



Hardness and algorithms of equitable tree-coloring problem in chordal graphs [☆]



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ARTICLE INFO

Article history:

Received 20 October 2020

Received in revised form 23 November 2020

Accepted 7 December 2020

Available online 16 December 2020

Keywords:

$W[1]$ -hardness

Linear-time algorithm

Approximation algorithm

Equitable tree-coloring

Chordal graphs

ABSTRACT

An equitable tree- k -coloring of a graph is a vertex k -coloring such that each color class induces a forest and the size of any two color classes differs by at most one. In this work, we show that every interval graph G has an equitable tree- k -coloring for any integer $k \geq \lceil (\Delta(G) + 1)/2 \rceil$, solving a conjecture of Wu, Zhang and Li (2013) for interval graphs, and furthermore, give a linear-time algorithm for determining whether a proper interval graph admits an equitable tree- k -coloring for a given integer k . For disjoint union of split graphs, or $K_{1,r}$ -free interval graphs with $r \geq 4$, we prove that it is $W[1]$ -hard to decide whether there is an equitable tree- k -coloring when parameterized by number of colors, or by treewidth, number of colors and maximum degree, respectively. On the positive side, we propose a quadratic 2-approximation algorithm for the equitable tree-coloring problem in chordal graphs. Moreover, it is proved that there is no α -approximation algorithm for any $\alpha < \frac{3}{2}$.

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1. Introduction

A minimization model in graph theory so-called the equitable tree-coloring can be used to formulate a structure decomposition problem on the communication network with some security considerations [2,3]. Namely, an *equitable tree-coloring* of a (finite, simple and undirected) graph G is a mapping $c : V(G) \rightarrow \{1, 2, \dots, k\}$ so that $c^{-1}(i)$ induces a forest for each $1 \leq i \leq k$, and $||c^{-1}(i)| - |c^{-1}(j)|| \leq 1$ for each pair of $1 \leq i < j \leq k$. The notion of the equitable tree- k -coloring was introduced by Wu, Zhang and Li [4], who conjectured that every graph G has an equitable tree- k -coloring for any integer $k \geq \lceil (\Delta(G) + 1)/2 \rceil$. This conjecture (equitable vertex arboricity conjecture, EVAC for short) is known to have an affirmative answer in some cases including:

- G is complete or bipartite [4];
- $\Delta(G) \geq (|G| - 1)/2$ [5,6];
- $\Delta(G) \leq 3$ [7];
- G is 5-degenerate [8];
- G is d -degenerate with $\Delta(G) \geq 10d$ [2];
- G is IC-planar with $\Delta(G) \geq 14$ or $g(G) \geq 6$ [9];

[☆] Supported by the National Natural Science Foundation of China (11871055, 11701440) and the Youth Talent Support Plan of Xi'an Association for Science and Technology (2018-6). The conference version of the paper is shown in [1].

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- G is a d -dimensional grid with $d \in \{2, 3, 4\}$ [10].

EVAC is still widely open.

Algorithmically, the following EQUITABLE TREE COLORING is NP-complete [11].

EQUITABLE TREE COLORING

Instance: A graph G and the number of colors k .

Question: Is there an equitable tree- k -coloring of G ?

Correspondingly, the optimization version of EQUITABLE TREE COLORING is to find the minimum k such that there is an equitable tree- k -coloring of a given graph G . Recently in [12], the last two authors proved that EQUITABLE TREE COLORING problem is W[1]-hard when parameterized by treewidth, and that it is polynomial solvable in the class of graphs with bounded treewidth, and in the class of graphs of bounded vertex cover number.

This paper focuses on chordal graphs. A graph G is *chordal* if every cycle in G of length > 3 possesses a chord: namely, an edge joining nonconsecutive vertices on the cycle. An enumeration v_1, v_2, \dots, v_n of the vertices of a graph G is a *perfect elimination ordering* if for every $1 \leq i \leq n$, $N_G(v_i) \cap \{v_1, v_2, \dots, v_i\}$ induces a clique in G . A graph G is a chordal graph if and only if it has a simplicial order on $V(G)$, which can be found in linear time [13,14]. In particular, the class of *interval graphs* is a subclass of chordal graphs. A graph G is an *interval graph* if there exist an *interval representation* of G , i.e., a family $\{T_v | v \in V(G)\}$ of intervals on the real line such that u and v are adjacent vertices in G if and only if $T_u \cap T_v \neq \emptyset$. For any vertex $v \in V(G)$, $L(v)$ and $R(v)$ denote the left point and the right point of its corresponding interval T_v , respectively. For two vertices $u, v \in V(G)$, if $L(u) < L(v)$, or $L(u)=L(v)$ and $R(u) \leq R(v)$, then we write $u < v$. For any three vertices $u, v, w \in V(G)$, it is clear that

$$\text{if } u < v < w \text{ and } uw \in E(G), \text{ then } uv \in E(G). \tag{1}$$

Olariu [15] shows that a graph is an interval graph if and only if it has a *linear order* $<$ on $V(G)$ satisfying (1); and that the order can be found in linear time. Using this fact, we give the following result as a quick start of this paper, confirming EVAC for interval graphs.

Theorem 1. *Every interval graph G has an equitable tree- k -coloring for any integer $k \geq \lceil \frac{\Delta(G)+1}{2} \rceil$, where the lower bound of k is sharp.*

Proof. Sort the vertices of G into $v_0 < v_1 < \dots < v_{n-1}$ so that (1) holds, and for each $0 \leq i \leq n-1$, let $c(v_i) = i \pmod k$. It is clear that c is an equitable k -coloring of G . If there is a monochromatic cycle in color i , then there are three vertices $v_{i+\alpha k}, v_{i+\beta k}$ and $v_{i+\gamma k}$ with $0 \leq \alpha < \beta < \gamma$ such that $v_{i+\alpha k}v_{i+\beta k}, v_{i+\alpha k}v_{i+\gamma k} \in E(G)$. By (1), $v_{i+\alpha k}v_j \in E(G)$ for any $i + \alpha k < j \leq i + \gamma k$, which implies $d_G(v_{i+\alpha k}) \geq (\gamma - \alpha)k \geq 2k \geq \Delta(G) + 1$, a contradiction. Hence, there is no monochromatic cycle under c . This implies that c is an equitable tree- k -coloring of G . Consider the complete graph K_{2s} . It is an interval graph and it does not admit an equitable tree- k -coloring for any $k \leq s - 1$, because in any k -coloring of K_{2s} , there are three monochromatic vertices, which induces a cycle in it. So the lower bound of k in this result is sharp. \square

On the other hand, if we are given an integer $k < \lceil \frac{\Delta(G)+1}{2} \rceil$, determining whether an interval graph, which is also a chordal graph, admits an equitable tree- k -coloring is not easy. Precisely, the next two theorems will be proved in Section 2.

Theorem 2. *EQUITABLE TREE COLORING of the disjoint union of split graphs parameterized by number of colors is W[1]-hard.*

Theorem 3. *EQUITABLE TREE COLORING of $K_{1,r}$ -free interval graph with $r \geq 4$ parameterized by treewidth, number of colors and maximum degree is W[1]-hard.*

Here, a *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. Note that any vertex in the clique may be adjacent to any vertex in the independent set. A *$K_{1,r}$ -free graph* is a graph that does not contain the star $K_{1,r}$ as an induced subgraph.

However, the hardness situation is much better if we are working with an *proper interval graph*, that is an interval graph that has an interval representation in which no interval properly contains any other interval. Actually, we have the following theorem, which will be proved in Section 3.

Theorem 4. *There is a linear-time algorithm to determine whether a proper interval graph admits an equitable tree- k -coloring for a given integer k .*

For chordal graphs, based on a perfect elimination ordering, we could design efficient approximation algorithm for the problem.

Theorem 5. *There is a 2-approximation algorithm for the EQUITABLE TREE COLORING of chordal graphs.*

Our algorithm is not bad according to the following unapproximable result:

Theorem 6. *There is no α -approximation algorithm for EQUITABLE TREE COLORING of interval graphs for any $\alpha < \frac{3}{2}$, unless P=NP.*

To end this section, we collect some notations that will be used in the next sections. For any two graphs G and H , their sum $G \oplus H$ is the graph given by $V(G \oplus H) = V(G) \cup V(H)$ and $E(G \oplus H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$, and their union $G \cup H$ is the graph given by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. By nG , we denote the n disjoint copies of G , and $[k]$ stands for $\{1, 2, \dots, k\}$. Other undefined notations follow [16].

2. W[1]-hardness: the proofs of Theorems 2 and 3

All of our reductions involve the following BIN-PACKING problem, which is NP-hard in the strong sense [17], and is W[1]-hard when parameterized by the number of bins [18,19].

BIN-PACKING
Instance: A set of n items $A = \{a_1, a_2, \dots, a_n\}$ and a bin capacity B .
Parameter: The number of bins k .
Question: Is there a k -partition φ of A such that, $\forall i \in [k], \sum_{a_j \in \varphi_i} a_j = B$?

Proof of Theorem 2. Given an instance of BIN-PACKING as above, our strategy is to construct a disjoint union of split graph G such that the answer of the BIN-PACKING is YES if and only if G admits an equitable tree- k -coloring. Here,

$$G = \bigcup_{j \in [n]} H(a_j, k),$$

where a_1, a_2, \dots, a_n are arbitrarily given integers in the instance of BIN PACKING, and

$$H(a, k) = K_{2k-1} \oplus (a + 1)K_1$$

defines a split graph for integers a and k . For $j \in [n]$, let I_j be the independent set of size $a_j + 1$ in $H(a_j, k)$, and let c_j be a fixed vertex in the clique part K_{2k-1} of $H(a_j, k)$.

Suppose that there is a k -partition φ of $A = \{a_1, a_2, \dots, a_n\}$ such that, $\forall i \in [k], \sum_{a_j \in \varphi_i} a_j = B$. For any $i \in [k]$ and for any j satisfying $a_j \in \varphi_i$, color c_j and all vertices in I_j with color i . For any $j \in [n]$ such that $a_j \in \varphi_i$, color the $2k - 2$ vertices in $V(H(a_j, k)) \setminus (I_j \cup \{c_j\})$ with $k - 1$ distinct colors in $[k] \setminus i$ so that each color is used exactly twice. At this moment, for any $j \in [n]$, $H(a_j, k)$ has been colored with k colors so that the set of all vertices with color i (here i is the integer such that $a_j \in \varphi_i$) induces a forest, which is actually a star with center c_j , and each of another $k - 1$ colors besides i is used for exactly two vertices. Hence this gives a tree- k -coloring of $H(a_j, k)$ for any $j \in [n]$, and thus finally gives a tree- k -coloring of G . To see that this coloring is an equitable tree- k -coloring of G , we denote the set of vertices with color i as V_i for any color $i \in [k]$. Clearly,

$$|V_i| = \sum_{a_j \in \varphi_i} (a_j + 2) + \sum_{a_j \notin \varphi_i} 2 = \sum_{a_j \in \varphi_i} a_j + 2n = B + 2n$$

for any $i \in [k]$, which implies that such a coloring is equitable.

On the other direction, if G admits an equitable tree- k -coloring ψ , then in the clique part K_{2k-1} of each $H(a_j, k)$ with $j \in [n]$, there is a color i appearing on exactly one vertex, and each of another $k - 1$ colors appears on exactly two vertices. It follows that all vertices in I_j of $H(a_j, k)$ are colored with i , since any other color classes contain two vertices in the clique part K_{2k-1} , which are adjacent to all vertices in I_j . Therefore, taking any one vertex $v_j \in I_j$ together with the clique part K_{2k-1} induces a clique K_{2k} containing exactly two vertices in each color class. Let ψ_i be the vertices of G colored with i under the coloring ψ . We show that $\sum_{I_j \subseteq \psi_i} a_j = B$, which indicates that the answer for the BIN-PACKING is YES.

Since there are

$$n(2k - 1) + \sum_{j \in [n]} (a_j + 1) = k(2n + B)$$

vertices in G (note that in BIN-PACKING we always assume that $\sum_{j \in [n]} a_j = kB$), each color class of ψ contains exactly $2n + B$ vertices, which consists of, for each $j \in [n]$, two vertices in the clique K_{2k} of $H(a_j, k)$ as chosen above and a_j vertices in $I_j \setminus \{v_j\}$ if $I_j \subseteq \psi_i$. So

$$2n + B = |\psi_i| = 2n + \sum_{I_j \subseteq \psi_i} a_j,$$

which gives $\sum_{I_j \subseteq \psi_i} a_j = B$.

Lemma 7. [20,21] *A graph is an interval graph if and only if its maximal cliques can be ordered as M_1, M_2, \dots, M_k such that for any $v \in M_i \cap M_k$ with $i < k$, it holds that $v \in M_j$ for any $i \leq j \leq k$.*

Proof of Theorem 3. We prove the theorem for parameter number of colors; and the theorem for another two parameters can be proved in the same way. Given an instance of the BIN-PACKING, our strategy is to construct a $K_{1,r}$ -free interval graph G with $r \geq 4$ such that the answer of the BIN-PACKING is YES if and only if G admits an equitable tree- k -coloring. Here,

$$G = \bigcup_{j \in [n]} J(a_j, k)$$

with a_1, a_2, \dots, a_n being arbitrarily given integers in the instance of BIN PACKING, and

$$J(a, k) = \left(\bigcup_{i \in [a]} (Q_i \oplus y_i) \right) \cup \left(\bigcup_{i \in [a]} (Q'_i \oplus y_i) \right) \cup \left(\bigcup_{i \in [a-1]} (Q_{i+1} \oplus y_i) \right),$$

where $S = \{Q_1, Q'_1, \dots, Q_a, Q'_a\}$ is a set of cliques such that $Q_i \simeq Q'_i \simeq K_{2k-1}$ and $Y = \{y_1, \dots, y_a\}$ is a set of vertices. Note that the vertices of G with largest degree are the ones contained in $Y \setminus \{y_a\}$, which have degrees equal to $3(2k - 1)$, and the treewidth of G is $2k - 1$.

We claim first that G is an interval graph. Indeed, it is sufficient to show that $J(a, k)$ is an interval graph for any positive integer a . By the definition of $J(a, k)$, one can see that it has $3a - 1$ maximal cliques $M_1, M_2, \dots, M_{3a-1}$ such that

$$M_i = \begin{cases} Q_i \oplus y_i & \text{if } i \equiv 1 \pmod{3} \\ Q'_i \oplus y_i & \text{if } i \equiv 2 \pmod{3} \\ Q_{i+1} \oplus y_i & \text{if } i \equiv 0 \pmod{3}. \end{cases}$$

Since $M_i \cap M_j = \emptyset$ for any $i \equiv 1 \pmod{3}$ and $j \geq i + 3$, $M_i \cap M_j = \emptyset$ for any $i \not\equiv 1 \pmod{3}$ and $j \geq i + 2$, and $M_i \cap M_{i+1} = M_i \cap M_{i+2} = M_{i+1} \cap M_{i+2} = \{y_i\}$ for any $1 \leq i \leq 3a - 5$ with $i \equiv 1 \pmod{3}$, the ordering $M_1, M_2, \dots, M_{3a-1}$ satisfies the property described by Lemma 7, and therefore, $J(a, k)$ is an interval graph.

Suppose that there is a k -partition φ of A such that, $\forall i \in [k], \sum_{a_j \in \varphi_i} a_j = B$. For any $i \in [k]$ and for any j satisfying $a_j \in \varphi_i$, color all vertices of Y_j , corresponding to the vertex set Y in $J(a_j, k)$, with color i . For any $j \in [n]$ such that $a_j \in \varphi_i$, color each $(k - 1)$ -clique of S_j , corresponding to the set S of cliques in $J(a_j, k)$, so that the color i is used for exactly one vertex, and each of the remaining $k - 1$ colors in $[k] \setminus i$ are used for exactly two vertices. Clearly, this gives a tree- k -coloring of $J(a_j, k)$, where $3a_j$ vertices consisting of Y_j and one vertex in each clique in S_j , are colored with color i , and each of another $k - 1$ colors is used for exactly two vertices in each clique of S_j . Hence a tree- k -coloring of G is given now. To see that this is an equitable tree- k -coloring of G , we denote the set of vertices with color i as V_i for any color $i \in [k]$. The fact that

$$|V_i| = \sum_{a_j \in \varphi_i} 3a_j + \sum_{a_j \notin \varphi_i} 4a_j = 3B + 4(kB - B) = (4k - 1)B$$

for any $i \in [k]$ implies the equability of this coloring.

On the other direction, if G admits an equitable tree- k -coloring ψ , then in each clique of S_j for $j \in [n]$, there is a color appearing on exactly one vertex, and each of another $k - 1$ colors appears on exactly two vertices. Suppose that the color i appears on exactly one vertex of the first clique $Q_1 \in S_j$ for some $j \in [n]$. It follows that y_1 should be colored with i because y_1 is adjacent to every vertices of Q_1 and each color in $[k] \setminus i$ already appears on two vertices of Q_1 . Consequently, the color being used exactly once for the vertices of the second clique $Q'_1 \in S_j$ and the third clique $Q_2 \in S_j$ is indeed i , which implies that y_2 shall be colored with i . Following this process, we can conclude that each vertex of Y_j is colored with i , and in each clique of S_j , the color i appears on exactly one vertex, and each of another $k - 1$ colors in $[k] \setminus i$ appears on exactly two vertices. Let ψ_i be the vertices of G colored with i under the coloring ψ . We show that $\sum_{Y_j \subseteq \psi_i} a_j = B$, which indicates that the answer for the BIN-PACKING is YES.

Since there are

$$\sum_{j \in [n]} (a_j + 2a_j(2k - 1)) = kB + (4k - 2)kB = k(4k - 1)B$$

vertices in G , for each $i \in [k]$,

$$|\psi_i| = (4k - 1)B,$$

and by the way of the coloring ψ as described above, we also see that

$$|\psi_i| = \sum_{Y_j \subseteq \psi_i} 3a_j + \sum_{Y_j \cap \psi_i = \emptyset} 4a_j = \sum_{j \in [n]} 4a_j - \sum_{Y_j \subseteq \psi_i} a_j = 4kB - \sum_{Y_j \subseteq \psi_i} a_j.$$

Combining the two expression gives $\sum_{Y_j \subseteq \psi_i} a_j = B$.

3. Linear algorithm for proper interval graphs: the proof of Theorem 4

Lemma 8. Let G be a proper interval graph with $V(G) = \{v_1, v_2, \dots, v_n\}$, where $v_1 < v_2 < \dots < v_n$. If $v_i v_j \in E(G)$, then $\{v_i, v_{i+1}, \dots, v_{j-1}, v_j\}$ induces a clique of size $j - i + 1$.

Proof. By the definition of the proper interval graph, for any $i \leq s < \ell \leq j$, $L(v_s) < L(v_\ell)$, and if $v_i v_j \in E(G)$, then $L(v_i) \leq L(v_s) < L(v_\ell) \leq L(v_j) \leq R(v_i) \leq R(v_s) < R(v_\ell) \leq R(v_j)$. This implies that the interval $[L(v_s), R(v_s)]$ intersects the interval $[L(v_\ell), R(v_\ell)]$ and thus $v_s v_\ell \in E(G)$. Hence any two vertices among $\{v_i, v_{i+1}, \dots, v_{j-1}, v_j\}$ are adjacent and such a vertex set induces a clique. \square

Lemma 9. A proper interval graph has an equitable tree- k -coloring if and only if its maximum clique has size at most $2k$.

Proof. If c is an equitable tree- k -coloring of G , then there is no clique on at least $2k + 1$ vertices, because otherwise there is a color appearing at least three times on this clique, implying the existence of a monochromatic triangle, a contradiction. Hence the maximum clique of G has size at most $2k$.

On the other direction, if the maximum clique of G has size at most $2k$, then sort the vertices of G into v_0, v_1, \dots, v_{n-1} so that $v_i < v_j$ if $i < j$. Let $c(v_i) = i \pmod k$. It is clear that c is an equitable k -coloring of G . If there is a monochromatic cycle under c , then there are two adjacent vertices v_i and v_j with $j - i = \beta k$ with $\beta \geq 2$. By Lemma 8, G contains a clique of size $j - i + 1 = \beta k + 1 \geq 2k + 1$ as a subgraph, a contradiction. This implies that there is no monochromatic cycle under c , and thus c is an equitable tree- k -coloring of G . \square

Algorithm 1: Linear-time Algorithm for proper interval graphs.

Input: A proper interval graph $G = (V, E)$ on n vertices; a set of k colors $\{0, 1, \dots, k - 1\}$;

Output: *Ans*;

- 1 *Ans* \leftarrow YES;
 - 2 Sort the vertices of G into $v_0 < v_1 < \dots < v_{n-1}$, where $<$ is a linear order on $V(G)$;
 - 3 **for** $i = 0$ to $n - 1$ **do**
 - 4 Color vertex v_i with the color $c(i) = i \pmod k$;
 - 5 **if** there is a monochromatic cycle in any color class **then**
 - 6 Output NO;
 - 7 Output *Ans*.
-

Theorem 10. Given a proper interval graph G and an integer $k > 0$, Algorithm 1 outputs YES in linear time if and only if there exists an equitable tree- k -coloring of G ; moreover, if YES, it gives an equitable tree- k -coloring of G .

Proof. From Algorithm 1, one sees that the given coloring is an equitable tree- k -coloring of G if it outputs YES. If there is an equitable tree- k -coloring of G , then the size of the maximum clique of G is at most $2k$ by Lemma 9. In any iterative step of Algorithm 1, every color class induces disjoint unions of paths, because if not, there are two adjacent vertices $v_s v_\ell$ with $\ell - s \geq 2k$, which gives a clique of size $2k + 1$ by Lemma 8, a contradiction. So the algorithm outputs YES. The time complexity dominates by Line 2, which takes $O(|V| + |E|)$ time, see [15, Theorem 6]. \square

Proof of Theorem 4. This is an immediate corollary from Theorem 10.

Remark. Lemma 9 implies that EQUITABLE TREE COLORING of proper interval graphs is equivalent to determine whether $2k$ is the upper bound of its clique number. We know that to calculate all maximal cliques of a chordal graph $G = (V, E)$ can be done in $O(|V| + |E|)$ time [22, Theorem 4.17]. This also proves Theorem 4, however, without giving an equitable tree- k -coloring of G if the algorithm outputs YES. Moreover, it also confirms that any proper interval graph is equitable tree- $\lceil \frac{\omega(G)}{2} \rceil$ -colorable, where $\omega(G)$ is the size of a maximum clique of G .

4. Approximation algorithm for chordal graphs: the proofs of Theorems 5 and 6

Given a chordal graph $G = (V, E)$, let v_1, v_2, \dots, v_n be a perfect elimination ordering of G , i.e. the neighbors of v_i in $\{v_1, v_2, \dots, v_i\}$ for $1 \leq i \leq n$ induce a clique in G . A k -coloring of the vertices of a graph G is called *acyclic* if any color class is an independent set and every subgraph induced by any two color classes is acyclic, i.e., a forest. The relation of acyclic coloring and equitable tree-coloring has been shown in [23]: *any acyclic k -coloring of a graph can be modified into an equitable tree- $(k - 1)$ -coloring for any integer $k \geq 2$* . For the integrality and the convenience of the readers, we will prove it from an algorithmic point of view later. In the following Algorithm 2, we find an acyclic $\omega(G)$ -coloring of G in lines 3 – 10 based on a perfect elimination ordering of G ; and then change this coloring to an equitable tree- $(\omega(G) - 1)$ -coloring in lines 11 – 20.

Algorithm 2: 2-approximation algorithm for chordal graphs.

Input: A chordal graph $G = (V, E)$ on n vertices;
Output: A coloring of G ;

- 1 Compute the size of a maximum clique ω of G ;
- 2 Find a perfect elimination ordering v_1, v_2, \dots, v_n of G ;
- 3 **for** $j = 1$ **to** ω **do**
- 4 $A_j \leftarrow \{v_j\}$;
- 5 **for** $i = \omega + 1$ **to** n **do**
- 6 $b \leftarrow 0$;
- 7 **for** $j = 1$ **to** ω **do**
- 8 **if** $b = 0$ and $N_G(v_i) \cap A_j = \emptyset$ **then**
- 9 $A_j \leftarrow A_j \cup v_i$;
- 10 $b \leftarrow 1$;
- 11 **for** $h = 1$ **to** $\omega - 1$ **do**
- 12 Find $A_{\min} \in \{A_1, A_2, \dots, A_\omega\}$ such that $A_{\min} \neq \emptyset$ and $|A_{\min}| = \min_{j=1,2,\dots,\omega} |A_j|$;
- 13 Find $A_{\max} \in \{A_1, A_2, \dots, A_\omega\}$ such that $A_{\max} \neq \emptyset$ and $|A_{\max}| = \max_{j=1,2,\dots,\omega} |A_j|$;
- 14 **if** $h \leq n \pmod{\omega - 1}$ **then**
- 15 Take any subset $S \subseteq A_{\max}$ such that $|S| = \lceil \frac{n}{\omega - 1} \rceil - |A_{\min}|$;
- 16 **else**
- 17 Take any subset $S \subseteq A_{\max}$ such that $|S| = \lfloor \frac{n}{\omega - 1} \rfloor - |A_{\min}|$;
- 18 $V_h \leftarrow A_{\min} \cup S$;
- 19 **for** $j = 1$ **to** ω **do**
- 20 $A_j \leftarrow A_j \setminus V_h$;
- 21 Output $V_1, V_2, \dots, V_{\omega-1}$, which give an equitable tree- $(\omega - 1)$ -coloring of G .

Lemma 11. Algorithm 2 outputs an equitable tree- $(\omega - 1)$ -coloring of G in $O(n^2)$ time.

Proof. First, let us prove that from Line 3 to 10, the obtained $A_1, A_2, \dots, A_\omega$ give an acyclic- ω coloring of G . If $n = \omega$, then $|A_j| = 1$ for each $j = 1, 2, \dots, \omega$. So $A_1, A_2, \dots, A_\omega$ give an acyclic- ω coloring of G . For any integer $N > \omega$, assume that it is true for $n \leq N - 1$. Now we prove it for $n = N$. From Line 5 to 10 one sees that $v_i \in A_j$ only if v_i has no neighbor in $\{v_1, v_2, \dots, v_{i-1}\} \cap A_j$ for some $j = 1, 2, \dots, \omega$. We claim that this A_j exists. Since the enumeration v_1, v_2, \dots, v_n is a perfect elimination ordering, v_i and its neighbors in $\{v_1, v_2, \dots, v_{i-1}\}$ induce a clique in G . So v_i has at most $\omega - 1$ neighbors in $\{v_1, v_2, \dots, v_{i-1}\}$. One sees that v_i is put in only one of the A_j s. So A_j induces an independent set after adding v_i . Moreover, any two neighbors u, u' of v_i in $\{v_1, v_2, \dots, v_{i-1}\}$ are also adjacent, so they are not in the same class of $A_1, A_2, \dots, A_\omega$. Then v_i cannot belong to any cycle of $A_j \cup A_{j'}$ for any $j' = 1, 2, \dots, \omega$. So $A_1, A_2, \dots, A_\omega$ give an acyclic- ω coloring of G .

Next, we prove that from Line 11 to 20, the obtained $V_1, V_2, \dots, V_{\omega-1}$ give an equitable tree- $(\omega - 1)$ coloring of G . For $h = 1$, one sees that from Line 12 (resp. 13), A_{\min} (resp. A_{\max}) is one of the smallest (resp. largest) class in the above $A_1, A_2, \dots, A_\omega$. So $|A_{\min}| \leq \frac{n}{\omega} < \frac{n}{\omega - 1}$. Since $\sum_{A_j \neq A_{\min}} (|A_{\min}| + |A_j|) = n + (\omega - 1)|A_{\min}|$, we have that $|A_{\min}| + |A_{\max}| \geq \frac{n + (\omega - 1)|A_{\min}|}{\omega - 1} > \frac{n}{\omega - 1}$. So there exists a subset $S \subseteq A_{\max}$ such that $|S| = \lceil \frac{n}{\omega - 1} \rceil - |A_{\min}|$ in Line 15. From $V_1 = A_{\min} \cup S$ in Line 18, we get that $|V_1| = \lceil \frac{n}{\omega - 1} \rceil$ and that V_1 induces a forest in G . Then V_1 is deleted in Line 20 and the algorithm continues recursively. One sees that each V_h for $h = 1, 2, \dots, \omega - 1$ satisfies that $|V_h| = \lceil \frac{n}{\omega - 1} \rceil$ or $\lfloor \frac{n}{\omega - 1} \rfloor$ and that V_h induces a forest in G . So $V_1, V_2, \dots, V_{\omega-1}$ give an equitable tree- $(\omega - 1)$ coloring of G .

Finally, the time complexity of the algorithm is dominated by Lines 5- 10, which takes $O(n^2)$ because of $\omega \leq n$. The lemma is proved. \square

Now we are ready to prove Theorem 5.

Proof of Theorem 5. From Lemma 11, we can always find an equitable tree- $(\omega - 1)$ -coloring for any chordal graph G in quadratic time, where ω is the size of a maximum clique of G . On the other side, for a clique of size ω , at least $\lceil \frac{\omega}{2} \rceil$ colors are needed to have a tree-coloring. So any equitable tree-coloring of G requires at least $\lceil \frac{\omega}{2} \rceil$ colors. This proves that Algorithm 2 is a 2-approximation algorithm for the EQUITABLE TREE COLORING problem in chordal graphs.

In the rest of the paper, we prove the unapproximability of the EQUITABLE TREE COLORING problem by reduction from the PARTITION PROBLEM, which is NP-complete [17].

PARTITION PROBLEM

Instance: A set of n integers $S = \{a_1, a_2, \dots, a_n\}$.

Question: Is there a subset $T \subseteq S$ such that $\sum_{a_j \in T} a_j = \sum_{a_j \notin T} a_j$?

Proof of Theorem 6. Given an instance of PARTITION PROBLEM as above, our strategy is to construct a disjoint union of split graphs G such that if there is an α -approximation algorithm for the EQUITABLE TREE COLORING problem, then the answer of the PARTITION PROBLEM can be decided in polynomial time. Here, the graph G is similar as the one in the proof of Theorem 2 for $k = 2$. Let

$$G = \bigcup_{j \in [n]} H(a_j, 2),$$

where a_1, a_2, \dots, a_n are arbitrarily given integers in the instance of the PARTITION PROBLEM, and

$$H(a, 2) = K_3 \oplus (a + 1)K_1$$

defines a split graph for integer a . From the proof of Theorem 2, one can prove that G is equitable tree-2-colorable if and only if the answer for the PARTITION PROBLEM is YES. Moreover, we see that G is always equitable tree-3-colorable. This is because there is an acyclic-4-coloring of G by coloring each K_3 with 3 colors and all K_1 s with the left color. So we get that if the answer for the PARTITION PROBLEM is YES, then G has an equitable tree-2-coloring; and otherwise, G has equitable tree-3-coloring. Let t be the minimum such that G has an equitable tree- t -coloring. Then $2 \leq t \leq 3$.

Suppose that there is an α -approximation algorithm \mathcal{A} for the EQUITABLE TREE COLORING problem. Then one finds an equitable tree- a -coloring of G by Algorithm \mathcal{A} . Then $2 \leq t \leq a \leq \alpha t < \frac{9}{2}$. So $a \in \{2, 3, 4\}$.

- If $a = 2$, then $t = 2$.
- If $a = 3$, then $t \geq \frac{a}{\alpha} > 2$. So $t = 3$
- If $a = 4$, then $t \geq \frac{a}{\alpha} > \frac{8}{3}$. So $t = 3$.

This shows that $a = 2$ if and only if $t = 2$, which is equivalent with that the answer for the PARTITION PROBLEM is YES. So we could solve the PARTITION PROBLEM in polynomial time. The theorem is proved.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The last author would like to acknowledge the supports provided by China Scholarship Council (CSC) under the grant number 201906965003 and by Institute for Basic Science (IBS, South Korea) during a visit of him to Discrete Mathematics Group, IBS.

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