Theoretical Computer Science 871 (2021) 51–61

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

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Theoretical aspects of equitable partition of networks into sparse modules ☆

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ARTICLE INFO

Article history:
Received 8 December 2020
Received in revised form 6 March 2021
Accepted 11 April 2021
Available online 14 April 2021
Communicated by A. Casteigts

Keywords:
Equitable partition
Degenerate graph
Maximum average degree
Planar graph
Complex network

ABSTRACT

The problem of partitioning a large complex network equitably into sparse modules with given rules can be modeled by the equitable list d-degenerate coloring of graphs. This paper establishes theoretical results on such a coloring based on a newly proposed conjecture which states that every graph G is equitably d-degenerate k-colorable and equitably d-degenerate k-choosable for every integer $k \geq (\Delta(G)+1)/(d+1)$. This conjecture is strong as it implies the Hajnal-Szemerédi theorem on equitable coloring, the equitable list coloring conjecture (Kostochka, Pelsmajer, and West, 2003), the equitable vertex arboricity conjecture (Wu, Zhang, and Li, 2013), and the equitable list vertex arboricity conjecture (Zhang, 2016). In this paper, we confirm this unified conjecture for globally coupled networks, $(d+1)$-degenerate graphs, graphs with bounded maximum average degree, and planar graphs with large maximum degree. The equitable d-degenerate k-colorability part of this conjecture is also verified for interval graphs, generalizing a result of Niu, Li, and Zhang (2021).

1. Introduction

In computer science, it is valuable to consider the problem of partitioning a large complex network into small modules under some special conditions so that the number of modules is as small as possible and each module acts as a layer of the network. This helps us understand the complex network interactions in some sense.

While a network is being partitioned, we may be required to obey some requirements associated with the topological structure and the scale of the module. For example, we sometimes require that the difference between any two module scales is very small so as to maintain the entire complex network efficiently, and meanwhile desire that each module is sparse enough for security reasons. To measure how sparse a network is, the degeneracy is a powerful parameter, which is within a constant factor of other sparsity measures such as the arboricity. For instance, a network with degeneracy 0 is a collection of some independent nodes, and a network with degeneracy 1 is an acyclic network, which are mainly used by social networks and the World Wide Web etc [2].

This network partition problem can be modeled by a minimization problem in graph theory and established by the language of graph theory [6,7,25]. Precisely, if we consider the network as a graph G, then our task can be transferred to partition the vertex set of G into the smallest possible number of disjoint subsets so that the size of any two distinct subsets differ by at most a fixed constant and the graph induced by each subset has low degeneracy.

☆ This paper is partially supported by National Natural Science Foundation of China (No. 11871055).
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https://doi.org/10.1016/j.tcs.2021.04.010
0304-3975/

From now on, we use standard graph theory notions and notations as in [5] and only consider finite graphs. A graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d. A *d-degenerate k-coloring* of a graph G is a partition of $V(G)$ into k disjoint subsets $V(G_i)$ with $1 \le i \le k$ so that

($\mathscr{A}1$) $V(G_i)$ induces a d-degenerate subgraph for each $1 \le i \le k$.
A d-degenerate k-coloring of a graph G is *equitable* if we further require
($\mathscr{A}2$) $\big||V(G_i)| - |V(G_j)|\big| \le 1$ for every $1 \le i \le j \le k$.
Note that ($\mathscr{A}2$) is equivalent to the following property
($\mathscr{A}2$') $\lfloor |G|/k \rfloor \le |V(G_i)| \le \lceil |G|/k \rceil$ for each $1 \le i \le k$.

The minimum k such that G has an equitable d-degenerate k-coloring is the *equitable d-degenerate chromatic number* of G. In the literature, equitable 0-degenerate coloring is known as *equitable coloring* [16] while equitable 1-degenerate coloring is known as *equitable tree-coloring* [14,21].

Returning to the background of network partition problems, in practice, sometimes there is a given rule on which kinds of sparse module a node may belong to. In such a case, we need to find an equitable partition acceptable under this rule, and this task can be modeled by equitable list d-degenerate coloring of graphs [15], where the definition on the equitability shall be rewritten so as to make the scale of each module is bounded above (note that an accepted partition by this given rule may not always satisfies ($\mathscr{A}2$)). The formal definition of equitable list d-degenerate coloring is given below.

To each vertex $v \in V(G)$, we assign a list of colors $L(v)$, and call L a *list assignment* of G. If G has a d-degenerate coloring c such that

($\mathscr{B}1$) $c(v) \in L(v)$ for each $v \in V(G)$, and
($\mathscr{B}2$) $|c^{-1}(i)| \le \lceil |G|/k \rceil$ for each color $i \in \{c(v) \mid v \in V(G)\}$,

then we call c an *equitable d-degenerate L-coloring* of G. If G admits an equitable d-degenerate L-coloring for every list assignment L such that $|L(v)| = k$ for each $v \in V(G)$, then G is *equitably d-degenerate k-choosable*, and the minimum k such that G is equitably d-degenerate k-choosable is the *equitable d-degenerate choosability* of G. The equitable 0-degenerate choosability (also known as *equitable list chromatic number*) was first introduced by Kostochka, Pelsmajer, and West [13], and the equitable 1-degenerate choosability (also known as *equitable list vertex arboricity*) was first investigated by Zhang [22]. Recently, Drgas-Burchardt, Furmańczyk, and Sidorowicz [6,7] investigated the equitable d-degenerate choosability with $d \ge 2$.

An equitably d-degenerate k-choosable graph may not be equitably d-degenerate k-colorable. An easy example supporting this conclusion is the complete bipartite graph $K_{1,6}$, which is equitably 0-degenerate 3-choosable, and is not equitably 0-degenerate 3-colorable. On the other hand, an equitably d-degenerate k-choosable graph may not be equitably d-degenerate $(k+1)$-choosable. For example, $K_{1,9}$ is equitably 0-degenerate 4-choosable, and is not equitably 0-degenerate 5-choosable. Recently, Kaul, Mudrock, and Pelsmajer [11] used the combination of probabilistic and algorithmic arguments to show that $K_{11,17}$ is equitably 1-degenerate 3-choosable, and is not equitably 1-degenerate 4-choosable. Indeed, it is still interesting to check whether there are such examples with $d \ge 2$.

The *maximum average degree* $\mathrm{mad}(G)$ of a graph G is defined by

$$\max\left\{\frac{2|E(H)|}{|V(H)|}\,\middle|\, \emptyset \neq H \subseteq G\right\}.$$

Planar graphs are a well-established class of graphs with bounded maximum average degree (note that $\mathrm{mad}(G) < 6$ for any planar graph G).

In 2013, Wu, Zhang, and Li [21] conjectured that the equitable 1-degenerate chromatic number of any planar graph G is bounded by a constant independent of G. Two years later, Esperet, Lemoine, and Maffray [9] confirmed it by proving that every planar graph is equitably 1-degenerate 4-colorable. Recently, Kim, Oum, and Zhang [12] showed that every planar graph is equitably 2-degenerate 3-colorable, and is equitably d-degenerate 2-colorable for every $d \ge 3$. The list analogue of such kinds of results first appeared in [22], where the author proved that every planar graph G with maximum degree at least 8 is equitably 1-degenerate k-choosable for every $k \ge \lceil (\Delta(G)+1)/2 \rceil$. Meanwhile, Zhang [22] put forward the following conjecture.

Conjecture 1.1. *Every graph G is equitably 1-degenerate k-choosable for every $k \ge \lceil \frac{\Delta(G)+1}{2} \rceil$.*

Zhang [22] proved that this conjecture holds for many classes of graphs including complete graphs, 2-degenerate graphs, 3-degenerate claw-free graphs with maximum degree at least 4, and planar graphs with maximum degree at least 8. Recently, Kaul, Mudrock and Pelsmajer [11] confirmed it for powers of cycles.

Motivated by Conjecture 1.1, it is natural to ask whether we have similar properties for equitable d-degenerate choosability. The following proposition is a start, and it generalizes a result of Niu, Li, and Zhang [17,18].

Proposition 1.2. *The globally coupled network with n nodes (i.e., complete graph K_n) is equitably d-degenerate k-colorable and equitably d-degenerate k-choosable for any nonnegative integer d and $k \geq \lceil \frac{n}{d+1} \rceil$, and K_n is not equitably d-degenerate k-colorable for any $k < \lceil \frac{n}{d+1} \rceil$.*

Proof. The proof of the first part of this theorem will be presented in Section 2 (actually it is an immediate corollary from Theorems 2.3(1) and 2.3(3) of the next section) and now we prove the second part. Suppose for a contradiction that K_n has an equitable d-degenerate k-coloring c for some $k < \lceil n/(d+1) \rceil$. It follows that every color class of c has at least $\lfloor n/k \rfloor \geq d+1$ vertices, and furthermore, this number should be exactly $d+1$, because otherwise there would be a monochromatic K_{d+2}, which is not d-degenerate. This implies $n = k(d+1)$ and thus $k = \lceil n/(d+1) \rceil$, a contradiction. □

Since $\Delta(K_n) = n-1$, we conjecture, motivated by Proposition 1.2, the following.

Conjecture 1.3. *Let d be a nonnegative integer and let G be a graph.*

(1) *G is equitably d-degenerate k-colorable for every $k \geq \lceil \frac{\Delta(G)+1}{d+1} \rceil$;*
(2) *G is equitably d-degenerate k-choosable for every $k \geq \lceil \frac{\Delta(G)+1}{d+1} \rceil$.*

In the case that $d = 0$, Conjecture 1.3(1) coincides with an old conjecture of Erdős [8], which was confirmed by Hajnal and Szemerédi [10] in 1970, and Conjecture 1.3(2) is known as *Equitable List Coloring Conjecture*, which was put forward by Kostochka, Pelsmajer and West [13] in 2003 and is still open. On the other hand, Conjecture 1.3(1) with $d = 1$ is known as *Equitable Vertex Arboricity Conjecture* [21], which was put forward in 2013 and has been confirmed for some classes of graphs including subcubic graphs [23], graphs G with $\Delta(G) \geq (|G|-1)/2$ [19,26], 5-degenerate graphs [4] and IC-planar graphs with large maximum degree [24], and Conjecture 1.3(2) with $d = 1$ is exactly Conjecture 1.1. In this paper, we aim to confirm Conjecture 1.3 for certain classes of graphs.

The well-know Chen-Lih-Wu Conjecture [3] states that the only connected graphs with maximum degree at most r that are not equitably 0-degenerate r-colorable are $K_{r,r}$ and K_{r+1}. Kostochka, Pelsmajer, and West [13] also conjectured that every connected graph G with maximum degree at least 3 is equitably 0-degenerate k-choosable for every $k \geq \Delta(G)$ unless it is a complete graph or $K_{t,t}$ for some odd t. Recently, Kaul, Mudrock, and Pelsmajer [11] conjectured that every connected graph G is equitably 1-degenerate k-choosable for every $k \geq \Delta(G)/2$ provided G is neither a cycle nor a complete graph of odd order. Motivated by those conjectures, it may be natural to consider whether we can address slightly smaller values of k than Conjecture 1.3, or specifically, whether we can replace $(\Delta(G)+1)/(d+1)$ with $\Delta(G)/(d+1)$, sometimes with the connectedness condition and with some graphs forbidden. Theorem 3.5 is such a result and it is interesting to consider more in the future.

2. Complete graphs and interval graphs

Instead of proving Proposition 1.2 directly, we prove a stronger result for a larger class of graphs, called interval graphs. A graph G is an *interval graph* if each vertex $u \in V(G)$ has a representation as an interval I_u such that $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$. We use $L(u)$ and $R(u)$ to denote the left and the right endpoint of the interval I_u respectively for each $u \in V(G)$. For any two vertices $u, v \in V(G)$, if $L(u) < L(v)$, or $L(u) = L(v)$ and $R(u) \leq R(v)$, then we say $u < v$. The following proposition is straightforward.

Proposition 2.1 *(Folklore). If G is an interval graph with three vertices such that $u < v < w$ and $uw \in E(G)$, then $uv \in E(G)$.*

We can say more than Proposition 2.1 by the following lemma, which describes the structure of interval graphs well.

Lemma 2.2. *[20, Olariu] A graph is an interval graph if and only if it has a linear order $<$ as defined above such that $u < v < w$ and $uw \in E(G)$ imply $uv \in E(G)$ for any $u, v, w \in V(G)$.*

Theorem 2.3. *Every interval graph G on n vertices is*

(1): *equitably d-degenerate k-choosable for every $k \geq \left\lceil \frac{n}{d+1} \right\rceil$ and $d \geq 0$;*
(2): *equitably d-degenerate k-choosable for every $k \geq \left\lceil \frac{\Delta(G)}{d} \right\rceil$ and $d \geq 1$;*
(3): *equitably d-degenerate k-colorable for every $k \geq \left\lceil \frac{\Delta(G)+1}{d+1} \right\rceil$ and $d \geq 0$.*

Proof. Sort vertices of G by $v_1 < v_2 < \cdots < v_n$ such that $<$ is a linear order satisfying Lemma 2.2 and assume that $n = pk + q$, where p, q are nonnegative integers and $0 \leq q < k$. We color $v_1, v_2, \ldots, v_n$ in this order so that

$$c(v_{ik+j}) = \min\left\{\alpha \mid \alpha \in L(v_{ik+j}) \setminus \bigcup_{\ell=1}^{j-1}\{c(v_{ik+\ell})\}\right\}$$

for every pair of integers i and j with $0 \le i \le p$ and $1 \le j \le k$, where L is a k-uniform list assignment of G if we consider choosability, or denotes $\{1, 2, \dots, k\}$ if we consider colorability. Since $c(v_{ik+j_1}) \ne c(v_{ik+j_2})$ if $j_1 \ne j_2$, the number of vertices among $\{v_1, v_2, \dots, v_n\}$ in a same color is at most

$$\begin{cases} p & \text{if } q = 0, \\ p+1 & \text{otherwise,} \end{cases}$$

which is exactly the value of $\lceil n/k \rceil$. On the other hand, if $L(v_i) = \{1, 2, \dots, k\}$ for each $1 \le i \le n$, then the size of any color class of c is at least $p = \lfloor n/k \rfloor$. It is sufficient to prove that the vertices in each color class induce a d-degenerate subgraph.

If $k \ge \lceil n/(d+1) \rceil$, then $\lceil n/k \rceil \le d+1$, which implies $|c^{-1}(i)| \le d+1$ for each color $i \in \cup_{j=1}^{n}\{c(v_i)\}$ and thus $c^{-1}(i)$ induces a d-degenerate graph. This proves (1).

We now prove (2) and (3) together. Let $v_{\alpha_i k+\beta_i}$ with $1 \le i \le s$, $0 \le \alpha_1 < \alpha_2 < \dots < \alpha_s \le p$ and $0 \le \beta_i < k$ be s arbitrary vertices colored by a fixed color γ (note that there may be more than s vertices colored by γ under the coloring c) and we just need to show that those s vertices induce a graph H with minimum degree at most d. This result is trivial for $s \le d+1$ so we assume that $s \ge d+2$. Suppose, to the contrary, that $\delta(H) \ge d+1$. This implies that $v_{\alpha_1 k+\beta_1}$ has at least $d+1$ neighbors among $\{v_{\alpha_i k+\beta_i} \mid 2 \le i \le s\}$. In particular, $v_{\alpha_1 k+\beta_1} v_{\alpha_s k+\beta_s} \in E(G)$, which implies $v_{\alpha_1 k+\beta_1} v_j \in E(G)$ for any $\alpha_1 k + \beta_1 + 1 \le j \le \alpha_s k + \beta_s$ by Proposition 2.1. It follows that

$$\begin{aligned} \deg(v_{\alpha_1 k+\beta_1}) &\ge (\alpha_s - \alpha_1)k + (\beta_s - \beta_1) \\ &\ge (s-1)k + (\beta_s - \beta_1) \\ &\ge (d+1)k + (\beta_s - \beta_1) \end{aligned} \tag{2.1}$$

While proving (2), we have $\beta_s - \beta_1 \ge 1-k$. So (2.1) implies $(d+1)k + (1-k) \le \deg(v_{\alpha_1 k+\beta_1}) \le \Delta(G) \le dk$, a contradiction. While proving (3), we have $\beta_1 = \beta_2 = \dots = \beta_s$ by the definition of the coloring c. So (2.1) implies $(d+1)k \le \deg(v_{\alpha_1 k+\beta_1}) \le \Delta(G) \le (d+1)k - 1$, a contradiction. □

Since complete graphs are clearly interval graphs, Proposition 1.2 is an immediate corollary from Theorems 2.3(1) and 2.3(3). Furthermore, Theorem 2.3(3) implies Conjecture 1.3(1) for all interval graphs.

To end this section, we remark that we did not use any property of interval graphs while proving Theorem 2.3(1). This actually means that the result in Theorem 2.3(1) holds for all n-vertex graphs.

Proposition 2.4. *Every graph G on n vertices is equitably d-degenerate k-choosable for every $k \ge \left\lceil \frac{n}{d+1} \right\rceil$ and $d \ge 0$.* □

3. Technical lemmas

In this section we prove technical lemmas that will be applied in the next two sections. To begin with, we present two algorithms that will be involved in the next proofs.

In the first algorithm Gap-Filling, we have four inputs. The first one is a graph H, and the second to the fourth ones are three positive integers s, i, and j with $i \le j$. While applying Gap-Filling in the following arguments, we always choose the input graph H to be s-degenerate to guarantee that we can really find a required vertex by line 4 during each iteration.

Gap-Filling (H, s, i, j)

```
1  S' ← Ø
2  if i ≤ j
3     then for ℓ ← i to j
4              do find a vertex x_ℓ ∈ V(H) of degree at most s
5                 H ← H \ {x_ℓ}
6                 S' ← S' ∪ {x_ℓ}
7     else quite
```

In the second algorithm Color-Extension, we still have four inputs. The first one is a graph G, each of whose vertex v is assigned a k-uniform list $L(v)$, and thus we make the function $L: V(G) \to \{L(v) : v \in V(G)\}$ to be its second input. The third input of Color-Extension is a set $\{x_1, x_2, \dots, x_k\}$ of k distinct vertices of G. The last input of Color-Extension is an equitable d-degenerate L-coloring φ of the graph $G - \{x_1, x_2, \dots, x_k\}$.

Color-Extension $(G, L, \{x_1, x_2, \ldots, x_k\}, \varphi)$

1 **for** $i \leftarrow 1$ **to** k
$\triangleright$ H stands for the graph $G - \{x_1, x_2, \ldots, x_k\}$.
$\triangleright$ $C_H^d(v)$ is the set of colors under φ that are used at most d times among all neighbors of v in H.
2 **do** find a color $\alpha_{k+1-i} \in L(x_{k+1-i}) \cap C_H^d(x_{k+1-i}) \setminus \bigcup_{j=k+2-i}^{k}\{\alpha_j\}$
3 color x_{k+1-i} with α_{k+1-i}

Lemma 3.1. *Let d be a nonnegative integer and let S be a set of distinct vertices $x_1, \ldots, x_k$ of G so that $G - S$ is equitably d-degenerate k-choosable (resp. k-colorable). If*

$$|N(x_i) \setminus S| \leq (d+1)i - 1 \tag{3.1}$$

for every $1 \leq i \leq k$, then G is equitably d-degenerate k-choosable (resp. k-colorable).

Proof. It is sufficient to prove that Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ to G whenever every vertex v of G is assigned a k-uniform list $L(v)$. According to the algorithm, we color $x_k, x_{k-1}, \cdots, x_1$ in this order with $\alpha_k, \alpha_{k-1}, \cdots, \alpha_1$, respectively. Suppose now that x_{k+1-i} $(1 \leq i \leq k)$ is being colored. Since $|N(x_{k+1-i}) \setminus S| \leq (d+1)(k+1-i) - 1$, there is at least one color, say α_{k+1-i}, in $L(x_{k+1-i}) \setminus \bigcup_{j=k+2-i}^{k}\{\alpha_j\}$ (note that this set has at least $k+1-i$ elements) such that α_{k+1-i} is used at most d times among all neighbors of v besides those in S. Hence line 2 of the algorithm Color-Extension applies in each iteration, and moreover, executing line 3 results in a d-degenerate (list) coloring. Denote by ϕ the extended d-degenerate (list) coloring of G when the algorithm finishes.

If we consider colorability, then φ is a d-degenerate coloring such that the sizes of every two color classes differ by at most one, and so is ϕ, because $x_k, x_{k-1}, \cdots, x_1$ receive different colors. On the other hand, if we consider choosability, then by the same reason, each color of ϕ appears on at most $\lceil (|G| - k)/k \rceil + 1 = \lceil |G|/k \rceil$ vertices, and thus ϕ is still equitable. $\square$

In [1], Bollobás and Thomason gave the formal definition of monotone properties of graphs. A *property* $\mathscr{P}$ of graphs is an infinite class of graphs which is closed under isomorphism. A property $\mathscr{P}$ is *monotone* if every subgraph of every member of $\mathscr{P}$ is also in $\mathscr{P}$.

If G is not equitably d-degenerate k-choosable (resp. k-colorable) but any proper subgraph of G is equitably d-degenerate k-choosable (resp. k-colorable), then we call G an *equitably d-degenerate k-choosable-critical* (resp. *equitably d-degenerate k-colorable-critical*) graph. The following observations are straightforward.

Observation 3.2. *Let $\mathscr{P}$ be a monotone property. If there is a graph $G \in \mathscr{P}$ which is not equitable d-degenerate k-choosable (resp. k-colorable), then in $\mathscr{P}$ there is also an equitable d-degenerate k-choosable-critical (resp. k-colorable-critical) graph.*

Observation 3.3. *If $\mathscr{P}_1$ and $\mathscr{P}_2$ are two monotone properties such that $\mathscr{P}_1 \subset \mathscr{P}_2$, and G is an equitably d-degenerate k-choosable-critical (resp., k-colorable-critical) graph in $\mathscr{P}_1$, then G is also an equitably d-degenerate k-choosable-critical (resp., k-colorable-critical) graph in $\mathscr{P}_2$.*

In the following, if $\mathscr{P}$ is a monotone property, then instead of saying that G is an equitably d-degenerate k-choosable-critical (resp., k-colorable-critical) graph in $\mathscr{P}$, we say that G is an equitably d-degenerate k-choosable-critical (resp., k-colorable-critical) graph in the class of graphs which are defined by $\mathscr{P}$.

Lemma 3.4. *Let k, Δ, d and i be nonnegative integers. If $k \geq \max\{\Delta/(d+1), 2\}$ and G is an equitably d-degenerate k-choosable-critical (resp., k-colorable-critical) graph in the class of $(d+i)$-degenerate graphs with maximum degree at most Δ, then $i \geq 2$.*

Proof. Suppose for a contradiction that $i \leq 1$. Since G is $(d+1)$-degenerate (note that if $i = 0$ then G is d-degenerate, and d-degeneracy implies $(d+1)$-degeneracy), there is an edge uv with $\deg(u) \leq d+1$. Label u, v as x_1, x_k respectively and then execute Gap-Filling $(G \setminus \{x_1, x_k\}, d+1, 2, k-1)$. This results in a set $S = \{x_1, \cdots, x_k\}$.

Since $|N(x_1) \setminus S| \leq \deg(u) - 1 \leq d$, $|N(x_i) \setminus S| \leq d+1 < 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-1$, and $|N(x_k) \setminus S| \leq \Delta - 1 \leq (d+1)k - 1$, S satisfies the condition (3.1). This implies that the equitable d-degenerate L-coloring φ of $G - S$ can be extended to that of G by Lemma 3.1 (actually by executing Color-Extension (G, L, S, φ)) whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction. $\square$

By Lemma 3.4, we have the following theorem, verifying Conjecture 1.3 for $(d+1)$-degenerate graphs.

Theorem 3.5. *Every $(d+1)$-degenerate graph G is equitably d-degenerate k-choosable and equitably d-degenerate k-colorable for every nonnegative integer d and $k \geq \max\left\{\lceil \frac{\Delta(G)}{d+1} \rceil, 2\right\}$.* $\square$

The following lemma will be frequently used in the next sections.

Lemma 3.6. *Let k, Δ and d be nonnegative integers. If $k \geq (\Delta+1)/(d+1)$ and G is an equitably d-degenerate k-choosable-critical (resp., k-colorable-critical) graph in the class of $(2d+1)$-degenerate graphs with maximum degree at most Δ, then the following holds.*

(1): *If $k \geq 2$, then $\delta(G) \geq d+2$.*
(2): *If $k \geq 3$ and uv is an edge with $\deg(u) = d+2$, then $\deg(v) \geq (d+1)(k-1)+1$.*
(3): *If $k \geq 4$ and uv is an edge with $\deg(u) \leq (d+1)(k-1)+1$ and $\deg(v) = d+2$, then $\deg(w) \geq (d+1)(k-2)+1$ for any other neighbor w of u besides v.*
(4): *If $k \geq 4$ and uv is an edge with $\deg(u) \leq (d+1)(k-2)$ and $\deg(v) = d+3$, then $\deg(w) \geq (d+1)(k-1)+1$ for any other neighbor w of v besides u.*
(5): *If $k \geq 3$ and uvw is a triangle with $\deg(u) = d+2$, then $\min\{\deg(v), \deg(w)\} \geq (d+1)(k-1)+2$.*
(6): *If $k \geq 4$ and uvw is a triangle with $\deg(u) = d+3$, then $\min\{\deg(v), \deg(w)\} \leq (d+1)(k-2)+1$ implies $\max\{\deg(v), \deg(w)\} \geq (d+1)(k-1)+2$.*

Proof. (1) Suppose, to the contrary, that G has a vertex u of degree at most $d+1$. Let v be a neighbor of u. Label u, v as x_1, x_k respectively and then execute Gap-Filling $(G \setminus \{x_1, x_k\}, 2d+1, 2, k-1)$. This results in a set $S = \{x_1, \cdots, x_k\}$, satisfying the condition (3.1); i.e., $|N(x_1) \setminus S| \leq \deg(u) - 1 \leq d$, $|N(x_i) \setminus S| \leq 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-1$, and $|N(x_k) \setminus S| < \Delta(G) \leq (d+1)k - 1$. Hence Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ (which exists since G is critical) to G by Lemma 3.1 whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction.

(2) Suppose for a contradiction $\deg(v) \leq (d+1)(k-1)$. Label u, v, and a vertex in $N(u) \setminus \{v\}$ as x_1, x_{k-1}, and x_k respectively, and execute Gap-Filling $(G \setminus \{x_1, x_{k-1}, x_k\}, 2d+1, 2, k-2)$. This results in a set $S = \{x_1, \cdots, x_{k-1}, x_k\}$. Since $|N(x_1) \setminus S| \leq \deg(u) - 2 = d$, $|N(x_i) \setminus S| \leq 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-2$, $|N(x_{k-1}) \setminus S| \leq \deg(v) - 1 \leq (d+1)(k-1) - 1$, and $|N(x_k) \setminus S| < \Delta(G) \leq (d+1)k - 1$, S satisfies the condition (3.1). Hence Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ to G by Lemma 3.1 whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction.

(3) Suppose for a contradiction that $\deg(w) \leq (d+1)(k-2)$. Label v, w, u, and a vertex in $N(v) \setminus \{u, w\}$ as x_1, x_{k-2}, x_{k-1}, and x_k respectively, and execute Gap-Filling $(G \setminus \{x_1, x_{k-2}, x_{k-1}, x_k\}, 2d+1, 2, k-3)$. This results in a set $S = \{x_1, \cdots, x_{k-2}, x_{k-1}, x_k\}$. Since $|N(x_1) \setminus S| \leq \deg(v) - 2 = d$, $|N(x_i) \setminus S| \leq 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-3$, $|N(x_{k-2}) \setminus S| \leq \deg(w) - 1 \leq (d+1)(k-2) - 1$, $|N(x_{k-1}) \setminus S| \leq \deg(u) - 2 \leq (d+1)(k-1) - 1$, and $|N(x_k) \setminus S| < \Delta(G) \leq (d+1)k - 1$, S satisfies the condition (3.1). Hence Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ to G by Lemma 3.1 whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction.

(4) Suppose for a contradiction that $\deg(w) \leq (d+1)(k-1)$. Label v, u, w, and a vertex in $N(v) \setminus \{u, w\}$ as x_1, x_{k-2}, x_{k-1}, and x_k respectively, and execute Gap-Filling $(G \setminus \{x_1, x_{k-2}, x_{k-1}, x_k\}, 2d+1, 2, k-3)$. This results in a set $S = \{x_1, \cdots, x_{k-2}, x_{k-1}, x_k\}$. Since $|N(x_1) \setminus S| \leq \deg(v) - 3 = d$, $|N(x_i) \setminus S| \leq 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-3$, $|N(x_{k-2}) \setminus S| \leq \deg(u) - 1 \leq (d+1)(k-2) - 1$, $|N(x_{k-1}) \setminus S| \leq \deg(w) - 1 \leq (d+1)(k-1) - 1$, and $|N(x_k) \setminus S| < \Delta(G) \leq (d+1)k - 1$, S satisfies the condition (3.1). Hence Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ to G by Lemma 3.1 whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction.

(5) Suppose for a contradiction that $\deg(v) \leq (d+1)(k-1)+1$. Label u, v, and w as x_1, x_{k-1}, and x_k respectively, and then execute Gap-Filling $(G \setminus \{x_1, x_{k-1}, x_k\}, 2d+1, 2, k-2)$. This results in a set $S = \{x_1, \cdots, x_{k-1}, x_k\}$. Since $|N(x_1) \setminus S| \leq \deg(u) - 2 = d$, $|N(x_i) \setminus S| \leq 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-2$, $|N(x_{k-1}) \setminus S| \leq \deg(v) - 2 \leq (d+1)(k-1) - 1$ and $|N(x_k) \setminus S| < \Delta(G) \leq (d+1)k - 1$, S satisfies the condition (3.1). Hence Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ to G by Lemma 3.1 whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction.

(6) Suppose for a contradiction that $\deg(v) \leq (d+1)(k-2)+1$ and $\deg(w) \leq (d+1)(k-1)+1$. Label u, v, w, and a vertex in $N(u) \setminus \{v, w\}$ by x_1, x_{k-2}, x_{k-1}, and x_k respectively, and execute Gap-Filling $(G \setminus \{x_1, x_{k-2}, x_{k-1}, x_k\}, 2d+1, 2, k-2)$. This results in a set $S = \{x_1, \cdots, x_{k-2}, x_{k-1}, x_k\}$. Since $|N(x_1) \setminus S| \leq \deg(u) - 3 = d$, $|N(x_i) \setminus S| \leq 2d+1 \leq (d+1)i - 1$ for each $2 \leq i \leq k-3$, $|N(x_{k-2}) \setminus S| \leq \deg(v) - 2 \leq (d+1)(k-2) - 1$, $|N(x_{k-1}) \setminus S| \leq \deg(w) - 2 \leq (d+1)(k-1) - 1$, and $|N(x_k) \setminus S| < \Delta \leq (d+1)k - 1$, S satisfies the condition (3.1). Hence Color-Extension (G, L, S, φ) extends the equitable d-degenerate L-coloring φ of $G - S$ to G by Lemma 3.1 whenever every vertex v of G is assigned a k-uniform list $L(v)$, a contradiction. □

4. Graphs with bounded maximum average degree

In this section we confirm Conjecture 1.3 for graphs with bounded maximum average degree by the following theorems.

Theorem 4.1. *Every graph G with $\mathrm{mad}(G) < 4$ is equitably 1-degenerate k-choosable and equitably 1-degenerate k-colorable for every integer $k \geq \max\{\lceil \frac{\Delta(G)+1}{2} \rceil, 4\}$.*

Theorem 4.2. *Every graph G with $\mathrm{mad}(G) < d+4$ is equitably d-degenerate k-choosable and equitably d-degenerate k-colorable for every integer $k \geq \lceil \frac{\Delta(G)+1}{d+1} \rceil$ if*

- *$k \geq 4$ and $d \geq 3$ is an integer, or*
- *$k \geq 5$ and $d = 2$.*

A vertex of degree k, at least k, or at most k is a *k, k^+, or k^--vertex*, respectively. Instead of proving Theorem 4.1 directly, we prove the following slightly stronger theorem.

Theorem 4.3. *Every graph G with $\mathrm{mad}(G) < 4$ and $\Delta(G) \leq \Delta$ is equitably 1-degenerate k-choosable and equitably 1-degenerate k-colorable for every integer $k \geq \max\left\{\lceil \frac{\Delta+1}{2} \rceil, 4\right\}$, where Δ is an integer.*

Proof. Suppose for a contradiction that there exists an equitably 1-degenerate k-choosable (colorable)-critical graph G in the class of graphs with maximum average degree less than 4 and maximum degree at most Δ. Since graphs with maximum average degree less than 4 are 3-degenerate, G is also an equitably 1-degenerate k-choosable (colorable)-critical graph in the class of 3-degenerate graphs with maximum degree at most Δ. Hence Lemma 3.6 with $d = 1$ applies. Specifically, the following facts hold.

($\mathscr{F}$1): $\delta(G) \geq 3$ (by Lemma 3.6(1));
($\mathscr{F}$2): a 3-vertex is adjacent only to 7^+-vertices (by Lemma 3.6(2));
($\mathscr{F}$3): a 7^+-vertex is adjacent to at most one 3-vertex (by Lemma 3.6(3));

Let n_i and n_i^+ with $i \geq 3$ be the number of vertices of degree i and of degree at least i, respectively. By ($\mathscr{F}$2) and ($\mathscr{F}$3), $n_7^+ \geq 3n_3$. By ($\mathscr{F}$1), we have

$$\begin{aligned}\mathrm{mad}(G) &\geq \frac{3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7^+}{n_3 + n_4 + n_5 + n_6 + n_7^+} \\ &\geq \frac{24n_3 + 4n_4 + 5n_5 + 6n_6}{4n_3 + n_4 + n_5 + n_6} \geq 4,\end{aligned}$$

a contradiction. □

Again, instead of proving Theorem 4.2 directly, we prove the following slightly stronger theorem.

Theorem 4.4. *Every graph G with $\mathrm{mad}(G) < d+4$ and $\Delta(G) \leq \Delta$ is equitably d-degenerate k-choosable and equitably d-degenerate k-colorable for every integer $k \geq \lceil \frac{\Delta+1}{d+1} \rceil$ if*

- *$k \geq 4$ and $d \geq 3$ is an integer, or*
- *$k \geq 5$ and $d = 2$,*

where Δ is an integer.

Proof. Suppose for a contradiction that there exists an equitably d-degenerate k-choosable (colorable)-critical graph G in the class of graphs with maximum average degree less than $d+4$ and maximum degree at most Δ. Since graphs with maximum average degree less than $d+4$ are $(d+3)$-degenerate and $d+3 \leq 2d+1$ if $d \geq 2$, G is also an equitably d-degenerate k-choosable (colorable)-critical graph in the class of $(2d+1)$-degenerate graphs with maximum degree at most Δ. Hence Lemma 3.6 with $d \geq 2$ applies. Specifically, the following facts hold.

($\mathscr{F}$1): $\delta(G) \geq d+2$ (by Lemma 3.6(1));
($\mathscr{F}$2): a $(d+2)$-vertex is adjacent only to $\Big((d+1)(k-1)+1\Big)^+$-vertices (by Lemma 3.6(2));
($\mathscr{F}$3): a $(d+3)$-vertex is adjacent only to $\Big((d+1)(k-2)+1\Big)^+$-vertices, or is adjacent to at least $d+2$ $\Big((d+1)(k-1)+1\Big)^+$-vertices (by Lemma 3.6(4));
($\mathscr{F}$4): a $\Big((d+1)(k-1)+1\Big)$-vertex is adjacent to at most one $(d+2)$-vertices, and furthermore, it is not adjacent to any $(d+3)$-vertex if it is adjacent to a $(d+2)$-vertex (by Lemma 3.6(3)).

To each vertex $v \in V(G)$, we assign an initial charge $c(v) = \deg(v) - (d+4)$, and thus $\sum_{v\in V(G)} c(v) = \sum_{v\in V(G)} \deg(v) - (d+4)|G| \leq \mathrm{mad}(G)\cdot|G| - (d+4)|G| < 0$. We proceed the discharging by the following rules.

($\mathscr{R}1$): Every $\Big((d+1)(k-1)+1\Big)^+$-vertex sends to each of its adjacent $(d+2)$-vertices

$$\alpha(d)=\begin{cases}\frac{2}{5} & \text{if } d\geq 3,\\ \frac{1}{2} & \text{if } d=2,\end{cases}$$

($\mathscr{R}2$): Every $\Big((d+1)(k-2)+1\Big)^+$-vertex sends to each of its adjacent $(d+3)$-vertices

$$\beta(d)=\begin{cases}\frac{1}{5} & \text{if } d\geq 3,\\ \frac{1}{4} & \text{if } d=2.\end{cases}$$

By $c'(v)$ we define the final charge of each vertex $v\in V(G)$. Our goal is to prove that $c'(v)\geq 0$ for each vertex $v\in V(G)$ and then obtain the contradiction: $0>\sum_{v\in V(G)}c(v)=\sum_{v\in V(G)}c'(v)\geq 0$.

By ($\mathscr{F}1$), there is no $(d+1)^-$-vertex in G.

If $\deg(v)=d+2$, then by ($\mathscr{F}2$) and ($\mathscr{R}1$),

$$\begin{aligned}c'(v)&\geq (d+2)-(d+4)+\alpha(d)(d+2)\\&=-2+\alpha(d)(d+2)\\&=\begin{cases}\frac{2d-6}{5}\geq 0 & \text{if } d\geq 3,\\ \frac{d-2}{2}=0 & \text{if } d=2.\end{cases}\end{aligned}$$

If $\deg(v)=d+3$, then by ($\mathscr{F}3$) and ($\mathscr{R}2$),

$$\begin{aligned}c'(v)&\geq (d+3)-(d+4)+\min\{\beta(d)(d+3),\alpha(d)(d+2)\}\\&=-1+\beta(d)(d+3)\\&=\begin{cases}\frac{d-2}{5}>0 & \text{if } d\geq 3,\\ \frac{d-1}{4}>0 & \text{if } d=2.\end{cases}\end{aligned}$$

If $\deg(v)\in[d+4,(d+1)(k-2)]$, then $c'(v)=c(v)\geq (d+4)-(d+4)=0$.

If $\deg(v)\in[(d+1)(k-2)+1,(d+1)(k-1)]$, then by ($\mathscr{R}2$),

$$\begin{aligned}c'(v)&\geq \deg(v)-(d+4)-\beta(d)\deg(v)\\&\geq\Big(1-\beta(d)\Big)\Big((d+1)(k-2)+1\Big)-(d+4)\\&=\Big((1-\beta(d))(k-2)-1\Big)d+\Big((1-\beta(d))(k-1)-4\Big)\\&\geq\begin{cases}\frac{3d-8}{5}>0 & \text{if } d\geq 3 \text{ and } k\geq 4,\\ \frac{5d-4}{4}>0 & \text{if } d=2 \text{ and } k\geq 5.\end{cases}\end{aligned}$$

If $\deg(v)=(d+1)(k-1)+1$, then by ($\mathscr{F}4$), ($\mathscr{R}1$), ($\mathscr{R}2$), and by the previous calculation,

$$\begin{aligned}c'(v)&\geq (d+1)(k-1)+1-(d+4)-\max\{\alpha(d),\beta(d)((d+1)(k-1)+1)\}\\&=(d+1)(k-1)+1-(d+4)-\beta(d)((d+1)(k-1)+1)\\&>(d+1)(k-2)+1-(d+4)-\beta(d)((d+1)(k-2)+1)\\&\geq 0\end{aligned}$$

If $\deg(v)\geq (d+1)(k-1)+2$, then by ($\mathscr{R}1$) and ($\mathscr{R}2$), and by the fact that $\alpha(d)\geq\beta(d)$,

$$\begin{aligned}c'(v)&\geq \deg(v)-(d+4)-\alpha(d)\deg(v)\\&\geq\Big(1-\alpha(d)\Big)\Big((d+1)(k-1)+2\Big)-(d+4)\\&=\Big((1-\alpha(d))(k-1)-1\Big)d+\Big((1-\alpha(d))(k+1)-4\Big)\end{aligned}$$

$$\geq \begin{cases} \frac{4d-5}{5} > 0 & \text{if } d \geq 3 \text{ and } k \geq 4, \\ d-1 > 0 & \text{if } d = 2 \text{ and } k \geq 5. \end{cases}$$

This ends the proof. □

5. Planar graphs

A drawing of a planar graph on the plane such that no edges cross each other is called a *plane graph*. This drawing divides the plane into a set of regions, called *faces*. The *degree of a face* in a plane graph is the number of edges on its boundary, where cut edges are counted twice. For a plane graph G, we use $F(G)$ to denote the set of faces of G, and a face of degree k, at least k, or at most k of G is a *k, k^+, or k^--face*, respectively. In the section, we always think a planar graph as a plane graph.

Note again that Conjecture 1.3(1) with $d = 0$ is exactly the Hajnal-Szemerédi Theorem [10]. The result of Esperet, Lemoine, and Maffray [9] states that every planar graph G is equitably 1-degenerate k-colorable for every integer $k \geq 4$, and thus Conjecture 1.3(1) with $d = 1$ holds for planar graphs G with $\Delta(G) \geq 6$. On the other hand, since Conjecture 1.1 holds for all 5-degenerate graphs (and thus naturally for all graphs G with $\Delta(G) \leq 5$) due to the result of Chen et al. [4], Conjecture 1.3(1) with $d = 1$ holds for planar graphs G with $\Delta(G) \leq 5$, and thus for all planar graphs. According to the result of Kim, Oum, and Zhang [12] that every planar graph is equitably 2-degenerate k-colorable for every $k \geq 3$, and is equitably d-degenerate k-colorable for every $d \geq 3$ and $k \geq 2$, we conclude that Conjecture 1.3(1) with $d = 2$ holds for planar graphs G with $\Delta(G) \geq 8$, and Conjecture 1.3(1) with $d \geq 3$ holds for planar graphs G with $\Delta(G) \geq d + 1$. Therefore, the colorability version of Conjecture 1.3 had already been verified for planar graphs with large maximum degree.

In this section, we confirm the choosability version of Conjecture 1.3 for planar graphs with large maximum degree.

Theorem 5.1. *Every planar graph G is equitably d-degenerate k-choosable for every integer*

$$k \geq max\left\{\left\lceil \frac{\Delta(G)+1}{d+1} \right\rceil, 6-d\right\},$$

where $d \geq 1$ is an integer.

Note the Theorem 5.1 is only interesting for $d \leq 4$, because a planar graph itself is 5-degenerate. Zhang [22] proved Theorem 5.1 with $d = 1$. So in what follows we complete the proofs of Theorem 5.1 with $d = 2, 3, 4$. For convenience, we use three independent theorems to show this fact.

Theorem 5.2. *Every planar graph G with $\Delta(G) \leq \Delta$ is equitably 2-degenerate k-choosable for every integer $k \geq max\left\{\left\lceil \frac{\Delta+1}{3} \right\rceil, 4\right\}$, where Δ is an integer.*

Proof. Suppose for a contradiction that there exists an equitably 2-degenerate k-choosable-critical graph G in the class of planar graphs with maximum degree at most Δ. Since planar graphs are 5-degenerate, G is also an equitably 2-degenerate k-choosable-critical graph in the class of 5-degenerate graphs with maximum degree at most Δ. Hence Lemma 3.6 with $d = 2$ applies. Specifically, the following facts hold.

($\mathscr{F}$1): $\delta(G) \geq 4$ (by Lemma 3.6(1));
($\mathscr{F}$2): If uvw is a triangle with $\deg(u) = 4$, then $\min\{\deg(v), \deg(w)\} \geq 11$ (by Lemma 3.6(5));
($\mathscr{F}$3): If uvw is a triangle with $\deg(u) = 5$, then either $\max\{\deg(v), \deg(w)\} \geq 11$ or $\min\{\deg(v), \deg(w)\} \geq 8$ (by Lemma 3.6(6)).

To each element $x \in V(G) \cup F(G)$, we assign an initial charge $c(x) = \deg(x) - 4$, and thus

$$\begin{aligned} \sum_{x \in V(G) \cup F(G)} c(x) &= \sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) \\ &= (2|E(G)| - 4|V(G)|) + (2|E(G)| - 4|F(G)|) \\ &= -4(|V(G)| + |F(G)| - |E(G)|) \\ &= -8, \end{aligned}$$

by Euler's formula. We use one discharging rule.

($\mathscr{R}$): Every 3-face receives $\frac{\deg(u)-4}{\deg(u)}$ from each of its incident vertices u with $\deg(u) \geq 5$.

By $c'(x)$ we denote the final charge of $x \in V(G) \cup F(G)$ after $(\mathscr{R})$ is applied. It is clear that G has no 3^--vertex by $(\mathscr{F}1)$ and $c'(x) \geq 0$ if x is a 4-vertex or a 4^+-face.

For any vertex u with $\deg(u) \geq 5$, $(\mathscr{R})$ implies $c'(u) \geq \deg(u) - 4 - \deg(u) \cdot \frac{\deg(u)-4}{\deg(u)} = 0$.

If f is a 3-face incident with three vertices u, v, and w with $\deg(u) \leq \deg(v) \leq \deg(w)$, then $\deg(u) \geq 4$ by $(\mathscr{F}1)$.

If $\deg(u) = 4$, then by $(\mathscr{F}2)$ and $(\mathscr{R})$, we have $c'(f) \geq 3 - 4 + 2 \cdot \frac{11-4}{11} > 0$.

If $\deg(u) = 5$, then by $(\mathscr{F}3)$, we have two subcases. If $\deg(v) \geq 8$, then $c'(f) \geq 3 - 4 + \frac{5-4}{5} + 2 \cdot \frac{8-4}{8} > 0$ by $(\mathscr{R})$. If $\deg(w) \geq 11$, then $c'(f) \geq 3 - 4 + 2 \cdot \frac{5-4}{5} + \frac{11-4}{11} > 0$ by $(\mathscr{R})$.

If $\deg(u) \geq 6$, then it is easy to conclude that $c'(f) \geq 3 - 4 + 3 \cdot \frac{6-4}{6} = 0$ by $(\mathscr{R})$.

Therefore, $c'(f) \geq 0$ for every 3-face f and thus $c'(x) \geq 0$ for any $x \in V(G) \cup F(G)$. This implies $\sum_{x \in V(G)\cup F(G)} c'(x) \geq 0$, contradicting the fact that $\sum_{x \in V(G)\cup F(G)} c'(x) = \sum_{x \in V(G)\cup F(G)} c(x) = -8$. □

Theorem 5.3. *Every planar graph G with $\Delta(G) \leq \Delta$ is equitably 3-degenerate k-choosable for every integer $k \geq \max\{\lceil \frac{\Delta+1}{4} \rceil, 3\}$, where Δ is an integer.*

Proof. Suppose for a contradiction that there exists an equitably 3-degenerate k-choosable-critical graph G in the class of planar graphs with maximum degree at most Δ. Since planar graphs are 5-degenerate (and thus 7-degenerate), G is also an equitably 3-degenerate k-choosable-critical graph in the class of 7-degenerate graphs with maximum degree at most Δ. Hence Lemma 3.6 with $d = 3$ applies. Specifically, the following facts hold.

$(\mathscr{F}$1): $\delta(G) \geq 5$ (by Lemma 3.6(1));
$(\mathscr{F}$2): If uvw is a triangle with $\deg(u) = 5$, then $\min\{\deg(v), \deg(w)\} \geq 10$ (by Lemma 3.6(5)).

To each element $x \in V(G) \cup F(G)$, we assign an initial charge

$$c(x) = \begin{cases} 2\deg(x) - 10 & \text{if } x \in V(G), \\ 3\deg(f) - 10 & \text{if } x \in F(G), \end{cases}$$

and thus $\sum_{x \in V(G)\cup F(G)} c(x) = -20$ by Euler's formula. We use one discharging rule.

$(\mathscr{R})$: Every 3-face receives $\frac{2\deg(u)-10}{\deg(u)}$ from each of its incident vertices u with $\deg(u) \geq 5$.

By $c'(x)$ we denote the final charge of $x \in V(G) \cup F(G)$ after $(\mathscr{R})$ is applied. Assume that f is a 3-face incident with three vertices u, v, and w with $\deg(u) \leq \deg(v) \leq \deg(w)$, then $\deg(u) \geq 5$ by $(\mathscr{F}1)$. If $\deg(u) = 5$, then by $(\mathscr{F}2)$ and $(\mathscr{R})$, we have $c'(f) \geq 3 \cdot 3 - 10 + 2 \cdot \frac{2 \cdot 10 - 10}{10} > 0$. If $\deg(u) \geq 6$, then it is clear that $c'(f) \geq 3 \cdot 3 - 10 + 3 \cdot \frac{2 \cdot 6 - 10}{6} = 0$ by $(\mathscr{R})$. Therefore, $c'(f) \geq 0$ if f is a 3-face. On the other hand, if f is a 4^+-face, then $c'(f) = c(f) = 3\deg(f) - 10 > 0$, and if v is a vertex, then v must be a 5^+-vertex by $(\mathscr{F}1)$ and thus $c'(v) \geq 2\deg(v) - 10 - \deg(v) \cdot \frac{2\deg(u)-10}{\deg(u)} = 0$ by $(\mathscr{R})$. Hence we have $0 \leq \sum_{x \in V(G)\cup F(G)} c'(x) = \sum_{x \in V(G)\cup F(G)} c(x) = -20$, a contradiction. □

Theorem 5.4. *Every planar graph G with $\Delta(G) \leq \Delta$ is equitably 4-degenerate k-choosable for every integer $k \geq \max\{\lceil \frac{\Delta+1}{5} \rceil, 2\}$, where Δ is an integer.*

Proof. The possible counterexample to this theorem will be an equitably 4-degenerate k-choosable-critical planar graph with maximum degree at most Δ, and thus it has minimum degree at least 6 by Lemma 3.6(1). This is impossible since every planar graph contains a vertex of degree at most 5. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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