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k-forested choosability of planar graphs and sparse graphs☆

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ABSTRACT

A proper vertex coloring of a simple graph G is k-forested if the subgraph induced by the vertices of any two color classes is a k-forest, i.e. a forest with maximum degree at most k. The 2-forested coloring is also known as linear coloring, which has been extensively studied in the literature. In this paper, we aim to extend the study of 2-forested coloring to general k-forested colorings for every $k \geq 3$ by combinatorial means. Precisely, we prove that for a fixed integer $k \geq 3$ and a planar graph G with maximum degree $\Delta(G) \geq \Delta$ and girth $g(G) \geq g$, if $(\Delta, g) \in \{(k+1, 10), (2k+1, 8), (4k+1, 7)\}$, then the k-forested chromatic number of G is actually $\lceil \frac{\Delta(G)}{k} \rceil + 1$. Moreover, we also prove that the k-forested chromatic number of a planar graph G with maximum degree $\Delta(G) \geq k+1$ and girth $g(G) \geq 8$ is $\lceil \frac{\Delta(G)}{k} \rceil + 1$ provided $k \geq 7$. In addition, we show that the k-forested chromatic number of an outerplanar graph G is at most $\lceil \frac{\Delta(G)}{k} \rceil + 2$ for every $k \geq 2$. In fact, all these results are proved for not only planar graphs but also for sparse graphs, i.e. graphs having a low maximum average degree $\text{mad}(G)$, and we actually prove a choosability version of these results.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $d_G(v)$ (or $d(v)$ for brevity), we denote the *degree* of a vertex v in G. Throughout this paper, a k-, k^+- and k^--vertex is a vertex of degree k, at least k and at most k, respectively. The *maximum average degree* of G is defined by $\text{mad}(G) = \max\{2|E(H)|/|V(H)|, H \subseteq G\}$. The *girth* $g(G)$ of a graph G is the length of its smallest cycle or $+\infty$ if G is a forest. Any undefined notation follows that of West [17].

In 1973, Grünbaum [11] introduced *acyclic coloring*, which is a proper vertex coloring such that the union of any two color classes forms a forest. In 1997, Hind et al. [12] introduced *k-frugal coloring*, which is a proper coloring such that no color appears more than k times in the neighborhood of any vertex. These two kinds of colorings have been extensively studied in many papers (see [1–3] for example) and attracted more and more attention since Coleman et al. [5,6] identified acyclic coloring as the model for computing Hessian matrices via a substitution method and Hind et al. [12,13] investigated k-frugal coloring as a tool toward improving results about the total chromatic number of a graph.

In 1998, Yuster [18] mixed the notions of acyclic coloring and 2-frugal coloring and first introduced the concept of *linear coloring*, which is a proper coloring such that the graph induced by the vertices of any two color classes is the union of vertex-disjoint paths. As far as we know, there are many papers concerning linear coloring in the literature, such as [8,15,16].

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Of course, we can also combine the idea of acyclic coloring and general k-frugal coloring together to define a new kind of coloring, namely, k-forested coloring. A proper vertex coloring of a simple graph is *k-forested* if the subgraph induced by the vertices of any two color classes is a *k-forest*, i.e. a forest with maximum degree at most k. Clearly, a k-forested coloring is actually an acyclic k-frugal coloring and a 2-forested coloring is actually a linear coloring. The k-forested chromatic number $\chi_k^a(G)$ of a graph G denotes the least number of colors needed in a k-forested coloring. Here note that the interesting case for k-forested chromatic number is when $k \leq \Delta(G) - 1$, since $\chi_k^a(G)$ is equal to the acyclic chromatic number of G, provided $k \geq \Delta(G)$. Yuster [18] proved that $\chi_2^a(G) \leq C_1\Delta(G)^{3/2}$ for some constant C_1 and for sufficiently large $\Delta(G)$ and constructed graphs such that $\chi_2^a(G) \geq C_2\Delta(G)^{3/2}$ for some constant C_2. Asymptotically, Kang and Müller [14] showed that $\chi_k^a(G) = O(\Delta(G)^{4/3})$ for $k \geq 3$ and respectively constructed graphs for which $\chi_3^a(G) = \Omega(\Delta(G)^{4/3})$ and $\chi_k^a(G) = \Omega(\frac{\Delta(G)^{4/3}}{(\log k)^{1/3}})$.

These concepts may be generalized to their list versions. Let $L : V(G) \mapsto 2^{\mathbb{N}}$ be a list assignment of colors to each vertex $v \in V(G)$. We say G is *k-forested L-colorable* if G has a k-forested coloring where each vertex v in G is colored from its own list $L(v)$ and say G is *k-forested t-choosable* or *k-forested list t-colorable* if G is k-forested L-colorable whenever $|L(v)| = t$ for every vertex $v \in V(G)$. The k-forested choosability $ch_k^a(G)$ of a graph G denotes the minimum integer t such that G is k-forested t-choosable. The 2-forested choosability is also known as linear choosability, which was introduced by Esperet et al. [9] in 2008 and has been sequentially studied by Cohen and Havet [4], and Cranston and Yu [7].

Now, please pay attention to a trivial fact that $ch_k^a(G) \geq \chi_k^a(G) \geq \lceil\frac{\Delta(G)}{k}\rceil + 1$ since each color can appear on at most k neighbors of a vertex with maximum degree, and this lower bound on $ch_k^a(G)$ and $\chi_k^a(G)$ can be reached by trees (highly sparse graphs indeed) with maximum degree $\Delta(G)$. So an interesting research would be to investigate which classes of graphs, especially planar sparse graphs, have k-forested chromatic number or k-forested choosability equal to or close to the minimum possible value $\lceil\frac{\Delta(G)}{k}\rceil + 1$. The following results are dedicated to this theme.

Theorem 1.1 (*See [15,16]*). *Every planar graph G has* $\chi_2^a(G) = \lceil\frac{\Delta(G)}{2}\rceil + 1$ *if there is a pair* $(\Delta, g) \in \{(3, 13), (5, 10), (7, 9), (9, 8), (13, 7)\}$ *such that G satisfies* $\Delta(G) \geq \Delta$ *and* $g(G) \geq g$.

Theorem 1.2 (*See [7,20]*). *Every planar graph G has* $ch_k^a(G) = \lceil\frac{\Delta(G)}{k}\rceil + 1$ *if* $k \geq 2$, $\Delta(G) \geq k + 1$ *and* $g(G) \geq 12$.

Theorem 1.3 (*See [19]*). *Every planar graph G has* $\chi_k^a(G) = \lceil\frac{\Delta(G)}{k}\rceil + 1$ *if* $k \geq 3$, $\Delta(G) \geq k + 1$ *and* $g(G) \geq 10$.

Note that Theorem 1.2 partially improves Theorem 1.1 and the restrictions on $\Delta(G)$ in Theorems 1.2 and 1.3 cannot be relaxed since $ch_k^a(G) = \chi_k^a(G) = 3$ if G is an odd cycle.

In this paper, we aim to extend Theorem 1.1 and improve Theorems 1.2 and 1.3 by proving the following result.

Theorem 1.4. *Every graph G with* $\Delta(G) \geq \Delta$ *and* $\mathrm{mad}(G) < m$ *has* $ch_k^a(G) = \lceil\frac{\Delta(G)}{k}\rceil + 1$ *if* $k \geq 3$ *and* $(\Delta, m) \in \{(k+1, \frac{5}{2}), (2k+1, \frac{8}{3}), (4k+1, \frac{14}{5})\}$, *or* $k \geq 7$ *and* $(\Delta, m) = (k+1, \frac{8}{3})$.

Since Euler's formula implies that a planar graph G has a bounded maximum average degree in terms of its girth: $\mathrm{mad}(G) < \frac{2g(G)}{g(G)-2}$, we immediately have the following corollary.

Corollary 1.5. *Every planar graph G with* $\Delta(G) \geq \Delta$ *and* $g(G) \geq g$ *has* $ch_k^a(G) = \lceil\frac{\Delta(G)}{k}\rceil + 1$ *if* $k \geq 3$ *and* $(\Delta, g) \in \{(k+1, 10), (2k+1, 8), (4k+1, 7)\}$, *or* $k \geq 7$ *and* $(\Delta, g) = (k+1, 8)$.

Note that the lower bounds on $\Delta(G)$ in Theorem 1.4 and Corollary 1.5 cannot be improved to a number less than $k + 1$ because of odd cycles.

Besides, in this paper, we also consider k-forested colorings and k-forested list colorings of *outerplanar graphs*, i.e. graphs that can be embedded in the plane such that all vertices lie on the outer face, and prove the following theorem.

Theorem 1.6. *Every outerplanar graph G has* $ch_k^a(G) \leq \lceil\frac{\Delta(G)}{k}\rceil + 2$ *if* $k \geq 2$ *and* $\Delta(G) \geq k + 1$.

During the proofs of Theorems 1.4 and 1.6 in the following sections, we shall involve a useful concept, namely, k-forested t-critical graph. We say a graph G is *k-forested t-critical* if G is not k-forested t-choosable but any proper subgraph of G is so. In Section 2, we first obtain some structural information about k-forested critical graphs and show that certain configurations cannot occur in k-forested critical graphs. Whereafter, we will prove Theorems 1.4 and 1.6 in Sections 3 and 4 respectively.

2. Forbidden configurations

Before proving the following Lemma 2.1, we shall be clear with the notions involved in it. Let G be a graph and let c be a k-forested coloring of G with colors set C. Then $c(u)$ denotes the color used for a vertex $u \in V(G)$ and $C_i(u)$ denotes the set of colors that are each used by c on exactly i neighbors of u. Clearly, $\bigcup_{i=1}^{k} C_i(u)$ is the set of colors that are associated with u.

Lemma 2.1. *Let M, q be positive integers and G be a k-forested t-critical graph, where $t = \lceil \frac{M}{k} \rceil + q$ and $M \geq \Delta(G) > k$. Then the following claims hold.*

(1) *If $q \geq 1$ and $k \geq 2$, then G contains no 1-vertices;*
(2) *If $q \geq 1$ and $k \geq 3$, then G does not contain a 2-vertex u with $N(u) = \{v, w\}$ such that $d(v) = 2$ and $d(w) \leq k(t-2)$, or $d(v) \geq 3$ and $d(w) \leq k(t - d(v)) + d(v) - 1$;*
(3) *If $q \geq 1$ and $k \geq 3$, then G does not contain a 3-cycle $[uvw]$ such that $d(u) = d(v) = 2$ and $d(w) \leq k(t-2) + 1$*
(4) *If $q \geq 1$ and $k \geq 3$, then G does not contain a 3-vertex u that is adjacent to two 2-vertices v_1, v_2 and a $[k(t-2)]^-$-vertex v_3 such that $N(v_i) = \{u, w_i\}$ $(i = 1, 2)$, $d(w_1) \leq k(t-2)$ and $d(w_2) \leq k(t-2) + 1$;*
(5) *If $q \geq 1$ and $k \geq 3$, then G does not contain a $[k(t-2)+1]$-vertex u that is adjacent only to 2-vertices such that at least $(k-1)(t-2)$ of them are adjacent to a 2-vertex and at most one of them is adjacent to a $(k+1)^+$-vertex different from u;*
(6) *If $q \geq 2$ and $k \geq 2$, then G does not contain two adjacent 2-vertices u and v;*
(7) *If $q \geq 2$ and $k \geq 2$, then G does not contain a 3-cycle $[uvw]$ such that $d(u) = 2$ and $d(v) = 3$;*
(8) *If $q \geq 2$ and $k \geq 2$, then G does not contain two intersecting 3-cycles $[uv_1w_1]$ and $[uv_2w_2]$ such that $d(v_1) = d(v_2) = 2$ and $d(u) = 4$.*

Proof. We will prove this lemma by contradiction. In each proof below, first we arbitrarily give a t-uniform list assignment L to every vertex of G (note that $t \geq 2 + q \geq 3$ by its definition). Since G is k-forested t-critical, for any proper vertex subset $S \subseteq V(G)$, the graph $G' = G - S$ admits a k-forested t-coloring c such that $c(x) \in L(x)$ for every $x \in V(G')$. Our next effort is trying to extend c to a k-forested t-coloring of G such that $c(x) \in L(x)$ for every $x \in V(G)$. This forms a contradiction completing the proof.

(1) Suppose G contains a vertex u such that $N(u) = \{v\}$. Set $S = \{u\}$. Then we extend the k-forested t-coloring c of $G - S$ to G by assigning a color $c(u)$ to u such that $c(u) \in L(u) \setminus F$, where $F = c(v) \cup C_k(v)$. Note that $|F| \leq 1 + \lfloor \frac{d(v)-1}{k} \rfloor \leq 1 + \lfloor \frac{M-1}{k} \rfloor = \lceil \frac{M}{k} \rceil < t$. So such a coloring $c(u)$ exists and the extended coloring c of G is k-forested.

(2) Suppose G contains such a 2-vertex u as described in the Lemma. Without loss of generality, we assume that $d(v) \leq d(w)$. Set $S = \{u\}$. Then we extend the k-forested t-coloring c of $G - S$ to G as follows.

If $c(v) \neq c(w)$, then assign u a color $c(u) \in L(u) \setminus F$ where $F = \{c(v), c(w)\} \cup C_k(v) \cup C_k(w)$. Note that $|F| \leq 2 + \lfloor \frac{d(v)-1}{k} \rfloor + \lfloor \frac{d(w)-1}{k} \rfloor$. If $d(v) = 2$ and $d(w) \leq k(t-2)$, then $|F| \leq 2 + \lfloor \frac{d(w)-1}{k} \rfloor \leq t - 1$. If $d(v) \geq 3$ and $d(w) \leq k(t - d(v)) + d(v) - 1$, then $|F| \leq 2 + t - d(v) + \lfloor \frac{d(v)-1}{k} \rfloor + \lfloor \frac{d(v)-2}{k} \rfloor \leq t - 1$. So in each case, the coloring of u as assigned above does exist and the extended coloring c of G is k-forested.

If $c(v) = c(w)$, then let $B(v) = \bigcup_{i=1}^{k-1} C_i(v)$, $B(w) = \bigcup_{i=1}^{k-1} C_i(w)$, $F_1 = B(v) \cap B(w)$ and $F_2 = C_k(v) \cup C_k(w)$. One can calculate that $|F_1| \leq |B(v)| \leq d(v) - 1 - k|C_k(v)| \leq d(v) - 1$ and $|F_1 \cup F_2| \leq |F_1| + \lfloor \frac{d(v)-1-|B(v)|}{k} \rfloor + \lfloor \frac{d(w)-1-|B(w)|}{k} \rfloor \leq |F_1| + \lfloor \frac{d(v)-1-|F_1|}{k} \rfloor + \lfloor \frac{d(w)-1-|F_1|}{k} \rfloor \leq d(v) - 1 + \lfloor \frac{d(w)-d(v)}{k} \rfloor$ (note that this upper bound is reached at $|F_1| = d(v) - 1$). If $d(v) = 2$ and $d(w) \leq k(t-2)$, then $|F_1 \cup F_2| \leq 1 + \lfloor \frac{k(t-2)-2}{k} \rfloor \leq t - 2$. If $d(v) \geq 3$ and $d(w) \leq k(t - d(v)) + d(v) - 1$, then $|F_1 \cup F_2| \leq d(v) - 1 + \lfloor \frac{k(t-d(v))+d(v)-1-d(v)}{k} \rfloor = t - 2$. In each case, we assign to u a coloring $c(u)$ such that $c(u) \in L(u) \setminus F$, where $F = F_1 \cup F_2 \cup \{c(v)\}$. Note that $|F| < t$ by above arguments. So such a coloring $c(u)$ does exist. Of course, the extended coloring c of G is k-forested.

(3) Suppose G contains such a 3-cycle as described in the Lemma. Set $S = \{u\}$. Then we extend the k-forested t-coloring c of $G - S$ to G by coloring u as follows. If $c(v) \notin C_k(w)$, then $|C_k(w)| \leq \lfloor \frac{d(w)-2}{k} \rfloor \leq \lfloor \frac{k(t-2)-1}{k} \rfloor = t - 3$ and thus color u with $c(u) \in L(u) \setminus F_1$, where $F_1 = \{c(v), c(w)\} \cup C_k(w)$. If $c(v) \in C_k(w)$, then $|C_k(w)| \leq \lfloor \frac{d(w)-1}{k} \rfloor \leq \lfloor \frac{k(t-2)}{k} \rfloor = t - 2$ and thus color u with $c(u) \in L(u) \setminus F_2$, where $F_2 = \{c(w)\} \cup C_k(w)$. In each case, we can ensure that such a coloring $c(u)$ does exist and the extended coloring c of G is k-forested.

(4) Suppose G contains such a 3-vertex u as described in the Lemma. Note that $d(w_2) \leq k(t-2) + 1$. Without loss of generality, assume that $d(w_2) = k(t-2) + 1$ and $N(w_2) = \{v_2, x_1, \ldots, x_{k(t-2)}\}$. By Lemma 2.1(3), we shall also assume that $v_1v_2 \notin E(G)$. Set $S = \{v_1\}$. Then we extend the k-forested t-coloring c of $G - S$ to G as follows. If $c(u) \neq c(w_1)$, then let $c(v_1) \in L(v_1) \setminus F$, where $F = \{c(u), c(w_1)\} \cup C_k(w_1)$. If $c(u) = c(w_1)$ and $c(v_2) = c(v_3)$, then let $c(v_1) \in L(v_1) \setminus F$, where $F = \{c(u), c(v_2)\} \cup C_k(w_1)$. In each case above, we can ensure that $|F| \leq t - 1$ since $d(w_1) \leq k(t-2)$ and thus the extended coloring c of G is k-forested. So without loss of generality, we shall assume that $c(u) = c(w_1) = 1$, $c(v_2) = 2$ and $c(v_3) = 3$. In this case, if $A = L(v_1) \setminus (\{1, 2, 3\} \cup C_k(w_1)) \neq \emptyset$, then we can color v_1 with $c(v_1) \in A$. If $B = L(u) \setminus (\{1, 2, 3\} \cup C_k(v_3)) \neq \emptyset$, then we can recolor u with $c(u) \in B$ and then color v_1 with $c(v_1) \in L(v_1) \setminus (\{c(u), c(w_1)\} \cup C_k(w_1)) \neq \emptyset$. So $\{1, 2, 3\} \subseteq L(v_1) \cap L(u)$ and $\{1, 2, 3\} \cap (C_k(w_1) \cup C_k(v_3)) = \emptyset$ (note that $\max\{d(v_3), d(w_1)\} \leq k(t-2)$). Now if $c(w_2) \neq 1$ or $2 \notin \bigcup_{i=1}^{k(t-2)} \{c(x_i)\}$, then we can color v_1 by 2. So $c(w_2) = 1$ and $2 \in \bigcup_{i=1}^{k(t-2)} \{c(x_i)\}$. This implies that $|C_k(w_2) \setminus \{2\}| \leq \lfloor \frac{d(w_2)-2}{k} \rfloor \leq \lfloor \frac{k(t-2)-1}{k} \rfloor = t - 3$. Now recolor v_2 with $c(v_2) \in L(v_2) \setminus (\{1, 2\} \cup C_k(w_2)) \neq \emptyset$. After such a recoloring, if now $c(v_2) \neq 3$, then we can color v_1 by 2. Otherwise, first recolor u by 2 (to avoid a possible bichromatic cycle through the path $w_2v_2uv_3$) and then color v_1 by 3. In each case, the extended coloring c of G is k-forested.

(5) Suppose G contains such a vertex u as described in the Lemma. Without loss of generality, let $N(u) = \{v_0^1\} \cup \bigcup_{j=1}^{k} \bigcup_{i=1}^{t-2} \{v_i^j\}$, $N(v_i^j) = \{u, w_i^j\}$ for $i \in \{1, \ldots, t-2\}$ and $j \in \{1, \ldots, k\}$, $N(w_i^j) = \{v_i^j, x_i^j\}$ for $i \in \{1, \ldots, t-2\}$ and $j \in \{2, \ldots, k\}$ and $\max_{1 \leq i \leq t-2} d(w_i^1) \leq k$. By Lemma 2.1(3), we shall assume that no two neighbors of u are

adjacent in G. Set $S = \{u\} \cup \bigcup_{j=1}^{k} \bigcup_{i=1}^{t-2} \{v_i^j\}$. Then we extend the k-forested t-coloring c of $G - S$ to G by coloring $v_1^1, \ldots, v_{t-2}^1, u, v_1^2, \ldots, v_{t-2}^2, \ldots, v_1^k, \ldots, v_{t-2}^k$ greedily with

$$c(v_1^1) \in L(v_1^1) \setminus \{c(v_0^1), c(w_1^1)\},$$

which is of size at least $t - 2 \geq 1$,

$$c(v_i^1) \in L(v_i^1) \setminus \{c(v_0^1), c(w_i^1)\} \cup \bigcup_{s=1}^{i-1} \{c(v_s^1)\},$$

which is of size at least $t - (i + 1) \geq 1$ for $i \in \{2, \ldots, t - 2\}$,

$$c(u) \in L(u) \setminus \{c(v_0^1)\} \cup \bigcup_{s=1}^{t-2} \{c(v_s^1)\},$$

which is of size $t - (t - 1) = 1$,

$$c(v_1^j) \in \begin{cases} L(v_1^j) \setminus \{c(u), c(w_1^j)\}, & \text{if } c(u) \neq c(w_1^j); \\ L(v_1^j) \setminus \{c(u), c(x_1^j)\}, & \text{if } c(u) = c(w_1^j), \end{cases}$$

which is of size at least $t - 2 \geq 1$ for $j \in \{2, \ldots, k\}$, and

$$c(v_i^j) \in \begin{cases} L(v_i^j) \setminus \left(\{c(u), c(w_i^j)\} \cup \bigcup_{s=1}^{i-1} \{c(v_s^j)\} \right), & \text{if } c(u) \neq c(w_i^j); \\ L(v_i^j) \setminus \left(\{c(u), c(x_i^j)\} \cup \bigcup_{s=1}^{i-1} \{c(v_s^j)\} \right), & \text{if } c(u) = c(w_i^j), \end{cases}$$

which is of size at least $t - (i + 1) \geq 1$ for $i \in \{2, \ldots, t - 2\}$ and $j \in \{2, \ldots, k\}$.

It is easy to check that the extended coloring is acyclic and the vertices with the same superscript receive different colors. So after this extending, the k-frugality on u would also be preserved. Hence, this gives a desired k-forested coloring of G.

(6) Suppose G contains two such vertices u and v as described in the Lemma. Let $N(u) = \{v, u_1\}$, $N(v) = \{u, v_1\}$ and set $S = \{u\}$. Then we extend the k-forested t-coloring c of $G - S$ to G by coloring u with $c(u) \in L(u) \setminus F$, where $F = \{c(u_1), c(v)\} \cup C_k(u_1)$ if $c(u_1) \neq c(v)$, and $F = \{c(u_1), c(v_1)\} \cup C_k(u_1)$ if $c(u_1) = c(v)$. Note that $|C_k(u_1)| \leq \lfloor \frac{d(u_1)-1}{k} \rfloor \leq \lfloor \frac{M-1}{k} \rfloor = \lceil \frac{M}{k} \rceil - 1 \leq t - 3$. So in each case, we have $|F| \leq t - 1$, which implies the existence of such an extending. Meanwhile, one can easily check that the extended coloring c of G is k-forested.

(7) Suppose G contains such a 3-cycle as described in the Lemma. Set $S = \{u\}$. Then we extend the k-forested t-coloring c of $G - S$ to G by coloring u with $c(u) \in L(u) \setminus F$, where $F = \{c(v), c(w)\} \cup C_k(w)$. Similarly as in (6), we have $|F| \leq t - 1$ and one can check that the extended coloring c of G is k-forested.

(8) Suppose G contains two such intersecting 3-cycles as described in the Lemma. Set $S = \{v_1\}$. Then we extend the k-forested t-coloring c of $G - S$ to G by coloring u with $c(u) \in L(u) \setminus F$, where $F = \{c(u), c(w_1)\} \cup C_k(w_1) \cup C_k(u)$. Similarly as in (6), we have $|C_k(w)| \leq t - 3$. Note that $v_2w_2 \in E(G - S)$ and $d(u) = 4$, we shall have $C_k(u) \subseteq \{c(w_1)\}$. So we again obtain $|F| \leq t - 1$, which implies the existence of such an extending. Meanwhile, one can easily check that the extended coloring c of G is k-forested. □

From Lemma 2.1, one can easily deduce the following Observation 2.2, which will be used frequently in the forthcoming proofs. In the following, an (r, s)-type 2-vertex in G is a 2-vertex with one neighbor of degree r and the other of degree s. The $(\geq r, s)$- and $(r, \geq s)$-type 2-vertices are defined similarly. Without loss of generality, we always set $r \leq s$ in these definitions.

Observation 2.2. Let M be a positive integer and G be a k-forested t-critical graph, where $t = \lceil \frac{M}{k} \rceil + 1$ and $M \geq \Delta(G) > k \geq 3$. Then we have the following observations.

(O1) $\delta(G) \geq 2$;
(O2) If a 2-vertex u in G is adjacent to a 2-vertex, then u is also adjacent to a $[k(t - 2) + 1]^+$-vertex;
(O3) If a 2-vertex u in G is adjacent to a d-vertex where $t \geq d \geq 3$, then u is also adjacent to a $[k(t - d) + d]^+$-vertex;
(O4) If a 3-vertex u in G is adjacent to two 2-vertices, then u is adjacent to either a $[k(t - 2) + 1]^+$-vertex or at least one $(3, \geq k(t - 2) + 1)$-type 2-vertex;
(O5) If a 3-vertex u in G is adjacent to three 2-vertices, then u is adjacent to either three $(3, \geq k(t - 2) + 1)$-type 2-vertices or two $(3, \geq k(t - 2) + 2)$-type 2-vertices;
(O6) Suppose u is a $k(t - 2) + 1$-vertex in G that is adjacent only to 2-vertices. Denote by n_1 and n_2, respectively, the number of $(2, k(t - 2) + 1)$-type vertices and the number of $(3, k(t - 2) + 1)$-type vertices that are adjacent to u. If $n_1 \geq (k - 1)(t - 2)$, then $n_1 + n_2 \leq k(t - 2) - 1$.

3. Proof of Theorem 1.4

In this section, we prove a slightly stronger theorem than Theorem 1.4 as follows.

Theorem 1.4′. *Every graph G with $M \geq \Delta(G) \geq \Delta$ and* $\mathrm{mad}(G) < m$ *has* $ch_k^a(G) \leq \lceil \frac{M}{k} \rceil + 1$ *if* $k \geq 3$ *and* $(\Delta, m) \in \{(k+1, \frac{5}{2}), (2k+1, \frac{8}{3}), (4k+1, \frac{14}{5})\}$, *or* $k \geq 7$ *and* $(\Delta, m) = (k+1, \frac{8}{3})$.

Of course, the interesting case of Theorem 1.4′ is when $M = \Delta(G)$. Indeed, Theorem 1.4′ is only a technical strengthening of Theorem 1.4, without which we would get complications when considering a subgraph $H \subset G$ such that $\Delta(H) < \Delta(G)$ (see [7]).

The proof of Theorem 1.4′ will be processed by contradiction and divided into four parts according to the bounds on k, $\Delta(G)$ and $\mathrm{mad}(G)$. The beginning of the proof of each part is almost the same. First, set $t = \lceil \frac{M}{k} \rceil + 1$ and without loss of generality, suppose G is a k-forested t-critical graph. Then, we apply a discharging procedure to G by originally assigning each vertex $u \in V(G)$ an initial charge $c(u) = d(u) - m$, which implies that

$$\sum_{u \in V(G)} c(u) = \sum_{u \in V(G)} d(u) - m|V(G)| \leq (\mathrm{mad}(G) - m)|V(G)| < 0.$$

Later, we will redistribute the charge of the vertices of G according to some defined discharging rules, which only move charge around but do not affect the total charges, so that after discharging the final charge $c'(u)$ of each vertex, $u \in V(G)$ is nonnegative. Thus, this leads to a contradiction completing the proof in final.

Following the strategy as described above, one may quickly find that the only differences among the proof of each part are the definition of the discharging rules and the estimation on the final charge $c'(u)$ of each vertex u in G.

Part 1. $k \geq 3$, $\Delta(G) \geq k+1$ and $\mathrm{mad}(G) < \frac{5}{2}$.

Note that $t \geq 3$ in this case. Our discharging rules are stated as follows.

Charge to 2-vertex. Let u be a 2-vertex with $N(u) = \{v, w\}$.

(R1) If $d(v) = 2$ and $d(w) \geq k(t-2)+1$, then w sends $\frac{1}{2}$ to u.

(R2) If $d(v) = 3$ and $d(w) \leq k(t-2)$, then both v and w send $\frac{1}{4}$ to u.

(R3) If $d(v) = 3$ and $d(w) = k(t-2)+1$, then v and w respectively sends $\frac{1}{6}$ and $\frac{1}{3}$ to u.

(R4) If $d(v) = 3$ and $d(w) \geq k(t-2)+2$, then w sends $\frac{1}{2}$ to u.

(R5) If $\min\{d(v), d(w)\} \geq 4$, then each of v and w sends $\frac{1}{4}$ to u.

Now we check that the final charge of each vertex $u \in V(G)$ is nonnegative. Let u be a vertex in G. Then by O1 (recall Observation 2.2), we have $d(u) \geq 2$. Suppose $d(u) = 2$. Then $c(u) = -\frac{1}{2}$ and u totally receives $\frac{1}{2}$ from its two neighbors by O2 and R1–R5. This implies that $c'(u) = 0$. Suppose $d(u) = 3$. Then by O2, u does not send out charge by R1. If u is adjacent to at most two 2-vertices, then by R2–R4, we have $c'(v) \geq 3 - \frac{5}{2} - 2 \times \frac{1}{4} = 0$. If u is adjacent to three 2-vertices, then by O5, either all of them are $(3, \geq k(t-2)+1)$-type implying that $c'(u) \geq 3 - \frac{5}{2} - 3 \times \frac{1}{6} = 0$ by R3 and R4, or two of them are $(3, \geq k(t-2)+2)$-type implying that $c'(u) \geq 3 - \frac{5}{2} - \frac{1}{4} > 0$ by R2–R4. Suppose $4 \leq d(u) \leq k(t-2)$ (of course this case may happen only if $k \geq 4$ or $t \geq 4$). Then by O2, R2 and R5, $c'(u) \geq d(u) - \frac{5}{2} - \frac{1}{4}d(u) \geq \frac{1}{2} > 0$. Suppose $d(u) = k(t-2)+1$. If u is adjacent to at least one 3^+-vertex, then by R1, R3 and R5, we have $c'(u) = d(u) - \frac{5}{2} - \frac{1}{2}(d(u)-1) = \frac{d(u)-4}{2} = \frac{k(t-2)-3}{2} \geq 0$ for $k \geq 3$ and $t \geq 3$. If u is adjacent only to 2-vertices and $n_1 \geq (k-1)(t-2)$ (recall the definitions of n_i in Observation 2.2), then by R1, R3, R5 and O6, $c'(v) \geq d(u) - \frac{5}{2} - \frac{1}{2}n_1 - \frac{1}{3}n_2 - \frac{1}{4}(d(u) - n_1 - n_2) \geq \frac{3}{4}d(u) - \frac{5}{2} - \frac{1}{4}(n_1+n_2) \geq \frac{k(t-2)-3}{2} \geq 0$ for $k \geq 3$ and $t \geq 3$. If u is adjacent only to 2-vertices but $n_1 \leq (k-1)(t-2) - 1$, then by R1, R3 and R5, we have $c'(u) \geq d(u) - \frac{5}{2} - \frac{1}{2}n_1 - \frac{1}{3}n_2 - \frac{1}{4}(d(u) - n_1 - n_2) = \frac{3}{4}d(u) - \frac{5}{2} - \frac{1}{12}(n_1+n_2) - \frac{1}{6}n_1 \geq \frac{2}{3}d(u) - \frac{5}{2} - \frac{1}{6}n_1 \geq \frac{(3k+1)(t-2)-10}{6} \geq 0$ for $k \geq 3$ and $t \geq 3$. Suppose $d(u) \geq k(t-2)+2$. Then by R1, R4 and R5, we have $c'(u) \geq d(u) - \frac{5}{2} - \frac{1}{2}d(u) \geq \frac{k(t-2)-3}{2} \geq 0$ for $k \geq 3$ and $t \geq 3$ in final.

Part 2. $k \geq 7$, $\Delta(G) \geq k+1$ and $\mathrm{mad}(G) < \frac{8}{3}$.

Note that $t \geq 3$ in this case. Our discharging rules are stated as follows.

Charge to 2-vertex. Let u be a 2-vertex with $N(u) = \{v, w\}$.

(R1) If $d(v) \leq k(t-2)$ and $d(w) \geq k(t-2)+1$, then w sends $\frac{2}{3}$ to u.

(R2) If $\max\{d(v), d(w)\} \leq k(t-2)$ or $\min\{d(v), d(w)\} \geq k(t-2)+1$, then v and w respectively sends $\frac{1}{3}$ to u.

Charge to 3-vertex. Let u be a 3-vertex.

(R3) If u is adjacent to a $[k(t-2)+1]^+$-vertex v, then v sends $\frac{1}{3}$ to u.

Now we check that the final charge $c'(u)$ of each vertex $u \in V(G)$ is nonnegative. Suppose $d(u) = 2$. Then $c(u) = -\frac{2}{3}$ and u totally receives $\frac{2}{3}$ from its two neighbors by R1 and R2. Suppose $d(u) = 3$. If u is adjacent to at most one 2-vertex, then by R2, $c'(u) \geq d(u) - \frac{8}{3} - \frac{1}{3} = 0$. If u is adjacent to exactly two 2-vertices, then by O4, either u is adjacent to a $[k(t-2)+1]^+$-vertex implying that $c'(u) \geq d(u) - \frac{8}{3} - 2 \times \frac{1}{3} + \frac{1}{3} = 0$ by R2 and R3, or u is adjacent to at least one $(3, \geq k(t-2)+1)$-type

2-vertex implying that $c'(u) \geq d(u) - \frac{8}{3} - \frac{1}{3} = 0$ by R1 and R2. If u is adjacent to exactly three 2-vertices, then by O5, u is adjacent to at most one $(3, \leq k(t-2))$-type 2-vertex, which implies that $c'(u) \geq d(u) - \frac{8}{3} - \frac{1}{3} = 0$ by R1 and R2. Suppose $4 \leq d(u) \leq k(t-2)$. Then by R1 and R2, $c'(u) \geq d(u) - \frac{8}{3} - \frac{1}{3}d(u) \geq 0$. Suppose $d(u) \geq k(t-2)+1$. Then by R1–R3, we also have $c'(u) \geq d(u) - \frac{8}{3} - \frac{2}{3}d(u) \geq \frac{k(t-2)-7}{3} \geq 0$ for $k \geq 7$ and $t \geq 3$ in final.

Part 3. $k \geq 3$, $\Delta(G) \geq 2k+1$ and $\mathrm{mad}(G) < \frac{8}{3}$.

Note that $t \geq 4$ in this case. Our discharging rules are stated as follows.

Charge to 2-vertex. Let u be a 2-vertex with $N(u) = \{v, w\}$.

(R1) If $d(v) = 2$ and $d(w) \geq k(t-2)+1$, then w sends $\frac{2}{3}$ to u.
(R2) If $d(v) = 3$ and $d(w) \geq k(t-3)+3$, then v and w respectively sends $\frac{1}{9}$ and $\frac{5}{9}$ to u.
(R3) If $\min\{d(v), d(w)\} \geq 4$, then each of v and w sends $\frac{1}{3}$ to u.

Now we check that the final charge $c'(u)$ of each vertex $u \in V(G)$ is nonnegative. Suppose $d(u) = 2$. Then $c(u) = -\frac{2}{3}$ and u totally receives $\frac{2}{3}$ from its two neighbors by O2, O3 and R1–R3. This implies that $c'(u) = 0$. Suppose $d(u) = 3$. Then by R2, $c'(u) \geq 3 - \frac{8}{3} - 3 \times \frac{1}{9} = 0$. Suppose $4 \leq d(u) \leq k(t-3)+2$. Then by R3, $c'(u) \geq d(u) - \frac{8}{3} - \frac{1}{3}d(u) \geq 0$. Suppose $k(t-3)+3 \leq d(u) \leq k(t-2)$. Then by R2 and R3, $c'(u) \geq d(u) - \frac{8}{3} - \frac{5}{9}d(u) \geq \frac{4k(t-3)-12}{9} > 0$ for $k \geq 3$ and $t \geq 4$. Suppose $d(u) = k(t-2)+1$. If u is adjacent to at least one 3^+-vertex, then by R1–R3, we have $c'(u) \geq d(u) - \frac{8}{3} - \frac{2}{3}(d(u)-1) = \frac{k(t-2)-5}{3} > 0$ for $k \geq 3$ and $t \geq 4$. If u is adjacent only to 2-vertices and $n_1 \geq (k-1)(t-2)$, then by R1–R3 and O6, $c'(u) \geq d(u) - \frac{8}{3} - \frac{2}{3}n_1 - \frac{5}{9}n_2 - \frac{1}{3}(d(u)-n_1-n_2) \geq \frac{2}{3}d(u) - \frac{8}{3} - \frac{1}{3}(n_1+n_2) \geq \frac{k(t-2)-5}{3} > 0$ for $k \geq 3$ and $t \geq 4$. If u is adjacent only to 2-vertices but $n_1 \leq (k-1)(t-2)-1$, then by R1–R3, we have $c'(u) \geq d(u) - \frac{8}{3} - \frac{2}{3}n_1 - \frac{5}{9}n_2 - \frac{1}{3}(d(u)-n_1-n_2) = \frac{2}{3}d(u) - \frac{8}{3} - \frac{2}{9}(n_1+n_2) - \frac{1}{9}n_1 \geq \frac{4}{9}d(u) - \frac{8}{3} - \frac{1}{9}n_1 \geq \frac{(3k+1)(t-2)-19}{9} > 0$ for $k \geq 3$ and $t \geq 4$. Suppose $d(u) \geq k(t-2)+2$. Then by R1–R3, we also have $c'(u) \geq d(u) - \frac{8}{3} - \frac{2}{3}d(u) \geq \frac{k(t-2)-6}{2} \geq 0$ for $k \geq 3$ and $t \geq 4$ in final.

Part 4. $k \geq 3$, $\Delta(G) \geq 4k+1$ and $\mathrm{mad}(G) < \frac{14}{5}$.

Note that $t \geq 6$ in this case. Our discharging rules are stated as follows.

Charge to 2-vertex. Let u be a 2-vertex with $N(u) = \{v, w\}$.

(R1) If $d(v) = 2$ and $d(w) \geq k(t-2)+1$, then w sends $\frac{4}{5}$ to u.
(R2) If $d(v) = 3$ and $d(w) \geq k(t-3)+3$, then v and w respectively sends $\frac{1}{15}$ and $\frac{11}{15}$ to u.
(R3) If $d(v) = 4$ and $d(w) \geq k(t-4)+4$, then v and w respectively sends $\frac{3}{10}$ and $\frac{1}{2}$ to u.
(R4) If $\min\{d(v), d(w)\} \geq 5$, then both v and w send $\frac{2}{5}$ to u.

Now we check that the final charge $c'(u)$ of each vertex $u \in V(G)$ is nonnegative. Suppose $d(u) = 2$. Then $c(u) = -\frac{4}{5}$ and u totally receives $\frac{4}{5}$ from its two neighbors by O2, O3 and R1–R4. This implies that $c'(u) = 0$. Suppose $d(u) = 3$. Then by R2, $c'(u) \geq d(u) - \frac{14}{5} - 3 \times \frac{1}{15} = 0$. Suppose $d(u) = 4$. Then by R3, $c'(u) \geq d(u) - \frac{14}{5} - 4 \times \frac{3}{10} = 0$. Suppose $5 \leq d(u) \leq k(t-4)+3$. Then by R4, $c'(u) \geq d(u) - \frac{14}{5} - \frac{2}{5}d(v) > 0$. Suppose $k(t-4)+4 \leq d(u) \leq k(t-3)+2$. Then by R3 and R4, $c'(u) \geq d(u) - \frac{14}{5} - \frac{1}{2}d(u) \geq \frac{5k(t-4)-8}{10} > 0$ for $k \geq 3$ and $t \geq 6$. Suppose $k(t-3)+3 \leq d(u) \leq k(t-2)$. Then by R2–R4, $c'(u) \geq d(u) - \frac{14}{5} - \frac{11}{15}d(u) \geq \frac{4k(t-3)-30}{15} > 0$ for $k \geq 3$ and $t \geq 6$. Suppose $d(u) = k(t-2)+1$. If u is adjacent to at least one 3^+-vertex, then by R1–R4, we have $c'(u) \geq d(u) - \frac{14}{5} - \frac{4}{5}(d(u)-1) \geq \frac{k(t-2)-9}{5} > 0$ for $k \geq 3$ and $t \geq 6$. If u is adjacent only to 2-vertices and $n_1 \geq (k-1)(t-2)$, then R1–R4 and O6 imply that $c'(u) \geq d(u) - \frac{14}{5} - \frac{4}{5}n_1 - \frac{11}{15}n_2 - \frac{1}{2}(d(u)-n_1-n_2) \geq \frac{1}{2}d(u) - \frac{14}{5} - \frac{3}{10}(n_1+n_2) \geq \frac{k(t-2)-10}{5} > 0$ for $k \geq 3$ and $t \geq 6$. If u is adjacent only to 2-vertices but $n_1 \leq (k-1)(t-2)-1$, then by R1–R4, we have $c'(u) \geq d(u) - \frac{14}{5} - \frac{4}{5}n_1 - \frac{11}{15}n_2 - \frac{1}{2}(d(u)-n_1-n_2) \geq \frac{1}{2}d(u) - \frac{14}{5} - \frac{7}{30}(n_1+n_2) - \frac{1}{15}n_1 \geq \frac{4}{15}d(u) - \frac{14}{5} - \frac{1}{15}n_1 \geq \frac{(3k+1)(t-2)-37}{15} > 0$ for $k \geq 3$ and $t \geq 6$. Suppose $d(u) \geq k(t-2)+2$. Then by R1–R4, we also have $c'(u) \geq d(u) - \frac{14}{5} - \frac{4}{5}d(u) \geq \frac{k(t-2)-12}{5} \geq 0$ for $k \geq 3$ and $t \geq 6$ in final.

4. Proof of Theorem 1.6

Similarly, instead of proving Theorem 1.6 directly, we prove the following slightly stronger result.

Theorem 1.6′. *Let M, k be positive integers and let G be an outerplanar graph. If $k \geq 2$ and $M \geq \Delta(G) \geq k+1$, then $ch_k^a(G) \leq \lceil \frac{M}{k} \rceil + 2$.*

Proof. We prove it by induction and contradiction. Set $t = \lceil \frac{M}{k} \rceil + 2$ and without loss of generality, suppose that G is a k-forested t-critical graph. Note that G is still outerplanar, so it contains at least one of the following configurations (see [10, Lemma 3]):

(C1) a 1-vertex u;
(C2) two adjacent 2-vertices u and v;

(C3) a 3-cycle $[uvw]$ such that $d(u) = 2$ and $d(v) = 3$;
(C4) two intersecting 3-cycles $[uv_1w_1]$ and $[uv_2w_2]$ such that $d(v_1) = d(v_2) = 2$ and $d(u) = 4$.

However, by Lemma 2.1(1), (6)–(8), the k-forested t-critical outerplanar graph G cannot contain any configuration among (C1)–(C4). This contradiction completes the proof of the theorem. □

5. Discussions

Recall the results we obtained in this paper, one can find that all planar graphs considered in this paper have girth at least 7. Indeed, it was proved in [7] that every planar graph with girth at least 5 has $ch_2^a(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 4$. We think this result can also be generated to the k-forested choosability of planar graph with girth at least 5 for every $k \geq 3$ only by Lemma 1 in [7]. The interested readers can check this by themselves. However, we think to find some sufficient conditions such that planar graphs with girth at least 5 has $\chi_k^a(G) = \lceil \frac{\Delta(G)}{k} \rceil + 1$ or $ch_k^a(G) = \lceil \frac{\Delta(G)}{k} \rceil + 1$ seems more interesting and challenging. Actually, we want to do the following conjecture.

Conjecture 5.1. *Every planar graph G with girth at least g and large enough maximum degree $\Delta(G)$ has $ch_k^a(G) = \lceil \frac{\Delta(G)}{k} \rceil + 1$, provided $k \geq 2$ and $g \geq 5$.*

Note that the complete bipartite graph $K_{2,n}$ is a planar graph with girth 4 and $ch_k^a(K_{2,n}) = \lceil \frac{\Delta(K_{2,n})}{k} \rceil + 2$. So the lower bound 5 for g in this conjecture is best possible if it is true.

In this study, we also considered outerplanar graph and found that the upper bound for its k-forested choosability in general case is very close to the naive lower bound $\lceil \frac{\Delta(G)}{k} \rceil + 1$. In fact, we conjecture that the following result holds.

Conjecture 5.2. *Every outerplanar graph G has $ch_k^a(G) = \lceil \frac{\Delta(G)}{k} \rceil + 1$ if $k \geq 2$ and $\Delta(G) \geq 2k + 1$.*

Of course, another class of sparse graphs, series–parallel graphs, can also be considered in the future. However, we can only conjecture the following conclusion and say the bound for $ch_k^a(G)$ in that conjecture is best possible if it is true, since the complete bipartite graph $K_{2,n}$ is a series–parallel graph with $ch_k^a(K_{2,n}) = \lceil \frac{\Delta(K_{2,n})}{k} \rceil + 2$ for every $n \geq 2$ and $k \geq 2$.

Conjecture 5.3. *Every series–parallel graph G has $ch_k^a(G) \leq \lceil \frac{\Delta(G)}{k} \rceil + 2$ if $k \geq 2$ and $\Delta(G) \geq k + 1$.*

References

[1] O. Amini, L. Esperet, J. van den Heuvel, Frugal colouring of graphs, arXiv:0705.0422, available on line at http://arxiv.org/pdf/0705.0422v1.
[2] O.V. Borodin, On acyclic colorings of planar graphs, Discrete Mathematics 25 (1979) 211–236.
[3] O.V. Borodin, A.V. Kostochka, D.R. Woodall, Acyclic colourings of planar graphs with large girth, Journal of the London Mathematical Society 60 (1999) 344–352.
[4] N. Cohen, F. Havet, Linear and 2-frugal choosability of graphs of small maximum average degree, Graphs and Combinatorics 27 (6) (2011) 831–849.
[5] T.F. Coleman, J.-Y. Cai, The cyclic coloring problem and estimation of spare Hessian matrices, SIAM Journal on Algebraic and Discrete Methods 7 (2) (1986) 221–235.
[6] T.F. Coleman, J.J. Moré, Estimation of spare Hession matrices and graph coloring problems, Mathematical Programming 28 (1984) 243–270.
[7] D.W. Cranston, G. Yu, Linear choosability of sparse graphs, Discrete Mathematics 311 (2011) 1910–1917.
[8] W. Dong, B. Xu, X. Zhang, Improved bounds on linear coloring of planar graphs, Science China Mathematics (2010) doi:10.1007/s11425-010-0073-z.
[9] L. Esperet, M. Montassier, A. Raspaud, Linear choosability of graphs, Discrete Mathematics 308 (2008) 3938–3950.
[10] L. Esperet, P. Ochem, Oriented colorings of 2-outerplanar graphs, Information Processing Letters 101 (2007) 215–219.
[11] B. Grünbaum, Acyclic colorings of planar graphs, Israel Journal of Mathematics 14 (1973) 390–408.
[12] H. Hind, M. Molloy, B. Reed, Colouring a graph frugally, Combinatorica 17 (1997) 469–482.
[13] H. Hind, M. Molloy, B. Reed, Total coloring with $\Delta + poly(\log \Delta)$ colors, SIAM Journal on Computing 28 (1999) 816–821.
[14] R.J. Kang, T. Müller, Frugal, acyclic and star colourings of graphs, Discrete Applied Mathematics 159 (2011) 1806–1814.
[15] A. Raspaud, W. Wang, Linear coloring of planar graphs with large girth, Discrete Mathematics 309 (2009) 5678–5686.
[16] W. Wang, C. Li, Linear coloring of graphs embeddable in a surface of nonegative characteristic, Science in China Series A 52 (5) (2009) 991–1003.
[17] D.B. West, Introduction to Graph Theory, second ed., Pearson Education, Inc., 2001.
[18] R. Yuster, Linear coloring of graphs, Discrete Mathematics 185 (1998) 293–297.
[19] X. Zhang, G. Liu, J.-L. Wu, k-forested coloring of planar graphs with large girth, Proceedings of the Japan Academy Series A Mathematical Sciences 86 (10) (2010) 169–173.
[20] X. Zhang, G. Liu, J.-L. Wu, k-forested choosability of graphs with bounded maximum average degree, Bulletin of the Iranian Mathematical Society, http://www.iranjournals.ir/ims/bulletin/?_action=articleInfo&article=244 (published online).