Journal of the Operations Research Society of China (2021) 9:181–193
https://doi.org/10.1007/s40305-019-00243-2

Linear Arboricity of Outer-1-Planar Graphs

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Received: 23 May 2018 / Revised: 15 December 2018 / Accepted: 7 March 2019 / Published online: 8 April 2019

Abstract

A graph is outer-1-planar if it can be drawn in the plane so that all vertices are on the outer face and each edge is crossed at most once. Zhang et al. (Edge covering pseudo-outerplanar graphs with forests, Discrete Math 312:2788–2799, 2012; MR2945171) proved that the linear arboricity of every outer-1-planar graph with maximum degree Δ is exactly $\lceil \Delta/2 \rceil$ provided that $\Delta = 3$ or $\Delta \geq 5$ and claimed that there are outer-1-planar graphs with maximum degree $\Delta = 4$ and linear arboricity $\lceil (\Delta+1)/2 \rceil = 3$. It is shown in this paper that the linear arboricity of every outer-1-planar graph with maximum degree 4 is exactly 2 provided that it admits an outer-1-planar drawing with crossing distance at least 1 and crossing width at least 2, and moreover, none of the above constraints on the crossing distance and crossing width can be removed. Besides, a polynomial-time algorithm for constructing a path-2-coloring (i.e., an edge 2-coloring such that each color class induces a linear forest, a disjoint union of paths) of such an outer-1-planar drawing is given.

Keywords Outer-1-planar graph · Crossing · Linear arboricity · Polynomial-time algorithm

Mathematics Subject Classification 05C10 · 05C15

Xin Zhang was supported by the Fundamental Research Funds for the Central Universities (No. JB170706), the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2017JM1010), and the National Natural Science Foundation of China (Nos. 11871055 and 11301410). Bi Li was supported by the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2017JQ1031), and the National Natural Science Foundation of China (Nos. 11701440 and 11626181).

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1 Introduction and Definitions

In this paper, all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges that are incident with v in G. By $N_G(v)$, we denote the set of neighbors of v in G. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of G, respectively. The *distance* $\text{dist}_G(u, w)$ between two vertices u and w of a connected graph G is the minimum length of the path (i.e., the number of edges on the path) connecting them. For $U, W \subseteq V(G)$, $\text{dist}_G(U, W) = \min\{\text{dist}_G(u, w) \mid u \in U, w \in W\}$ denotes the distance between two vertex sets U and W. If $U = \{u\}$, we write $\text{dist}_G(u, W)$ instead of $\text{dist}_G(\{u\}, W)$. For undefined concepts, we refer the readers to [1].

A *linear forest* is a forest in which every connected component is a path. The *linear arboricity* $\text{la}(G)$ of G, introduced by Harary [2], is the minimum number of colors that can be used to color the edges of G so that each color class induces a linear forest of G. In 1980, Akiyama et al. [3] conjectured the following:

Conjecture 1.1 (Linear Arboricity Conjecture) *For any graph G,* $\lceil \frac{\Delta(G)}{2} \rceil \le \text{la}(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$.

Although Conjecture 1.1 has been proved to be true for all planar graphs [4,5], finding planar graphs G with

$$\text{la}(G) = \lceil \Delta(G)/2 \rceil \tag{1.1}$$

is stlll interesting. Wu [4] proved (1.1) for planar graphs with maximum degree at least 13, and this bound 13 was later improved to 9 by Cygan et al. [6]. Wu [7] also proved (1.1) for all series–parallel graphs (hence also for all outerplanar graphs) with maximum degree at least 3.

From the view of the computational complexity perspective, Peroche [8] claimed that determining whether a given graph has linear arboricity k for a given integer k is, however, NP-complete, even for graphs with maximum degree 4. For planar graphs, Cygan et al. [6] conjectured that it is NP-complete to determine whether a given planar graph with maximum degree 4 has linear arboricity 2.

A graph is *outer*-1-*planar* if it can be drawn in the plane so that all vertices are on the outer face and each edge crosses at most one other edge. Outer-1-planar graphs were first introduced by Eggleton [9] who called them *outerplanar graphs with edge crossing number one* and were also investigated under the notion of *pseudo-outerplanar graphs* by Zhang et al. [10,11]. Actually, every outer-1-planar graph is planar. This fact was released in [10] without detailed proof, and a formal proof was given by Auer et al. [12].

A drawing of an outer-1-planar graph in the plane such that its outer-1-planarity is satisfied is an *outer*-1-*plane graph* or *outer*-1-*planar drawing*.

Let G be an outer-1-plane graph. The *associated plane graph* $G^\times$ of G is the plane graph obtained from G by turning all crossings into new vertices of degree four. If u is a crossing (not a real vertex) in G, then we define $\mathcal{N}_G(u)$ to be $N_{G^\times}(u)$, called the *cluster* of u in G.

Let $\mathfrak{C}(G)$ be the set of crossings in an outer-1-plane graph G. The *crossing distance* of an outer-1-plane graph G is defined by

$$CD(G) = \begin{cases} \min\limits_{u,v\in\mathfrak{C}(G),u\neq v} \mathrm{dist}_G\big(\mathcal{N}_G(u), \mathcal{N}_G(v)\big), & \text{if } |\mathfrak{C}(G)| \geq 2; \\ +\infty, & \text{if } |\mathfrak{C}(G)| \leq 1. \end{cases}$$

For a vertex u in G, its *crossing distance* is defined by

$$CD_G(u) = \begin{cases} \min\limits_{v\in\mathfrak{C}(G)} \mathrm{dist}_G\big(u, \mathcal{N}_G(v)\big), & \text{if } |\mathfrak{C}(G)| \geq 1; \\ +\infty, & \text{if } |\mathfrak{C}(G)| = 0. \end{cases}$$

The *crossing width* of an outer-1-plane graph G is defined by

$$CW(G) = \begin{cases} \min\limits_{u\in\mathfrak{C}(G)} \max\limits_{v,w\in\mathcal{N}_G(u)} d_G(v, w), & \text{if } |\mathfrak{C}(G)| \geq 1; \\ +\infty, & \text{if } |\mathfrak{C}(G)| = 0. \end{cases}$$

It has been shown recently by Dehkordi and Eades [13] that every outer-1-planar graph has a right angle crossing drawing that preserves its outer-1-planarity. Auer et al. [14] confirmed that the recognition of outer-1-planarity can process in linear time, and the same result was also independently obtained by Hong et al. [15].

The partition problems on the outer-1-plane graphs were also investigated in the literatures. For example, Zhang et al. [10] showed that each outer-1-plane graph admits edge decompositions into a linear forest and an outerplane graph, or a star forest and an outerplane graph, or two forests and a matching, or $\max\{\Delta(G), 4\}$ matchings, or $\max\{\lceil\Delta(G)/2\rceil, 3\}$ linear forests if $\Delta(G) \geq 4$ and two linear forests if $\Delta(G) \leq 3$. From the last result of the above, we conclude the following

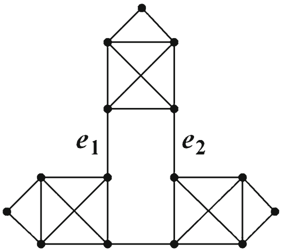
Observation 1.2 *If G is an outer-1-plane graph, then $\mathrm{la}(G) = \lceil\Delta(G)/2\rceil$ provided that $\Delta(G) = 3$ or $\Delta(G) \geq 5$, and $\mathrm{la}(G) \leq 3$ while $\Delta(G) = 4$.*

For an outer-1-plane graph G with $\Delta(G) = 4$, it may have $\mathrm{la}(G) = 3$. Graphs (a) and (b) in Fig. 1 are such examples (this fact can be easily checked if one has noticed that the two edges e_1 and e_2 must be in different colors while only two colors are available). One can see that graph (a) in Fig. 1 has crossing distance 1 and crossing width 1, while graph (b) in Fig. 1 has crossing distance 0 and crossing width 2.

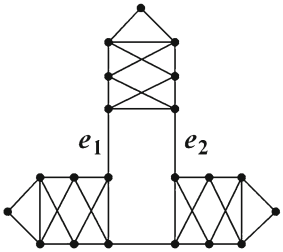
Actually, if we forbid graph G to be with crossing distance at least 1 and crossing width at least 2, we can avoid all exceptions. In other words, we have the following

Theorem 1.3 *If G is an outer-1-plane graph with $\Delta(G) = 4$, $CD(G) \geq 1$ and $CW(G) \geq 2$, then $\mathrm{la}(G) = 2$.*

Theorem 1.3 is best possible in the sense that there exist graphs G with $\mathrm{la}(G) = 3$ and with $CD(G) \geq 1$, $CW(G) = 1$, or $CD(G) = 0$, $CW(G) \geq 2$ (see Fig. 1 for examples).



(a) $CD(G)$=1, $CW(G)$=1



(b) $CD(G)$=0, $CW(G)$=2

Fig. 1 Outer-1-planar graphs with maximum degree 4 and linear arboricity 3

2 Useful Lemmas

In this section, we release some lemmas on the structures of an outer-1-plane graph G. First, we assume that G is 2-connected. By $v_1, \cdots, v_{|G|}$, we denote the vertices of G with clockwise ordering on the boundary of its outer-1-planar drawing. Here $|G|$ is the order of G, i.e., the number of vertices in G.

Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \cdots, v_j\}$ (listing in a clockwise order) and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j]\backslash\{v_i, v_j\}$, where $i \neq j$ and the subscripts are taken modulo $|G|$. By $G[v_i, v_j]$ and $G(v_i, v_j)$, we denote the subgraph of G induced by $\mathcal{V}[v_i, v_j]$ and $\mathcal{V}(v_i, v_j)$, respectively.

A vertex set $\mathcal{V}[v_i, v_j]$ with $i \neq j$ is a *non-edge* if $j = i + 1$ and $v_i v_j \notin E(G)$ and is a *path* if $v_k v_{k+1} \in E(G)$ for all $i \leq k < j$. An edge $v_i v_j$ in G is a *chord* if $j - i \neq 1$ or $1 - |G|$. By $C[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$.

Lemma 2.1 [10, Claim 1] *Let v_a and v_b be vertices of a 2-connected outer-1-plane graph G. If there is no crossed chord in $C[v_a, v_b]$ and no edge between $\mathcal{V}(v_a, v_b)$ and $\mathcal{V}(v_b, v_a)$, then $\mathcal{V}[v_a, v_b]$ is either a non-edge or a path.*

In any figure of this section, the degree of a solid (or hollow) vertex is exactly (or at least) the number of edges that are incident with it, respectively, and a solid vertex is distinct to every another vertex but two hollow vertices may be identified unless stated otherwise.

In what follows, when mentioning the configuration G_i with $1 \leq i \leq 8$ we always refer to the corresponding picture in Fig. 2.

Saying that an outer-1-plane graph G *contains* G_i with $1 \leq i \leq 8$, we mean that G contains a subgraph isomorphic to G_i such that the degree in G of any solid (resp. hollow) vertex in that picture is exactly (resp. at least) the number of edges that are incident with it. Specially, saying G contains G_i with $3 \leq i \leq 8$, we also mean that the corresponding picture is a partial drawing of G such that all marked vertices are consecutive on the outer boundary of G.

Let v_a and v_b be two vertices on the outer boundary of an outer-1-plane graph G. Saying $G[v_a, v_b]$ *properly contains* G_i for some i, we mean that $G[v_a, v_b]$ contains G_i such that v_a and v_b do not correspond to the solid vertices or hollow vertices with degree restrictions in the picture of G_i. Note that if $a = 1$ and $b = |G|$, then $G[v_a, v_b]$

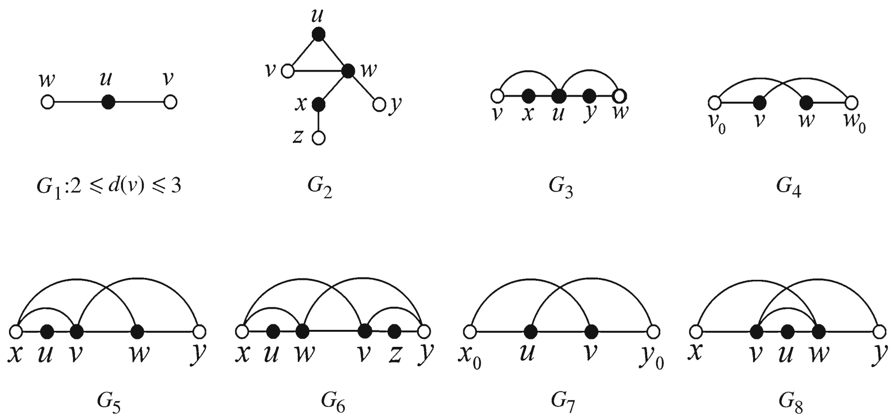


Fig. 2 Local structures in outer-1-planar graph with $\Delta(G) = 4$, $CD(G) \geq 1$ and $CW(G) \geq 2$

is clearly the graph G. However, by the definition of the proper containment, we cannot say that G properly contains G_i, but saying that $G[v_1, v_{|G|}]$ properly contains G_i is permitted. Actually, the proper containment plays an important role when we generate results from the 2-connected case to the connected case. One can see this in the proof of Lemma 2.5.

Lemma 2.2 *Let $\mathcal{V}[v_a, v_b]$ with $b - a \geq 3$ (i.e. $|\mathcal{V}[v_a, v_b]| \geq 4$) be a path in a 2-connected outer-1-plane graph G. If $\Delta(G) \leq 4$ and there is no crossed chord in $\mathcal{C}[v_a, v_b]$ and no edge between $\mathcal{V}(v_a, v_b)$ and $\mathcal{V}(v_b, v_a)$, then $G[v_a, v_b]$ properly contains G_1 or G_3.*

Proof If $\mathcal{C}[v_a, v_b] \setminus \{v_a v_b\} = \varnothing$ (note that the chord $v_a v_b$ may not really exist), then $d(v_{a+1}) = d(v_{a+2}) = 2$ and G_1 is properly contained. If there is at least one chord in $\mathcal{C}[v_a, v_b] \setminus \{v_a v_b\}$, then choose one, say $v_{a'} v_{b'}$ with $a' < b'$, so that there is no other chord in $\mathcal{C}[v_{a'}, v_{b'}]$. If $b' - a' \geq 3$, then $d(v_{a'+1}) = d(v_{a'+2}) = 2$ and G_1 is properly contained. If $b' - a' = 2$, then $d(v_{a'+1}) = 2$. Choose $t \in \{a', b'\}$ such that $v_t \neq v_a, v_b$. If $d(v_t) \leq 3$, then G_1 is properly contained. If $d(v_t) = 4$, then there is another one chord $v_t v_{c'}$ with $c' \neq a', b'$. If $|c' - t| = 2$, then $d(v_{t-1}) = d(v_{t+1}) = 2$, and thus G_3 is properly contained. If $|c' - t| \geq 3$, then let $a := \min\{c', t\}$, $b := \max\{c', t\}$ and come back to the first line of this proof. Since $t' \neq a, b$, $|c' - t| < |b - a|$, which implies that this iterative process will terminate.

Actually, the proof of Lemma 2.2 can be rewritten into the following Algorithm 1. It takes at most $O(|E|)$ time to run each of Steps 2, 4, 5, $O(|V|)$ time to run each of Steps 10, 12, 13, and $O(1)$ time to run Steps 3, 6–9, 11, 14–17. Hence at most $O(|E|)$ time is needed for one iteration, and the running time of Finding-Outerplanar-Configuration is at most $O(|V| \cdot |E|)$, since there occurs at most $|V|$ iterations.

Lemma 2.3 *Let $v_i v_j$ cross $v_k v_l$ in a 2-connected outer-1-plane graph G with $i < k < j < l$ so that there are no other crossed chords besides $v_i v_j$ and $v_k v_l$ in $\mathcal{C}[v_i, v_l]$. Suppose that $\Delta(G) \leq 4$.*

Algorithm 1: FINDING-OUTERPLANAR-CONFIGURATION(v_a, v_b)

Input: Vertices v_a and v_b of a 2-connected outer-1-plane graph $G = (V, E)$ so that conditions in Lemma 2.2 are satisfied;
Output: Special configuration that $G[v_a, v_b]$ properly contained;

Step 1 **while** $b - a \geq 3$ **do**
Step 2   **if** $C[v_a, v_b] \setminus \{v_a v_b\} = \varnothing$ **then**
Step 3   Find two adjacent vertices v_{a+1} and v_{a+2} of degree 2 in $G(v_a, v_b)$, and output G_1;
Step 4   **else**
Step 5   Choose a chord $v_{a'} v_{b'}$ with $a \leq a' < b' \leq b$ so that there is no other chord in $C[v_{a'}, v_{b'}]$;
Step 6   **if** $b' - a' \geq 3$ **then**
Step 7   Find two adjacent vertices $v_{a'+1}$ and $v_{a'+2}$ of degree 2 in $G(v_{a'}, v_{b'})$, and output G_1;
Step 8   **else**
Step 9   Find $v_t \in \{v_{a'}, v_{b'}\} \setminus \{v_a, v_b\}$;
Step 10   **if** $d(v_t) \leq 3$ **then**
Step 11   Find the configuration G_1 that is properly contained in $G[v_{t-1}, v_{b'}]$ (resp. $G[v_{a'}, v_{t+1}]$) if $t = a'$ (resp. $t = b'$), and output G_1;
Step 12   **else**
Step 13   Find a chord $v_t v_{c'}$ with $c' \neq a', b'$;
Step 14   **if** $|c' - t| \geq 3$ **then**
Step 15 $a \leftarrow \min\{c', t\}$;
Step 16 $b \leftarrow \max\{c', t\}$;
Step 17   **else**
Step 18   Find the configuration G_3 that is properly contained in $G[v_{t-2}, v_{t+2}]$, and output G_3.

(1) *If* $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \geq 4$, *then* $G[v_i, v_l]$ *properly contains* G_1 *or* G_3;
(2) *If* $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \leq 3$, *then* $G[v_i, v_l]$ *properly contains one of the configurations among* $G_1, G_2, G_4, G_5, G_6, G_7$ *and* G_8.

Proof (1) If $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \geq 4$, then assume, without loss of generality, that $|\mathcal{V}[v_i, v_k]| \geq 4$. By Lemma 2.1, $\mathcal{V}[v_i, v_k]$ is a path, and by Lemma 2.2, $G[v_i, v_k]$ properly contains G_1 or G_3.

(2) If $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \leq 3$, then we assume, among $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$, that at most one of them is a non-edge. Otherwise we have two cases by symmetry. If $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_k, v_j]$ are non-edges, then $d_G(v_k) = 1$ contradicting the 2-connectivity of G. If $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_j, v_l]$ are non-edges, then $\mathcal{V}[v_k, v_j]$ is not a non-edge for otherwise $d_G(v_k) = 1$, which again contradicts the 2-connectivity of G. By Lemma 2.1, $\mathcal{V}[v_k, v_j]$ is a path. If $|\mathcal{V}[v_k, v_j]| = 2$, then v_k and v_j are two adjacent vertices of degree two in G, and if $|\mathcal{V}[v_k, v_j]| = 3$, then $d_G(v_k) \leq 3$ and $d_G(v_{k+1}) = 2$. In any case, G_1 is properly contained in $G[v_i, v_l]$. Hence, by Lemma 2.1, among $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$, at least two of them are paths. By symmetry, we consider two cases.

Case 1 $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_k, v_j]$ are paths.

Subcase 1.1 $|\mathcal{V}[v_i, v_k]| = 2$ (implying $v_i v_k \in E(G)$).

If $|\mathcal{V}[v_k, v_j]| = 2$, then $v_k v_j \in E(G)$ and $d_G(v_k) = 3$, which implies that $\mathcal{V}[v_j, v_l]$ is a path, because otherwise it is a non-edge by Lemma 2.1 and thus $d_G(v_j) = 2$, which implies that G_1 is properly contained. If $|\mathcal{V}[v_j, v_l]| = 2$, then $v_j v_l \in E(G)$ and G_7 is properly contained. If $|\mathcal{V}[v_j, v_l]| = 3$, then $G[v_i, v_l]$ properly contains G_1 if $v_j v_l \notin E(G)$, and G_5 otherwise.

Therefore we assume that $|\mathcal{V}[v_k, v_j]| = 3$. If $v_k v_j \notin E(G)$, then $d_G(v_k) = 3$ and $d_G(v_{k+1}) = 2$, and thus G_1 is properly contained. If $v_k v_j \in E(G)$, then $\mathcal{V}[v_j, v_l]$ is a path, because otherwise it is a non-edge by Lemma 2.1 and thus $d_G(v_j) = 3$, which implies that G_1 is properly contained. Since $\Delta(G) \leq 4$, $v_j v_l \notin E(G)$. Hence $G[v_i, v_l]$ properly contains G_2 if $|\mathcal{V}[v_j, v_l]| = 3$ (note that $d_G(v_{j+1}) = 2$), and G_8 otherwise.

Subcase 1.2 $|\mathcal{V}[v_i, v_k]| = 3$ (implying $d_G(v_{k-1}) = 2$).

If $|\mathcal{V}[v_k, v_j]| = 2$, then $v_k v_j \in E(G)$. If $v_i v_k \notin E(G)$, then $d_G(v_k) = 3$ and thus G_1 is properly contained. Hence we assume that $v_i v_k \in E(G)$. If $\mathcal{V}[v_j, v_l]$ is a non-edge, then $d_G(v_j) = 2$ and G_2 is properly contained. If $\mathcal{V}[v_j, v_l]$ is not a non-edge, then it is a path by Lemma 2.1. If $|\mathcal{V}[v_j, v_l]| = 2$, then G_5 is properly contained. If $|\mathcal{V}[v_j, v_l]| = 3$, then $v_j v_l \in E(G)$, because otherwise $d_G(v_j) = 3$ and $d_G(v_{j+1}) = 2$, and this G_1 is properly contained. At this stage, one can easily see that G_6 is properly contained in $G[v_i, v_l]$.

Therefore we assume that $|\mathcal{V}[v_k, v_j]| = 3$. This implies that $d_G(v_{k+1}) = 2$. If $v_i v_k \in E(G)$ or $v_k v_j \in E(G)$, then G_2 is properly contained. If $v_i v_k \notin E(G)$ and $v_k v_j \notin E(G)$, then v_k has degree 3 and thus G_1 is properly contained.

Case 2 $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_j, v_l]$ are paths.

Here we assume that $\mathcal{V}[v_k, v_j]$ is not a path (because otherwise we come back to Case 1). By Lemma 2.1, $\mathcal{V}[v_k, v_j]$ is a non-edge. If $|\mathcal{V}[v_i, v_k]| = 3$ (resp. $|\mathcal{V}[v_j, v_l]| = 3$), then $d_G(v_{k-1}) = 2$ (resp. $d_G(v_{j+1}) = 2$) and $d_G(v_k) \leq 3$ (resp. $d_G(v_j) \leq 3$), which implies that G_1 is properly contained. If $|\mathcal{V}[v_i, v_k]| = |\mathcal{V}[v_j, v_l]| = 2$, then one can easily see that G_4 is properly contained in $G[v_i, v_l]$.

Lemma 2.4 *Let G be a 2-connected outer-1-plane graph with $\Delta(G) \leq 4$. We clockwise label the vertices of G on its outer boundary with $v_1, v_2, \cdots, v_n$, where $n = |G|$. If $n \geq 4$, then $G[v_1, v_n]$ properly contains one of the configurations among $G_1, G_2, \cdots, G_8$.*

Proof If there is no crossing in G, then $\mathcal{V}[v_1, v_n]$ forms a path since G is 2-connected. Under this condition, one can easily show, by Lemma 2.2, that $G[v_1, v_n]$ properly contains G_1 or G_3, since $n \geq 4$. Hence in the following we assume that there is at least one crossing in G. Choose two crossed chords $v_i v_j$ and $v_k v_l$ in G such that (1) $1 \leq i < k < j < l \leq n$, and (2) $l - i$ is minimum. By (2), there are no crossed chords in $C[v_i, v_j]$ besides $v_i v_j$ and $v_k v_l$. Therefore, Lemma 2.3 implies the required conclusion.

Lemma 2.5 *If G is a connected outer-1-plane graph with $2 \leq \delta(G) \leq \Delta(G) \leq 4$, then G contains one of the configurations among $G_1, G_2, \cdots, G_8$.*

Proof If G is 2-connected, then set $H := G$, otherwise set H to be an end-block of G, i.e., a 2-connected component of G that contains only one cut-vertex of G. Let $v_1, v_2, \cdots, v_m$ be vertices lying clockwise on the outer boundary of H so that v_1 is a cut-vertex of G if G is not 2-connected. If $|H| = 2$ (note that $|H| \geq 2$), then one can easily find a vertex of degree 1 in G, contradicting the fact that $\delta(G) \geq 2$. If $|H| = 3$, then by the 2-connectivity of H, $H \cong K_3$, which implies the appearances of two adjacent vertices of degree 2 in G, and then the containment of G_1. If $|H| \geq 4$, then by Lemma 2.4, $H[v_1, v_m]$ properly contains one of the configurations among $G_1, G_2, \cdots, G_8$. Recall the definition of the proper containment, we can immediately conclude that G contains one of those configurations.

Let Ω be the class of all outer-1-plane graphs G so that $\Delta(G) \leq 4$, $\mathcal{CD}(G) \geq 1$ and $\mathcal{CW}(G) \geq 2$. Clearly, $G' \in \Omega$ if G' is a subgraph of G. The following lemma shows that the class Ω is closed under some special graph operations. In the following, $G - S$, where S is a subset of $V(G)$, stands for the graph obtained from G by removing all vertices in S. Specially, if $S = \{v\}$, we write $G - v$ instead of $G - \{v\}$ for convenience. $G + E$, where E is a set of edges, stands for the graph derived form G by adding edges in E that do not exist in G. Again, we do not distinguish $G + uv$ with $G + \{uv\}$.

Lemma 2.6 *If $G \in \Omega$ contains the configuration G_4 or G_7 or G_8, then $G - v + v_0w_0 \in \Omega$ or $G - v + x_0y_0 \in \Omega$ or $G - \{v, w\} + \{ux, uy, xy\} \in \Omega$, respectively.*

Proof Suppose that G contains G_4 and let $G' = G - v + v_0w_0$. It is easy to see that G' is an outer-1-plane graph with $\Delta(G') \leq 4$. Since $\mathcal{CD}(G) \geq 1$, we have $\mathcal{CD}_{G'}(v_0) \geq 1$ and $\mathcal{CD}_{G'}(w_0) \geq 1$, which implies that $\mathcal{CD}(G') \geq 1$. If $\mathcal{CW}(G') = 1$, then there exists a crossing u in G' (and thus in G) so that $v_0, w_0 \in \mathcal{N}_{G'}(u) = \mathcal{N}_G(u)$. Since u is a crossing in G that is deferent from the one, denoted by u', appearing in the picture of G_4, and $\mathcal{N}_G(u) \cap \mathcal{N}_G(u') = \{v_0, w_0\}$, we have $\mathcal{CD}(G) = 0$, a contradiction. This implies that $\mathcal{CW}(G') \geq 2$, and thus $G' \in \Omega$. Another two cases can be similarly proved.

To end this section, we combine the proofs of Lemmas 2.2–2.5 into Algorithm 2, and claim that for any graph $G \in \Omega$, there is a polynomial-time algorithm to find one of the configurations among $G_0, G_1, G_2, \cdots, G_8$, where the configuration G_0 refers to an edge uv with $d(u) = 1$.

In Algorithm 2, it takes at most $O(|V|)$ time to run each of the Steps 1–4, $O(|E|^2)$ time to run each of the Steps 5,10,11, $O(|V| \cdot |E|)$ time to run Step 9 or Step 14 by Algorithm 1, and $O(1)$ time to run each of the Steps 6–8,12,13,15 and 16. Hence the running time of FINDING-OUTER-1-PLANAR-CONFIGURATION is at most $O(|E|^2)$.

3 The Proof of Theorem 1.3

A *path-t-coloring* of G is a function c from $E(G)$ to $\{1, 2, \cdots, t\}$ so that the graph induced by $c^{-1}(i)$ is a linear forest for any $i \in \{1, 2, \cdots, t\}$. Therefore, the linear arboricity $\mathrm{la}(G)$ of G is actually the minimum integer t so that G admits a path-t-coloring.

Algorithm 2: FINDING-OUTER-1-PLANAR-CONFIGURATION(G)

Input: A connected outer-1-plane graph $G = (V, E)$ with $|E| \geq 3$, $\Delta(G) \leq 4$, $CD(G) \geq 1$ and $CW(G) \geq 2$;
Output: Special configuration that G contained;
Step 1 **if** $\delta(G) \leq 1$ **then**
Step 2 Find a 1-valent vertex in G and output G_0;
Step 3 **else**
Step 4 Find an end-block H of G with vertices $v_1, \cdots, v_{|H|}$ lying clockwise on the outer boundary of H and v_1 being a cut-vertex of G;
Step 5 **if** *there is no crossing in* H **then**
Step 6 **if** $|H| = 3$ **then**
Step 7 Find two adjacent 2-valent vertices v_2 and v_3, and output G_1;
Step 8 **else**
Step 9 FINDING-OUTERPLANAR-CONFIGURATION($v_1, v_{|H|}$);
Step 10 **else**
Step 11 Choose a pair of crossed chords $v_i v_j$ and $v_k v_l$ with $1 \leq i < k < j < l$ so that there is no other crossed chords besides $v_i v_j$ and $v_k v_l$ in $C[v_i, v_l]$;
Step 12 **if** $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \geq 4$ **then**
Step 13 Choose one, say $|\mathcal{V}[v_i, v_k]|$, from them so that $|\mathcal{V}[v_i, v_k]| \geq 4$;
Step 14 FINDING-OUTERPLANAR-CONFIGURATION(v_i, v_k);
Step 15 **else**
Step 16 Find one of the configurations among $G_1, G_2, G_4, G_5, G_6, G_7$ and G_8 that is properly contained in $H[v_i, v_l]$, and output this configuration.

In what follows, we prove that every outer-1-plane graph $G \in \Omega$ is path-2-colorable by the induction on $|E(G)|$. Clearly, this result holds for every outer-1-plane graph $G \in \Omega$ with $|E(G)| \leq 1$. Following the induction procedure, we assume it holds for every outer-1-plane graph $G' \in \Omega$ with $|E(G')| < |E(G)|$ when an outer-1-plane graph $G \in \Omega$ is considered.

By Lemma 2.5, for every outer-1-plane graph $G \in \Omega$, either it is disconnected, or $\delta(G) = 1$, or it contains one of the configurations among $G_1, G_2, \cdots, G_8$ as a subgraph.

Claim 3.1 *If G is disconnected or $\delta(G) = 1$, then G is path-2-colorable.*

Proof If G is disconnected, then every component of G belongs to Ω, and thus has a path-2-coloring by the induction hypothesis. This implies that G is path-2-colorable.

If G has a vertex u of degree one, then $G - uv \in \Omega$, where v is the neighbor of u in G. By the induction hypothesis, $G - uv$ has a path-2-coloring c, which can be extended to a path-2-coloring of G by assigning uv a color that has been used at most once on the edges incident with v in $G - uv$. Note that v has degree at most 3 in $G - uv$.

Claim 3.2 *If G contains G_1 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_1 as a subgraph, then $G - uv \in \Omega$, thus by the induction hypothesis, it has a path-2-coloring c. If $N_G(v) = \{u, v_1, v_2\}$ and $c(vv_1) = c(vv_2)$, then color uv with a color different from $c(vv_1)$, otherwise color uv with a color different from $c(uw)$. In any case we get a path-2-coloring of G.

Claim 3.3 *If G contains G_2 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_2 as a subgraph, then $G' = G - uw$ has a path-2-coloring c by the induction hypothesis, since $G' \in \Omega$.

If $c(wy) = c(wx) = 2$, then color uw with 1. If the resulting coloring of G is not a path-2-coloring, then $c(uv) = 1$, in which case we just need exchange the colors on uw and wx.

If $c(wy) = 1$ and $c(wx) = 2$, then we consider two subcases.

If $c(vw) = 1$, then color uw with 2. If the resulting coloring of G is not a path-2-coloring, then $c(uv) = c(xz) = 2$, in which case we exchange the colors on uv and vw, and recolor wx with 1.

If $c(vw) = 2$, then color uw with 1. If the resulting coloring of G is not a path-2-coloring, then $c(uv) = 1$ and there is a path initialing from w and ending with v such that all its incident edges are colored with 1 under the coloring c of G'. This implies that there cannot exist a path initialing from w and ending with x such that all its incident edges are colored with 1 under the coloring c of G'. Hence we just need exchange the colors on uw and wx so as to get a path-2-coloring of G.

Claim 3.4 *If G contains G_3 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_3 as a subgraph, then by the induction hypothesis, $G' = G - xv$ has a path-2-coloring c, since $G' \in \Omega$. By Claim 3.2, we shall assume that $d_G(v) = d_G(w) = 4$. Let v_1, v_2 and w_1, w_2 be other two neighbors of v and w in G, respectively.

If $c(vv_1) = c(vv_2) = 1$, then color xv with 2. The resulting coloring is a path-2-coloring of G unless $c(xu) = 2$. If this special case occurs, then we exchange the colors on xu and uy, and a path-2-coloring of G is constructed.

If $c(vv_1) = 1$ and $c(vv_2) = 2$ (without loss of generality, we also assume that $c(uv) = 1$), then color xv with 2. The resulting coloring is a path-2-coloring of G unless $c(xu) = 2$ and $\{c(ww_1), c(ww_2)\} = \{1, 2\}$. If $c(uw) = 2$, then exchange the colors on uv and uw, and on xv and yw. If $c(uw) = 1$, then exchange the colors on uv and uy, and recolor xv with 1. In any case, we obtain a path-2-coloring of G.

Claim 3.5 *If G contains G_4 as a subgraph, then G is path-2-colorable.*

Proof If $v_0w_0 \notin E(G)$, then the graph G' obtained from G by deleting v and adding a new edge v_0w_0 belongs to Ω by Lemma 2.6. By the induction hypothesis, G' has a path-2-coloring c. Restrict c to G and color vv_0 and vw_0 with $c(v_0w_0)$. We get a path-2-coloring of G.

On the other hand, if $v_0w_0 \in E(G)$, then $G' = G - v \in \Omega$, and thus, by the induction hypothesis, G' has a path-2-coloring c. Assign vv_0 and vw_0 a color that has been used at most once on the edges incident with v_0 and w_0 in G', respectively. The resulting coloring of G, still denoted by c, is a path-2-coloring unless $c(vv_0) = c(vw_0) = c(v_0w_0)$ or $c(vv_0) = c(vw_0) = c(wv_0) = c(ww_0)$. If the former case occurs, then exchange the colors on vv_0 and wv_0. If the latter case occurs, then recolor vv_0 and ww_0 with $c(v_0w_0)$, and v_0w_0 with $c(vv_0)$. In any case we obtain a path-2-coloring of G.

Claim 3.6 *If G contains G_5 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_5 as a subgraph, then delete u, v and w from G and denote the resulting graph by G'. Clearly, $G' \in \Omega$, thus by the induction hypothesis, G' has a path-2-coloring c.

By Claim 3.2, only the case that $d_G(x) = 4$ and $d_G(y) \geq 3$ shall be considered. Without loss of generality, assume that $d_G(y) = 4$ (note that the case when $d_G(y) = 3$ can be dealt with much more easily). Let x_1 be the fourth neighbor of x and let y_1, y_2 be the remaining two neighbors of y in G. Suppose that $c(xx_1) = 1$. If $c(yy_1) = c(yy_2) = 1$, then color xw, uv, vw with 1 and xu, xv, vy, wy with 2. If $c(yy_1) = c(yy_2) = 2$, then color xw, uv, vy, wy with 1 and xu, xv, vw with 2. If $c(yy_1) = 1$ and $c(yy_2) = 2$, then color xw, uv, vy with 1 and xu, xv, vw, wy with 2. In any case we obtain a path-2-coloring of G.

Claim 3.7 *If G contains G_6 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_6 as a subgraph, then delete u, v, w and z from G and denote the resulting graph by G'. Clearly, $G' \in \Omega$, and thus by the induction hypothesis, G' has a path-2-coloring c.

By Claim 3.2, we can assume that $d_G(x) = d_G(y) = 4$. Let x_1 and y_1 be the fourth neighbor of x and y in G, respectively. Suppose that $c(xx_1) = 1$. If $c(yy_1) = 1$, then color xw, uw, vy, vz with 1 and xu, xv, wv, wy, yz with 2. If $c(yy_1) = 2$, then color xw, wv, vy, yz with 1 and xu, xv, uw, wy, vz with 2. In any case we obtain a path-2-coloring of G.

Claim 3.8 *If G contains G_7 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_7 as a subgraph, then $x_0y_0 \notin E(G)$, since $\mathcal{CW}(G) \geq 2$. Construct a graph G' from G by deleting the vertex v and adding an edge x_0y_0. By Lemma 2.6, $G' \in \Omega$.

By the induction hypothesis, G' admits a path-2-coloring c. Here we only analyze the case that $d_G(x_0) = d_G(y_0) = 4$, since the case that $\min\{d_G(x_0), d_G(y_0)\} = 3$ is easier. Note that Claim 3.2 implies $\min\{d_G(x_0), d_G(y_0)\} \geq 3$, because otherwise we are done.

Let x_1, x_2 and y_1, y_2 be the remaining two neighbors of x_0 and y_0 in G, respectively.

If $c(x_0x_1) = c(x_0x_2) = 1$, then $\{c(y_0y_1), c(y_0y_2)\} = \{1, 2\}$. We construct a path-2-coloring of G by restricting c to G and coloring vx_0, vy_0 with 2 and uv with 1.

If $\{c(x_0x_1), c(x_0x_2)\} = \{c(y_0y_1), c(y_0y_2)\} = \{1, 2\}$, then assume, without loss of generality, that $c(x_0y_0) = 1$ and $c(ux_0) = c(uy_0) = 2$. Note that there does not exist in $G' - x_0y_0$ a path initialing from x_0 and ending with y_0 such that all its incident edges are colored with 1 under the coloring c. We construct a path-2-coloring of G by restricting c to G, coloring vx_0, uv with 1 and vy_0 with 2, and recoloring uy_0 with 1.

Claim 3.9 *If G contains G_8 as a subgraph, then G is path-2-colorable.*

Proof If G contains G_8 as a subgraph, then $xy \notin E(G)$, because otherwise $\mathcal{CW}(G) = 1$, a contradiction. Remove v, w from G and add edges ux, uy and xy. Denote by G' the resulting graph. Lemma 2.6 implies that $G' \in \Omega$.

By the induction hypothesis, G' has a path-2-coloring c. By Claim 3.3, only the case that $\min\{d_G(x), d_G(y)\} \geq 3$ shall be considered. Assume, without loss of generality, that $d_G(x) = d_G(y) = 4$ (the case that $\min\{d_G(x), d_G(y)\} = 3$ can be considered much more easily). Let x_1, x_2 and y_1, y_2 be another two neighbors of x and y in G, respectively.

If $c(xx_1) = c(xx_2) = 1$, then $\{c(yy_1), c(yy_2)\} = \{1, 2\}$. We extend c to G by coloring vy, vw and uw with 1, and vx, wx, wy and uv with 2.

If $\{c(xx_1), c(xx_2)\} = \{c(yy_1), c(yy_2)\} = \{1, 2\}$, then there does not exist in $G' - u$ a path initialing from x and ending with y such that all its incident edges are colored with 1 or 2 under the coloring c. Hence we can extend c to G by coloring vx, vw and wy with 1, and vy, wx, uv and uw with 2.

In each case we obtain a path-2-coloring of G.

4 Conclusions

In Sect. 3, we have proved that if G is an outer-1-plane graph with $\Delta(G) = 4$, $CD(G) \geq 1$ and $CW(G) \geq 2$, then $\mathrm{la}(G) \leq 2$. Actually, for such an outer-1-plane graph G there is a polynomial-time algorithm, according to the proofs in Sect. 3, to construct a path-2-coloring.

First of all, we consider the connected case. In Algorithm 3, Steps 1–3 and 6–9 run in $O(1)$ time, and Step 5 runs in $O(|E|^2)$ time by Algorithm 2. Therefore, after $O(|E|^3)$ time we can come to Step 10, which just runs in $O(1)$ time. Since Step 12 runs in $O(1)$ time by the proofs of Claims 3.1–3.9, we shall use another $O(|E|)$ time to complete the algorithm. Hence the running time of PATH-COLORING-CONNECTED-CASE is $O(|E|^3)$.

Algorithm 3: PATH-COLORING-CONNECTED-CASE(G)

Input: A connected outer-1-plane graph $G = (V, E)$ with $\Delta(G) = 4$, $CD(G) \geq 1$ and $CW(G) \geq 2$;
Output: A path-2-coloring of G;

Step 1 $i \leftarrow 1$;
Step 2 $E_i \leftarrow \varnothing$;
Step 3 $H_0 \leftarrow G$;
Step 4 **while** $|E(G)| > 2$ **do**
Step 5 FINDING-OUTER-1-PLANAR-CONFIGURATION(G);
Step 6 According to the found configuration that G contained, construct a subgraph $H_i \in \Omega$ as in the proof of the corresponding claim in Sect. 3;
Step 7 $E_i \leftarrow E(G) \setminus E(H_i)$;
Step 8 $G \leftarrow H_i$;
Step 9 $i \leftarrow i + 1$;
Step 10 Find a path-2-coloring c_0 of G;
Step 11 **for** $j = 1$ *to* $i - 1$ **do**
Step 12 Find a path-2-coloring c_j of H_{i-j-1} by coloring E_{i-j} according to c_{j-1};
Step 13 Output c_{i-1}.

For the general case, even if G has many component $\Psi_i = (V_i, E_i)$ with $i = 1, 2, \cdots t$, we can independently apply Algorithm 3 to each of Ψ_i. Hence after $O(|E_1|^3) + O(|E_2|^3) + \cdots + O(|E_t|^3) = O(|E|^3) = O(|V|^3)$ time (note that $|E| \leq \frac{5}{2}|V| - 4$ for every outer-1-planar graph $G = (V, E)$, see [12, Theorem 3]), we can obtain a path-2-coloring of G.

In conclusion, there is a polynomial-time algorithm to construct a path-2-coloring of an outer-1-plane graph G with $\Delta(G) = 4$, $\mathcal{CD}(G) \geq 1$ and $\mathcal{CW}(G) \geq 2$.

Note that in this paper we just show the existence of a polynomial-time algorithm for the problem we investigate but do not optimize the time complexity. Maybe there is a faster algorithm for this problem.

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