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Equitable list tree-coloring of bounded treewidth graphs

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ABSTRACT

The equitable list tree-coloring model is an useful tool to formulate a structure decomposition problem on the complex network with some security considerations. In this paper, it is proved that the equitable list vertex arboricity of every graph with treewidth ω is at most $\lceil\Delta(G)/2\rceil+\omega-2$ whenever $\Delta(G)\geq 4\omega+1$, and moreover, if such a graph does not contain $K_{3,3}$ as a topological minor, then its equitable list vertex arboricity is at most $\lceil\Delta(G)/2\rceil$ provided that $\omega\in\{2,3,4\}$ and $\Delta(G)\geq 6\omega-3$.

1. Introduction

The minimization problem in graph theory so-called the equitable list tree-coloring problem can be used to formulate a structure decomposition problem on the complex network with some security considerations. Precisely, our task is to partition the network into many smaller disjoint pieces so that each piece has a tree-like property and thus we may identify the possible node failures efficiently. Typically, the fewer number of the pieces the better, and sometimes we try to make as small difference between the scales of any two pieces as possible so as to manage each piece of the network with some uniform policies. Moreover, if we are given beforehand a rule for every node in the network on which kinds of tree-like pieces may it belongs to, then we can further use the equitable list tree-coloring model that was introduced by Zhang [16] in 2016 to formulate it.

We begin with some graph-based notations and definitions. For a graph G, $V(G)$ and $\Delta(G)$ denote the set of vertices in G and maximum degree of G, respectively. By $|G|$, we denote the value of $|V(G)|$. Two subsets $U,W\subseteq V(G)$ are disjoint if $U\cap W=\emptyset$, and for two disjoint subsets $U,W\subseteq V(G)$, $e(U,W)$ denotes the number of edges that have one end-vertex in U and the other in W. A graph G is *ω-degenerate* if every subgraph of G contains a vertex of degree at most ω. For a graph H, if we replace some edges of H with new paths so that the inner vertices of those paths have degrees 2, then we result in a *subdivision* of H. If a graph G contains a subdivision of a graph H as a subgraph, then we say that H is a *topological minor* of G. For the unmentioned notations and definitions, we refer the readers to the classic book due to Diestel [5].

Let k be a positive integer and let L be a function on $V(G)$ such that $|L(v)|=k$ for each vertex $v\in V(G)$, where $L(v)$ is a list of colors available for v. We call such a function L a *k-uniform list assignment* of G. A graph G has an *equitable k-list*

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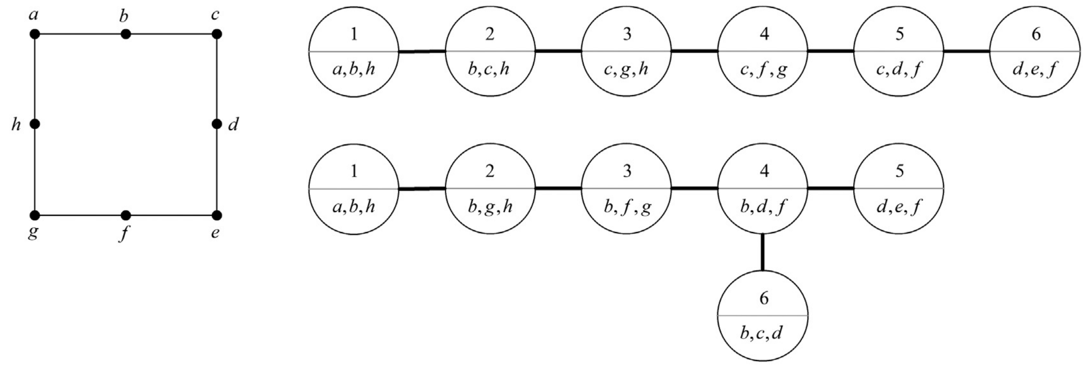

Fig. 1. Two distinct tree-decompositions of C_8 with width 2.

tree-coloring, or is *equitable tree k-choosable* if, for each k-uniform list assignment L of G, we can choose for each $v \in V(G)$ a color from its list $L(v)$ so that the resulting coloring of G satisfies (i) each color class (i.e., the set of vertices with a same color) induces a forest, and (ii) the size of any color class is at most $\lceil |G|/k \rceil$. The minimum integer k such that G is equitable tree k-choosable is the *equitable list vertex arboricity* of G, denoted by $\rho_l^=(G)$. The notions of the equitable list tree-coloring and the equitable list vertex arboricity were first introduced by Zhang [16], who put forward the following conjecture.

Conjecture 1. *$\rho_l^=(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$ for every simple graph G.*

This conjecture was settled by Zhang [16] for complete graphs, 2-degenerate graphs, 3-degenerate claw-free graphs with maximum degree at least 4, and planar graphs with maximum degree at least 8. Drgas-Burchardt *et al.* [6] verified Conjecture 1 for d-dimensional grids with $d \in \{2, 3, 4\}$ and for graphs with (edge) arboricity 2. Moreover, they proved that

$$\rho_l^=(G) \le \max\left\{\frac{8\Delta(G)-2}{7}, \frac{7\Delta(G)-7}{6}\right\}$$

for every simple graph G. This is an interesting result because it is the first one giving an upper bound on $\rho_l^=(G)$ in a general case, although it is a bit far away from Conjecture 1. Recently, Kaul, Mudrock and Pelsmajer [9] verified it for powers of cycles.

On the other hand, many large networks including social network and the communication network are often thought of as having tree-like structure, see [1,2,10,11,13], and there are many relative papers in the literature on the structures of the bounded treewidth networks, such as metabolic networks [4] and Bayesian networks [7]. This motivates us to investigate the equitable list tree-coloring problem for bounded treewidth networks, i.e., graphs with bounded treewidth.

A *tree-decomposition* of a graph $G = (V, E)$ is a pair $(T, \mathcal{X})$ where T is a tree, and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a family of subsets of V, called *bags*, such that

- $\bigcup_{i \in V(T)} X_i = V$;
- for any edge $uv \in E$, there is a bag X_i (for some node $i \in V(T)$) containing both u and v;
- for any vertex $v \in V$, the set $\{i \in V(T) \mid v \in X_i\}$ induces a subtree of T.

The *width* of a tree-decomposition $(T, \mathcal{X})$ is $\max_{i \in V(T)} |X_i| - 1$, and the *treewidth* of G, denoted by $tw(G)$, is the minimum width over all possible tree-decompositions of G.

For example, Fig. 1 gives two distinct tree-decompositions of the cycle C_8 on 8 vertices with width 2. In each picture, the nodes of the tree T are marked by numbers as to make $V(T) = \{1, 2, 3, 4, 5, 6\}$. Furthermore, for the tree-decomposition $(T, \mathcal{X})$ as shown in the upper right corner, we have $X_1 = \{a, b, h\}$, $X_2 = \{b, c, h\}$, $X_3 = \{c, g, h\}$, $X_4 = \{c, f, g\}$, $X_5 = \{c, d, f\}$, and $X_6 = \{d, e, f\}$. It is easy to check the first two rules in the definitions of tree-decomposition are definitely satisfied. For the last rule, we shall verify it for every vertex of C_8. For example, for the vertex $c \in V(C_8)$, the set $\{i \in V(T) \mid c \in X_i\}$ is exactly equal to $\{2, 3, 4, 5\} \subseteq V(T)$, which induces a subtree of T.

Treewidth is commonly used as a parameter in the parameterized complexity analysis of graph algorithms. The graphs with treewidth at most ω are also called partial ω-trees [8,14]. Here a partial ω-tree is a subgraph of a ω-tree, which defines a graph formed by starting with a complete graph on $\omega + 1$ vertices and then repeatedly adding vertices in such a way that each added vertex v has exactly ω neighbors U such that U induces a complete graph on ω vertices. By the definition, it is easy to say that graphs with treewidth at most ω is ω-degenerate. It is known that there is a linear time algorithm [12] to decide whether a given graph G is ω-degenerate for a given variable ω, however, it is NP-complete [3] to determine whether a given graph G has treewidth at most ω for a given variable ω.

In the next sections, we give an upper bound on $\rho_l^=(G)$ that is very close to the one in Conjecture 1 for graphs with bounded treewidth (see Theorem 6 and Corollary 8), and meanwhile, confirm Conjecture 1 for $K_{3,3}$-topological-minor-free graphs G with treewidth $w \in \{2,3,4\}$ and maximum degree at least $6w-3$ (see Theorem 7 and Corollary 9).

2. Structural lemmas

For a vertex $v \in V(G)$, $N(v)$ denotes the set of neighbors of v in G. For a subset $U \subseteq V(G)$, let

$$N(U) = \{v \in V(G) - U : N(v) \cap U \neq \emptyset\}$$

and

$$N[U] = N(U) \cup U.$$

Clearly, by the definition of $N(U)$, we have

$$N(U) \cap U = \emptyset.$$

To begin with, we introduce an useful lemma that was given by Zhang [16].

Lemma 2. *[16, Lemma 2.2] If there exists a set $S = \{z_1, \cdots, z_k\} \subseteq V(G)$ of k distinct vertices such that $G - S$ is equitable tree k-choosable and $|N(z_i) - S| \leq 2i - 1$ for each $1 \leq i \leq k$, then G is equitable tree k-choosable.*

In this paper, we use a symmetric form of the above lemma, which is easier to be applied in the remaining arguments.

Lemma 3. *If there exists a set $S = \{x_1, \cdots, x_k\} \subseteq V(G)$ of k distinct vertices such that $G - S$ is equitable tree k-choosable and $|N(x_i) - S| \leq 2(k-i)+1$ for each $1 \leq i \leq k$, then G is equitable tree k-choosable.*

Proof. Let $z_i = x_{k+1-i}$ with $1 \leq i \leq k$. Since $|N(x_i) - S| \leq 2(k-i)+1$, $|N(z_i) - S| = |N(x_{k+1-i}) - S| \leq 2(k-(k+1-i))+1 = 2i-1$ and thus the result holds by Lemma 2. □

The following lemma due to Pelsmajer [15] describes a local structure of graphs with treewidth $\omega \geq 2$.

Lemma 4. *[15, Lemma 7] If G has treewidth $\omega \geq 2$ and $|G| \geq 3\omega - 1$, then there is a subset $U \subseteq V(G)$ such that $\omega \leq |U| \leq 2(\omega-1)$ and $|N(U)| \leq \omega + 1$. Moreover, if $|N(U)| = \omega + 1$, then no vertex of U is adjacent to all of the vertices in $N(U)$.*

The following structural lemma is an useful tool for our latter proofs of the main theorem.

Lemma 5. *Let G be an ω-degenerate graph with maximum degree at most Δ and let U be a subset of $V(G)$ such that $\omega \leq |U| \leq 2(\omega-1)$ and $|N(U)| \leq \omega+1$. Suppose that h, k are two integers such that $h \geq 0$, $k \geq \lceil \frac{\Delta+h}{2} \rceil$ and $k \geq 3\omega - 1$. If $|N(U)| \leq \lfloor \frac{h}{2} \rfloor + 1$ or there are sets*

$$Z \subseteq N(U)$$
$$\left\{z_j : 2 \leq j \leq |Z| - \left\lfloor \frac{h}{2} \right\rfloor \right\} \subseteq Z$$

and

$$\left\{u_j : 0 \leq j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\} \subseteq U$$

such that

(1) $|N(z_j) \cap (Z \cup U)| \geq 2j - 1$, and
(2) $|N(u_j) \cap (N(U) - Z)| \leq 2j + 1$,

then there exists $S = \{x_1, ..., x_k\} \subseteq V(G)$ such that $|N(x_i) - S| \leq 2(k-i)+1$ for each $1 \leq i \leq k$.

Proof. We divide the proof into two cases as follows.

Case 1. $|N(U)| \leq \lfloor \frac{h}{2} \rfloor + 1$.

Since $|N(U)| + |U| \leq 3\omega - 1 \leq k$ and $N(U) \cap U = \emptyset$, we can assume that $N(U) = \{x_1, x_2, ..., x_{|N(U)|}\}$ and $U = \{x_{k-|U|+1}, ..., x_k\}$. Let $H_{|N(U)|+1} = G - N[U]$ and let $x_{|N(U)|+1}$ be a vertex of degree at most ω in $H_{|N(U)|+1}$ if this graph is not empty (note that such a vertex always exists since G is ω-degenerate). For $|N(U)|+1 \le i < k-|U|$ (if exists), we let $H_{i+1} = H_i - \{x_i\}$ and let x_{i+1} be a vertex that has degree at most ω in H_{i+1}. At this stage we have assigned k distinct vertices to $S = \{x_1, ..., x_k\}$ and our goal is to check that $|N(x_i) - S| \le 2(k-i)+1$ for each $1 \le i \le k$.

Since $G - S$ is a subgraph of H_i for each $|N(U)|+1 \le i \le k-|U|$ and x_i has degree at most ω in H_i, $|N(x_i) - S| \le \omega \le |U| < 2(k-(k-|U|))+1 \le 2(k-i)+1$ for $|N(U)|+1 \le i \le k-|U|$.

For $1 \le i \le |N(U)|$, since $|N(x_i) \cap U| \ge 1$ and $k \ge \lceil \frac{\Delta+h}{2} \rceil \ge \frac{\Delta+h}{2}$, we conclude that $|N(x_i) - S| \le \Delta - 1 \le 2k-(h+1) \le 2k - (2|N(U)|-1) \le 2(k-i)+1$.

For $k-|U|+1 \le i \le k$, it is clear that $|N(x_i) - S| = 0 \le 2(k-i)+1$ since $N(U) \subseteq S$.

Case 2. $|N(U)| > \lfloor \frac{h}{2} \rfloor + 1$.

If $|Z| \le \lfloor \frac{h}{2} \rfloor$, then add vertices from $N(U) - Z$ to Z until $|Z| = \lfloor \frac{h}{2} \rfloor + 1$. This operation still preserves (1) and (2) and thus we can proceed the proof using this updated set Z. Therefore, we can assume, in advance, that $|Z| \ge \lfloor \frac{h}{2} \rfloor + 1$.

Let $Z = \{x_1, \cdots, x_{\lfloor h/2 \rfloor + 1}, x_{\lfloor h/2 \rfloor + 2}, \cdots, x_{|Z|}\}$, where $x_i = z_{i - \lfloor h/2 \rfloor}$ for $\lfloor \frac{h}{2} \rfloor + 2 \le i \le |Z|$ (if such i exists), and let $U = \{x_{k-|U|+1}, \cdots, x_{k - \lceil (|N(U)-Z|-1)/2 \rceil}, x_{k - \lceil (|N(U)-Z|-1)/2 \rceil + 1}, \cdots, x_k\}$, where $x_i = u_{k-i}$ for $k - \lceil \frac{|N(U)-Z|-1}{2} \rceil < i \le k$ (if such i exists). Since $(k-|U|) - |Z| \ge k - (|U| + |N(U)|) \ge k - (3\omega - 1) \ge 0$, Z and U are well represented by the above two sets as they satisfy the condition $Z \cap U = \emptyset$. Next, we complete the set S by choosing x_i for $|Z| < i \le k - |U|$ (if such i exists) from $G - (Z \cup U)$ by the w-degeneracy of G such that $|N(x_i) - S| \le w \le |U| < 2(k-(k-|U|))+1 \le 2(k-i)+1$. The detailed analysis of the step-by-step choices can be proceeded similarly as what we have done in Case 1, so we do not repeat it here. In the following, we remain to check that $|N(x_i) - S| \le 2(k-i)+1$ for each $1 \le i \le |Z|$ and for each $k-|U|+1 < i \le k$. Note that $Z \cup U \subset S$.

For $1 \le i \le \lfloor \frac{h}{2} \rfloor + 1$,

$$\begin{aligned} |N(x_i) - S| &\le |N(x_i) - (Z \cup U)| = |N(x_i)| - |N(x_i) \cap (Z \cup U)| \\ &\le \Delta - |N(x_i) \cap (Z \cup U)| \le (2k - h) - |N(x_i) \cap (Z \cup U)| \\ &\le 2k - h - 1 \le 2k - (2i-2) - 1 = 2(k-i)+1. \end{aligned}$$

Recall that $x_i \in Z \subseteq N(U)$ in this case and thus $N(x_i) \cap U \ne \emptyset$.

For $\lfloor \frac{h}{2} \rfloor + 1 < i \le |Z|$, since $x_i = z_{i - \lfloor h/2 \rfloor}$, we conclude by (1) that

$$\begin{aligned} |N(x_i) - S| &= |N(z_{i-\lfloor h/2 \rfloor}) - S| \\ &\le |N(z_{i-\lfloor h/2 \rfloor}) - (Z \cup U)| = |N(z_{i-\lfloor h/2 \rfloor})| - |N(z_{i-\lfloor h/2 \rfloor}) \cap (Z \cup U)| \\ &\le \Delta - \left(2\left(i - \left\lfloor \frac{h}{2} \right\rfloor\right) - 1\right) \le (2k-h) - (2i-h-1) = 2(k-i)+1. \end{aligned}$$

For $k-|U|+1 < i \le k$, $N(x_i) - S \subseteq N(x_i) \cap (N(U) - Z)$, since $x_i \in U$, $Z \cup U \subseteq S$, and $Z \subseteq N(U)$. If $i \le k - \lceil \frac{|N(U)-Z|-1}{2} \rceil$, then

$$|N(x_i) - S| \le |N(x_i) \cap (N(U) - Z)| \le |N(U) - Z| \le 2(k-i)+1.$$

On the other hand, if $i > k - \lceil \frac{|N(U)-Z|-1}{2} \rceil$, then $x_i = u_{k-i}$ and thus we conclude by (2) that

$$|N(x_i) - S| \le |N(x_i) \cap (N(U) - Z)| = |N(u_{k-i}) \cap (N(U) - Z)| \le 2(k-i)+1.$$

This completes the proof. □

3. Main results

In this section, we present our main theorems as follows.

Theorem 6. *A graph G is equitable tree k-choosable for every*

$$k \ge \max\left\{\left\lceil \frac{\Delta(G) + 2w - 4}{2} \right\rceil, 3\omega - 1\right\}$$

if G has treewidth $\omega \ge 2$.

Proof. Suppose that G is a counterexample to the result with fewest number of vertices. It follows that $|G| \ge 3\omega - 1$ (otherwise it is possible to color the vertices of G from their lists so that they receive $|G|$ distinct colors because $k \ge 3\omega - 1 > |G|$), and $G - S$ is equitable tree k-choosable for any set $S = \{x_1, \cdots, x_k\} \subseteq V(G)$ of k distinct vertices (this is because that $G - S$ has less vertices than G and thus $G - S$ is not a counterexample by the minimality of G). Note that graphs with treewidth ω are clearly ω-degenerate. By Lemma 4, there is a subset $U \subseteq V(G)$ such that $\omega \le |U| \le 2(\omega - 1)$ and $|N(U)| \le \omega + 1$.

In the following, we claim that

$$\text{there exists } S = \{x_1, ..., x_k\} \subseteq V(G) \text{ such that } |N(x_i) - S| \le 2(k-i)+1 \text{ for each } 1 \le i \le k. \tag{$*$}$$

Therefore, G is equitable tree k-choosable by Lemma 3, contradicting the fact that G is a counterexample.

Let $h = 2\omega - 4$. We may assume that $|N(U)| \ge \lfloor \frac{h}{2} \rfloor + 2 = \omega$, because otherwise ($*$) holds by Lemma 5. We now consider two major cases.

Case 1. *$N(U)$ contains a vertex, say z_2, such that it has at least three neighbors in U.*

Note that $\omega \le |N(U)| \le \omega + 1$. If $|N(U)| = \omega$, then let $Z = N(U)$. If $|N(U)| = \omega + 1$, then let $Z = N(U) - \{z_0\}$, where $z_0 \in N(U)$ and $z_0 \ne z_2$. In any of the above subcases,

$$Z \supseteq \{z_2\} = \left\{ z_j : 2 \le j \le |Z| - \left\lfloor \frac{h}{2} \right\rfloor \right\}$$
$$U \supseteq \emptyset = \left\{ u_j : 0 \le j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\}$$

and

$$|N(z_2) \cap (Z \cup U)| \ge |N(z_2) \cap U| \ge 3.$$

Therefore, ($*$) holds by Lemma 5.

Case 2. *Every vertex in $N(U)$ has at most two neighbors in U.*

Since $|N(U)| \le \omega + 1$ and $|U| \ge \omega$, $\frac{e(U,N(U))}{|U|} \le \frac{2|N(U)|}{|U|} \le \frac{2(\omega+1)}{\omega} < 3$. This implies that there exists $u_0 \in U$ such that $|N(u_0) \cap N(U)| \le 2$. Let $z_0 \in N(U)$ be a neighbor of u_0 if $N(u_0) \cap N(U) \ne \emptyset$, or let $z_0 \in N(U)$ be an arbitrary vertex otherwise.

If $|N(U)| = \omega$, then let $Z = N(U) - \{z_0\}$. If $|N(U)| = \omega + 1$, then let $Z = N(U) - \{z_0, z_1\}$, where $z_1 \in N(U) \setminus N(u_0)$. Note that such z_1 exists because $|N(u_0) \cap N(U)| \le 2$ and $|N(U)| = \omega + 1 \ge 3$. In each of the above subcases,

$$Z \supseteq \emptyset = \left\{ z_j : 2 \le j \le |Z| - \left\lfloor \frac{h}{2} \right\rfloor \right\}$$
$$U \supseteq \{u_0\} \supseteq \left\{ u_j : 0 \le j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\}$$

and

$$|N(u_0) \cap (N(U) - Z)| \le 1 \text{ (note that } N(u_0) \cap (N(U) - Z) \subseteq \{z_0\}).$$

Therefore, ($*$) holds by Lemma 5. □

Theorem 7. *A graph G is equitable tree k-choosable for every*

$$k \ge \max\left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, 3\omega - 1 \right\}$$

if G has treewidth $\omega \in \{2, 3, 4\}$ and $K_{3,3}$ is not its topological minor.

Proof. Choose the minimal (in terms of the number of vertices) counterexample G to the result and the same U as the one in the beginning of the proof of Theorem 6. Our final goal is to prove ($*$) again.

Case 1. *$N(U)$ contains a vertex, say z_2, such that it has at least three neighbors in U.*

Recall that $\omega \le |U| \le 2(\omega - 1)$ and $|N(U)| \le \omega + 1$.

If $|N(U)| \le 2$, then let $Z = N(U)$.

If $|N(U)| = 3$, then let $Z = N(U) - \{z_0\}$, where $z_0 \in N(U)$ and $z_0 \ne z_2$.

If $|N(U)| = 4$ (this case implies $\omega \ge 3$), then choose two distinct vertices $a, b \in N(U) - \{z_2\}$ and a vertex $u_0 \in U$ such that $|N(u_0) \cap \{a, b\}| \le 1$, and let $Z = N(U) - \{a, b\}$. Actually, such a, b and u_0 exist. If we fail to find them, then every vertex in U is adjacent to every vertex in $N(U) - \{z_2\}$. Since $|U| \ge \omega \ge 3$ and $|N(U) - \{z_2\}| = 3$, we find a copy of $K_{3,3}$ in G, contradicting the fact that G does not contain a $K_{3,3}$ as a topological minor.

If $|N(U)| = 5$ (this case implies $\omega = 4$), then choose three distinct vertices $a, b, c \in N(U) - \{z_2\}$ and a vertex $u_0 \in U$ such that $|N(u_0) \cap \{a, b, c\}| \le 1$, and let $Z = N(U) - \{a, b, c\}$. Actually, such a, b, c and u_0 exist. If we fail to find them, then every vertex in U is adjacent to at least three vertices in $N(U) - \{z_2\}$.

Let $\{v_1, v_2, v_3\} \subseteq N(z_2) \cap U$ and let $N(U) = \{z_2, s_1, s_2, s_3, s_4\}$. By Lemma 4 and the above conclusion, any vertex in $N(z_2) \cap U$ is adjacent to exactly three vertices in $\{s_1, s_2, s_3, s_4\}$. Assume, without loss of generality, that $N(v_1) \cap N(U) \setminus \{z_2\} =$

$\{s_1, s_2, s_3\}$. Since G does not contain $K_{3,3}$ as a subgraph, either $N(v_2) \cap N(U) \setminus \{z_2\} \neq \{s_1, s_2, s_3\}$ or $N(v_3) \cap N(U) \setminus \{z_2\} \neq \{s_1, s_2, s_3\}$, and we assume the former one. In such a case, we can assume that $N(v_2) \cap N(U) \setminus \{z_2\} = \{s_1, s_2, s_4\}$ by symmetry, which follows that $\{s_1, s_2\} \nsubseteq N(v_3) \cap N(U)$ (otherwise G contains a copy of $K_{3,3}$). By the symmetry of s_1 and s_2 at this stage, we assume that $N(v_3) \cap N(U) \setminus \{z_2\} = \{s_1, s_3, s_4\}$.

Let $v_4 \in U \setminus \{v_1, v_2, v_3\}$ (note that $|U| \geq \omega = 4$, and we do not mind whether v_4 is in $N(z_2)$ or not). Suppose first that $s_2 \in N(v_4)$. Since v_4 is adjacent to at least three vertices in $\{s_1, s_2, s_3, s_4\}$, either $s_3 \in N(v_4)$ or $s_4 \in N(v_4)$. If $s_3 \in N(v_4)$, then $s_2 v_4 s_3 v_3$ is a path from s_2 to v_3, and if $s_4 \in N(v_4)$, then $s_2 v_4 s_4 v_3$ is a path from s_2 to v_3. In each case, $\{v_1, v_2, v_3\} \subseteq N(z_2) \cap N(s_1)$, $\{v_1, v_2\} \subseteq N(s_2)$ and there is a path from s_2 to v_3 that does not pass the vertices among v_1, v_2, z_2 and s_1. So, there is a $K_{3,3}$-subdivision in G, a contradiction. Therefore we conclude that $s_2 \notin N(v_4)$ and thus $N(v_4) \cap N(U) \setminus \{z_2\} = \{s_1, s_3, s_4\}$. In this case, $\{s_1, s_3, s_4\} \subseteq N(v_3) \cap N(v_4)$, $\{s_1, s_4\} \subseteq N(v_2)$ and there is a path, say $v_2 z_2 v_1 s_3$ from v_2 to s_3. This results in a $K_{3,3}$-subdivision in G, a contradiction.

In each of the above subcases, $|Z| \leq 2$, $|N(U) - Z| \leq 1$ if $|N(U)| \leq 3$, and $2 \leq |N(U) - Z| \leq 3$ if $|N(U)| \geq 4$, which implies

$$Z \supseteq \{z_2\} \supseteq \left\{ z_j : 2 \leq j \leq |Z| \right\}$$
$$U \supseteq \emptyset = \left\{ u_j : 0 \leq j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\} \text{ if } |N(U)| \leq 3$$
$$U \supseteq \{u_0\} = \left\{ u_j : 0 \leq j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\} \text{ if } |N(U)| \geq 4.$$

Since we already have

$$|N(z_2) \cap (Z \cup U)| \geq |N(z_2) \cap U| \geq 3$$

and have proved (by the choice of u_0)

$$|N(u_0) \cap (N(U) - Z)| \leq 1$$

for the case $|N(U)| \geq 4$, we conclude that $(*)$ holds by Lemma 5.

Case 2. *Every vertex in $N(U)$ has at most 2 neighbors in U.*

Since $\frac{e(U, N(U))}{|U|} \leq \frac{2|N(U)|}{|U|} \leq \frac{2(w-1)}{w} < 3$, there is a vertex $u_0 \in U$ such that $|N(u_0) \cap N(U)| \leq 2$. If $|N(u_0) \cap N(U)| = 2$, then let $Z = \{z_0\}$, where $z_0 \in N(U)$ is a neighbor of u_0. If $|N(u_0) \cap N(U)| \leq 1$, then let $Z = \{z_0\}$, where z_0 is an arbitrary vertex chosen from $N(U)$. In each subcase, we have $|Z| = 1$, $|N(U)| \leq w + 1$ and thus $\lceil \frac{|N(U) - Z| - 1}{2} \rceil \leq \lceil \frac{w-1}{2} \rceil$.

If $\omega \in \{2, 3\}$, then

$$Z \supseteq \emptyset = \left\{ z_j : 2 \leq j \leq |Z| \right\}$$
$$U \supseteq \{u_0\} \supseteq \left\{ u_j : 0 \leq j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\}.$$

Since $|N(u_0) \cap (N(U) - Z)| \leq 1$ by the choices of u_0 and Z, $(*)$ holds by Lemma 5.

If $\omega = 4$, then

$$\frac{e(U - \{u_0\}, N(U))}{|U| - 1} \leq \frac{2N(U)}{|U| - 1} \leq \frac{2(w+1)}{w-1} = \frac{10}{3} < 4,$$

which implies that there is a vertex $u_1 \in U - \{u_0\}$ such that $|N(u_1) \cap N(U)| \leq 3$. Note that

$$Z \supseteq \emptyset = \left\{ z_j : 2 \leq j \leq |Z| \right\}$$
$$U \supseteq \{u_0, u_1\} \supseteq \left\{ u_j : 0 \leq j < \left\lceil \frac{|N(U) - Z| - 1}{2} \right\rceil \right\}.$$

Since $|N(u_0) \cap (N(U) - Z)| \leq 1$ by the choices of u_0 and Z, and $|N(u_1) \cap (N(U) - Z)| \leq |N(u_1) \cap N(U)| \leq 3$, we conclude that $(*)$ holds by Lemma 5. □

From Theorems 6 and 7, we immediately deduce the following corollaries.

Corollary 8. *If G has treewidth $\omega \geq 2$, then*

$$\rho_l^{=}(G) \leq \begin{cases} \lceil \frac{\Delta(G)}{2} \rceil + \omega - 2, & \text{if } \Delta(G) \geq 4\omega + 1; \\ 3w - 1, & \text{if } \Delta(G) \leq 4\omega. \end{cases}$$

Corollary 9. *If G has treewidth $\omega \in \{2, 3, 4\}$ and $K_{3,3}$ is not its topological minor, then*

$$\rho_l^{=}(G) \leq \begin{cases} \lceil \frac{\Delta(G)}{2} \rceil, & \text{if } \Delta(G) \geq 6\omega - 3; \\ 3w - 1, & \text{if } \Delta(G) \leq 6\omega - 4. \end{cases}$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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