)''$. For each $j \in [n]$, let

$$\varphi(w, w u_j) = \begin{cases} j & \text{if } 1 \le j \le \Delta(G)+1, \\ (j - \Delta(G) - 1)'' & \text{if } \Delta(G)+2 \le j \le 2\Delta(G)+1 \\ (j - 2\Delta(G) - 1)' & \text{if } j \ge 2\Delta(G)+2 \end{cases}$$

By similar arguments as that in the proof of Theorem 2.3, one can show that every two adjacent incidences of $M(G)$ receive different colors under φ, and thus φ is a proper incidence coloring.

By the construction of φ, the set of colors used by Type 1 and Type 3 incidences is

$$S_{1\&3} = \{1, 2, \dots, \chi_i(G)\},$$

the one used by Type 2 incidences is a subset of

$$S_2 = \{1'', 2'', \dots, \Delta(G)''\},$$

and the one used by Type 4 incidences is

$$S_4 = \{(\Delta(G)+1)''\}.$$

If $|G| \ge 2\Delta(G)+2$, then Type 5 incidences involve the set of colors

$$\begin{aligned} S_5 &= \{1, 2, \dots, \Delta(G)+1, 1'', 2'', \dots, \Delta(G)'', 1', \dots, (|G| - 2\Delta(G) - 1)'\} \\ &\subseteq S_{1\&3} \cup S_2 \cup \{1', \dots, (|G| - 2\Delta(G) - 1)'\}. \end{aligned}$$

This implies $\chi_i(M(G)) \le \chi_i(G) + \Delta(G) + 1 + (|G| - 2\Delta(G) - 1) = \chi_i(G) + |G| - \Delta(G)$.

On the other hand, if $|G| \le 2\Delta(G)+1$, then the set of colors used by Type 5 incidences is a subset of $S_{1\&3} \cup S_2$, which implies $\chi_i(M(G)) \le \chi_i(G) + \Delta(G) + 1$. $\square$

We show the sharpness of the two bounds in Theorem 3.2. First, let G be a graph with $2\Delta(G)+2 \le |G| \le 3\Delta(G)+1$ and $\chi_i(G) = \Delta(G)+1$. By Theorem 3.2, $\chi_i(M(G)) \le \chi_i(G) + |G| - \Delta(G) = |G|+1$. On the other hand, $\chi_i(M(G)) \ge \Delta(M(G)) + 1 = \max\{2\Delta(G), |G|\} + 1 = |G|+1$. Hence $\chi_i(M(G)) = |G|+1 = \chi_i(G) + |G| - \Delta(G)$. Second, if G is P_2, then it satisfies $|G| \le 2\Delta(G)+1$ and $M(G) = C_5$, and thus $\chi_i(M(G)) = 4 = 2+1+1 = \chi_i(G) + \Delta(G) + 1$.

To end this section, we prove the following theorem, which can be seen as an interesting corollary of Theorem 3.2, and a powerful evidence supporting our Conjecture 1.2.

Theorem 3.3. *Let G be a graph such that $|G| \le 3\Delta(G)+1$, then*

(1) *If* $|G| \geq 2\Delta(G)+1$ *and* $\chi_i(G) = \Delta(G)+1$, *then* $\chi_i(M(G)) = \Delta(M(G))+1$.
(2) *If* $|G| \geq 2\Delta(G)+1$ *and* $\chi_i(G) = \Delta(G)+2$, *or* $|G| \leq 2\Delta(G)$ *and* $\chi_i(G) = \Delta(G)+1$, *then* $\chi_i(M(G)) \leq \Delta(M(G))+2$.

Proof. If $|G| \geq 2\Delta(G)+2$, then $\Delta(M(G)) = \max\{2\Delta(G), |G|\} = |G|$. By Theorem 3.2,

$$\Delta(M(G))+1 \leq \chi_i(M(G)) \leq \chi_i(G)+|G|-\Delta(G) = \begin{cases} \Delta(M(G))+1 & \text{if } \chi_i(G) = \Delta(G)+1, \\ \Delta(M(G))+2 & \text{if } \chi_i(G) = \Delta(G)+2. \end{cases}$$

If $|G| = 2\Delta(G)+1$, then $\Delta(M(G)) = \max\{2\Delta(G), |G|\} = |G|$. By Theorem 3.2,

$$\Delta(M(G))+1 \leq \chi_i(M(G)) \leq \chi_i(G)+\Delta(G)+1 = \begin{cases} \Delta(M(G))+1 & \text{if } \chi_i(G) = \Delta(G)+1, \\ \Delta(M(G))+2 & \text{if } \chi_i(G) = \Delta(G)+2. \end{cases}$$

If $|G| \leq 2\Delta(G)$ and $\chi_i(G) = \Delta(G)+1$, then $\Delta(M(G)) = \max\{2\Delta(G), |G|\} = 2\Delta(G)$, and thus

$$\chi_i(M(G)) \leq \chi_i(G)+\Delta(G)+1 = 2\Delta(G)+2 = \Delta(M(G))+2$$

by Theorem 3.2. □

4. Applications

Combining Theorems 2.4 and 3.3, we are easy to conclude that Conjecture 1.2 satisfies if G is not a graph with

(1) $2\Delta(G)+1 \leq |G| \leq 3\Delta(G)+1$ and $\chi_i(G) \geq \Delta(G)+3$, or
(2) $|G| \leq 2\Delta(G)$ and $\chi_i(G) \geq \Delta(G)+2$.

In this section, we apply Theorems 2.4 and 3.2 to determine the incidence chromatic number of the Mycielskian of a cycle or a path.

4.1. Mycielskians of cycles

Lemma 4.1 *(Folklore).*

$$\chi_i(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \bmod 3, \\ 4 & \text{otherwise.} \end{cases}$$

Lemma 4.2. *$M(C_4)$ is not incidence 5-colorable.*

Proof. Suppose for a contradiction that φ is a proper incidence 5-coloring of $M(C_4)$. It follows that

(a) the weak incidences of every vertex of degree 4 are colored with a same color under φ.

Without loss of generality, assume $\varphi(v_2, v_1v_2) = 1, \varphi(v_2, v_2v_3) = 2, \varphi(v_2, u_1v_2) = 3, \varphi(v_2, u_3v_2) = 4$, and thus $\varphi(v_1, v_1v_2) = \varphi(v_3, v_2v_3) = \varphi(u_1, u_1v_2) = \varphi(u_3, u_3v_2) = 5$ by **(a)**. Since (v_2, v_1v_2) is a weak incidence of v_1 colored with 1 and v_1 has degree 4, $\varphi(v_4, v_1v_4) = \varphi(u_2, u_2v_1) = \varphi(u_4, u_4v_1) = 1$ by **(a)**. Similarly, $\varphi(v_4, v_3v_4) = \varphi(u_2, u_2v_3) = \varphi(u_4, u_4v_3) = 2$ by **(a)**, and thus $\varphi(u_3, u_3v_4) = 3$. It follows by **(a)** that $\varphi(v_3, v_3v_4) = \varphi(u_1, u_1v_4) = \varphi(v_1, v_1v_4) = 3$, which implies $\varphi(v_3, u_2v_3) = 4$ and $\varphi(v_3, u_4v_3) = 1$. This contradicts the fact that $\varphi(v_3, u_4v_3) \neq \varphi(u_4, u_4v_1) = 1$ (note that (v_3, u_4v_3) and (u_4, u_4v_1) are adjacent). □

Lemma 4.3. *$M(C_3)$ is not incidence 5-colorable.*

Proof. Suppose for a contradiction that φ is a proper incidence 5-coloring of $M(C_3)$. It follows that

(a) the weak incidences of every vertex of degree 4 are colored with a same color under φ;
(b) the weak incidences of every vertex of degree 3 are colored with at most two colors under φ.

Without loss of generality, assume $\varphi(v_2, v_1v_2) = 1, \varphi(v_2, v_2v_3) = 2, \varphi(v_2, u_1v_2) = 3, \varphi(v_2, u_3v_2) = 4$, and thus $\varphi(v_1, v_1v_2) = \varphi(v_3, v_2v_3) = \varphi(u_1, u_1v_2) = \varphi(u_3, u_3v_2) = 5$ by **(a)**. Since (v_2, v_1v_2) is a weak incidence of v_1 colored with 1 and v_1 has degree 4, $\varphi(v_3, v_1v_3) = \varphi(u_2, u_2v_1) = \varphi(u_3, u_3v_1) = 1$ by **(a)**. Similarly, $\varphi(v_1, v_1v_3) = \varphi(u_1, u_1v_3) = \varphi(u_2, u_2v_3) = 2$ by **(a)**. This implies $\varphi(u_1, u_1w) \in \{1, 4\}$, $\varphi(u_2, u_2w) \in \{3, 4, 5\}$, and $\varphi(u_3, u_3w) \in \{2, 3\}$. If $\varphi(u_2, u_2w) = 5$,

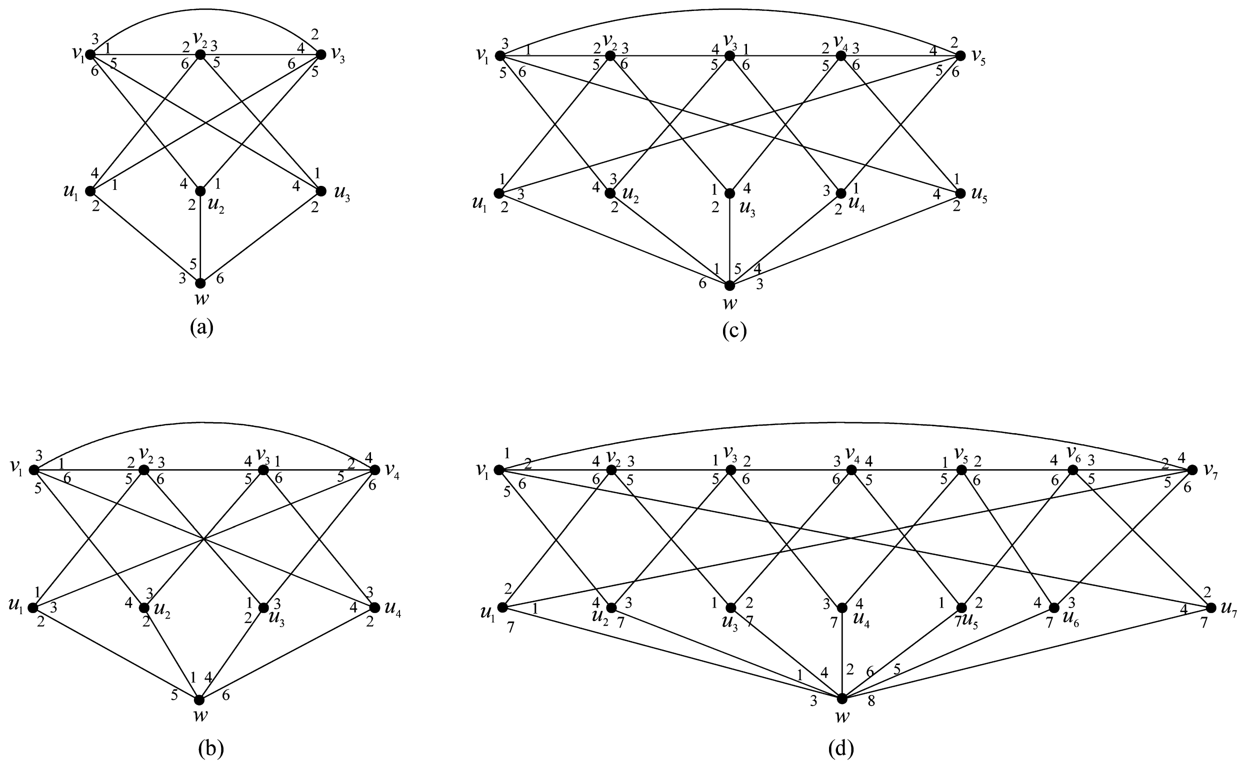

Fig. 1. Incidence coloring of $M(C_n)$ for $n \in \{3, 4, 5, 7\}$.

then $|\{\varphi(u_1, u_1w), \varphi(u_2, u_2w), \varphi(u_3, u_3w)\}| = 3$, contradicting **(b)**. Hence $\varphi(u_2, u_2w) \in \{3, 4\}$. By the symmetry of 3 and 4, we may assume $\varphi(u_2, u_2w) = 3$. It follows that $\varphi(v_1, u_2v_1) = \varphi(v_3, u_2v_3) = 4$ and $\varphi(v_1, u_3v_1) = \varphi(v_3, u_1v_3) = 3$, which implies $\varphi(u_3, u_3w) = 2$ and thus $|\{\varphi(u_1, u_1w), \varphi(u_2, u_2w), \varphi(u_3, u_3w)\}| = 3$, contradicting **(b)**. $\square$

Theorem 4.4.

$$\chi_i(M(C_n)) = \begin{cases} n+3 & \text{if } n = 3, \\ n+2 & \text{if } n = 4, \\ n+1 & \text{if } n \geq 5, \end{cases} = \begin{cases} \Delta(M(C_n)) + 2 & \text{if } 3 \leq n \leq 4, \\ \Delta(M(C_n)) + 1 & \text{if } n \geq 5. \end{cases}$$

Proof. Note that

$$\Delta(M(C_n)) = \max\{2\Delta(C_n), n\} = \begin{cases} n & \text{if } n \geq 4, \\ 4 & \text{if } n = 3. \end{cases}$$

If $n \geq 8$, then $n \geq 3\Delta(C_n) + 2$, implying $\chi_i(M(C_n)) = \Delta(M(C_n)) + 1 = n + 1$ by Theorem 2.4.

If $n = 7$, then $\chi_i(M(C_n)) \leq 8 = n + 1$, since Fig. 1(d) gives a proper incidence 8-coloring of $M(C_7)$. In the other direction, $\chi_i(M(C_n)) \geq \Delta(M(C_n)) + 1 = n + 1$. Hence $\chi_i(M(C_n)) = n + 1 = \Delta(M(C_n)) + 1$.

If $n = 6$, then $2\Delta(C_n) + 2 = n \leq 3\Delta(C_n) + 1$, implying

$$n + 1 = \Delta(M(C_n)) + 1 \leq \chi_i(M(C_n)) \leq \chi_i(C_n) + n - \Delta(C_n) = 3 + n - 2 = n + 1$$

by Lemma 4.1 and Theorem 3.2. Hence $\chi_i(M(C_n)) = n + 1 = \Delta(M(C_n)) + 1$.

If $n = 5$, then $\chi_i(M(C_n)) \leq 6 = n + 1$, since Fig. 1(c) gives a proper incidence 6-coloring of $M(C_5)$. In the other direction, $\chi_i(M(C_n)) \geq \Delta(M(C_n)) + 1 = n + 1$. Hence $\chi_i(M(C_n)) = n + 1 = \Delta(M(C_n)) + 1$.

If $n = 4$, then $\chi_i(M(C_n)) \leq 6 = n + 2$, since Fig. 1(b) gives a proper incidence 6-coloring of $M(C_4)$. In the other direction, $\chi_i(M(C_n)) \geq 6 = n + 2$ by Lemma 4.2. Hence $\chi_i(M(C_n)) = n + 2 = \Delta(M(C_n)) + 2$.

If $n = 3$, then $\chi_i(M(C_n)) \leq 6 = n + 3$, since Fig. 1(a) gives a proper incidence 6-coloring of $M(C_3)$. In the other direction, $\chi_i(M(C_3)) \geq 6 = n + 3$ by Lemma 4.3. Hence $\chi_i(M(C_n)) = n + 3 = \Delta(M(C_n)) + 2$. $\square$

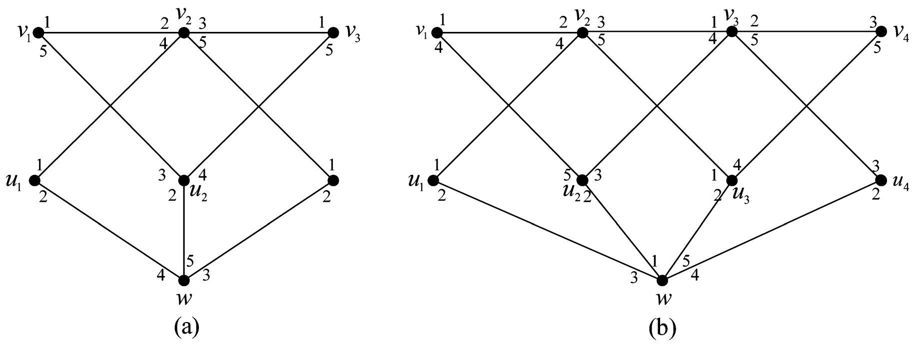

Fig. 2. Incidence coloring of $M(P_n)$ for $n \in \{3, 4\}$.

4.2. Mycielskians of paths

Lemma 4.5 *(Folkflore). $\chi_i(P_n) = \min\{n, 3\}$ for every $n \geq 2$.*

Theorem 4.6.

$$\chi_i(M(P_n)) = \begin{cases} n+2 & \text{if } 2 \leq n \leq 3, \\ n+1 & \text{if } n \geq 4, \end{cases} = \begin{cases} \Delta(M(P_n))+2 & \text{if } n=2, \\ \Delta(M(P_n))+1 & \text{if } n \geq 3. \end{cases}$$

Proof. Note that

$$\Delta(M(P_n)) = \max\{2\Delta(P_n), n\} = \begin{cases} n & \text{if } n \geq 4 \text{ or } n=2, \\ 4 & \text{if } n=3, \end{cases}$$

and $M(P_2) = C_5$.

If $n \geq 8$, then $n \geq 3\Delta(P_n)+2$, implying $\chi_i(M(P_n)) = \Delta(M(P_n))+1 = n+1$ by Theorem 2.4.

If $n = 6$ or 7, then $2\Delta(P_n)+2 \leq n \leq 3\Delta(P_n)+1$, implying

$$n+1 = \Delta(M(P_n))+1 \leq \chi_i(M(P_n)) \leq \chi_i(P_n)+n-\Delta(P_n) = 3+n-2 = n+1$$

by Lemma 4.5 and Theorem 3.2. Hence $\chi_i(M(P_n)) = n+1 = \Delta(M(P_n))+1$.

If $n = 5$, then $n = 2\Delta(P_n)+1$, implying

$$n+1 = \Delta(M(P_n))+1 \leq \chi_i(M(P_n)) \leq \chi_i(P_n)+\Delta(P_n)+1 = 3+2+1 = n+1$$

by Lemma 4.5 and Theorem 3.2. Hence $\chi_i(M(P_n)) = n+1 = \Delta(M(P_n))+1$.

If $n = 4$, then $\chi_i(M(P_n)) \leq 5 = n+1$, since Fig. 2(b) gives a proper incidence 5-coloring of $M(P_4)$. In the other direction, $\chi_i(M(P_n)) \geq \Delta(M(P_n))+1 = n+1$. Hence $\chi_i(M(P_n)) = n+1 = \Delta(M(P_n))+1$.

If $n = 3$, then $\chi_i(M(P_n)) \leq 5 = n+2$, since Fig. 2(a) gives a proper incidence 5-coloring of $M(P_3)$. In the other direction, $\chi_i(M(P_n)) \geq \Delta(M(P_n))+1 = n+2$. Hence $\chi_i(M(P_n)) = n+2 = \Delta(M(P_n))+1$.

If $n = 2$, then $\chi_i(M(P_n)) = \chi_i(C_5) = 4 = n+2 = \Delta(M(P_n))+2$ by Lemma 4.1. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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