Total Coloring of Outer-1-planar Graphs: The Cold Case

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Abstract. A graph is outer-1-planar if it has a drawing in the plane so that its vertices are on the boundary face and each edge is crossed at most once. Zhang (2013) proved that the total chromatic number of every outer-1-planar graph with maximum degree $\Delta \geq 5$ is $\Delta + 1$, and showed that there are graphs with maximum degree 3 and total chromatic number 5. For outer-1-planar graphs with maximum degree 4, Zhang (2017) confirmed that its total chromatic number is at most 5 if it admits an outer-1-planar drawing in the plane so that any two pairs of crossing edges share at most one common end vertex. In this paper, we prove that the total chromatic number of every Anicop graph with maximum degree 4 is at most 5, where an Anicop graph is an outer-1-planar graph that admits a drawing in the plane so that if there are two pairs of crossing edges sharing two common end vertices, then any of those two pairs of crossing edges would not share any end vertex with some other pair of crossing edges. This result generalizes the one of Zhang (2017) and moves a step towards the complete solving of the cold case.

Keywords: Outer-1-planar graph · Total coloring · Maximum degree

1 Introduction

A *total k-coloring* of a graph G is an assignment of k colors to all vertices and edges of G so that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of a graph G is the minimum integer k so that G has a total k-coloring. In any total coloring of a graph G with maximum degree Δ, it is easy to see that we shall use $\Delta + 1$ colors to color the vertex of degree Δ and its incident edges. This implies that $\chi''(G) \geq \Delta(G) + 1$ for every graph G. On the other hand, looking for a general upper bound in terms of $\Delta(G)$ for $\chi''(G)$ seems interesting and challenging. Actually, Behzad [3] and Vizing [10] independently conjectured at least fifty years ago that $\chi''(G) \leq \Delta(G) + 2$ for

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every graph G. This conjecture was now confirmed for graphs with maximum degree at most 3 by Rosenfeld [7] and Vijayaditya [9], 4 and 5 by Kostochka [5,6], and verified for planar graphs with maximum degree 7 by Sanders and Zhao [8], 8 by Andersen [1], and at least 9 by Borodin [4]. However, the conjecture itself is still quite open, even for planar graphs with maximum degree 6.

In the literature, there are some well-established subclasses of planar graphs including

- *outerplanar graphs:* graphs that can be drawn in the plane so that all the vertices are on the outer face (equivalently, graphs that do not contain $K_{2,3}$ or K_4 as a minor);
- *series-parallel graphs:* graphs that do not contain K_4 as a minor;
- *outer-1-planar graphs:* graphs that can be drawn in the plane so that all the vertices are on the outer face and each edge is crossed at most once.

Outplanar graphs and series-parallel graphs are planar due to the well-known Wagner's theorem which says that a graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor. But the planarity of outer-1-planar graphs is not trivially proved—such a proof was given by Auer et al. [2], who also pointed out that the class of outer-1-planar graphs is not minor-closed. A graph is *quasi-Hamiltonian* if each of its block is Hamiltonian. Zhang, Liu, and Wu [19] showed that the intersection of the class of quasi-Hamiltonian outer-1-planar graphs and the class of series-parallel graphs is indeed the class of outerplanar graphs.

Zhang, Zhang, and Wang [20] showed in 1988 that the $\chi''(G) = \Delta(G)+1$ for every outerplanar graph with maximum degree at least 3. The same result also holds for series-parallel graphs, which was proved in 2004 by Wu and Hu [12]. In 2011, Zhang and Liu [18] proved the total coloring conjecture for outer-1-planar graphs, and moreover, showed that $\chi''(G) = \Delta(G) + 1$ for every outer-1-planar graph with maximum degree at least 5, and this result was later generalized to its list version by Zhang [13] in 2013. In [13,18], the authors also pointed out that there are outer-1-planar graphs G with $\Delta(G) = 3$ and $\chi''(G) = 5$, and whether outer-1-planar graphs G with $\Delta(G) = 4$ satisfy $\chi''(G) = \Delta(G) + 1 = 5$ is unknown.

For this cold case, Zhang [15] considered the *Nicop graphs*, i.e., outer-1-plane graphs so that any two pairs of crossing edges share at most one common end vertex. Here, an *outer-1-plane graph* is a drawing of outer-1-planar graph in the plane so that its outer-1-planarity is preserved and the number of crossings is as small as possible. Zhang [15] proved the following

Theorem 1 [15]. *If G is a Nicop graph with $\Delta(G) = 4$, then $\chi''(G) = 5$.*

In this paper, we aim to generalize this result to a larger class of graphs $\mathcal{G}$. Here, a graph G belongs to $\mathcal{G}$ if and only if

- G is an outer-1-plane graph, and
- if there are two pairs of crossing edges sharing two common end vertices, then any of those two pairs of crossing edges would not share any end vertex with some other pair of crossing edges.

From now on, a graph $G \in \mathcal{G}$ is called an *outer-1-plane graph with almost-near-independent crossings*, or an *Anicop graph* for short. Our main result is stated as follows:

Theorem 2. *If G is an Anicop graph with $\Delta(G) = 4$, then $\chi''(G) = 5$.*

Since Nicop graphs are Anicop graphs, Theorem 2 implies Theorem 1. Actually, we believe that the same conclusion holds for every outer-1-planar graph with maximum degree 4, so we end this section with the following conjecture.

Conjecture 3. If G is an outer-1-planar graph with $\Delta(G) = 4$, then $\chi''(G) = 5$.

2 Reducibilities: The Proof of Theorem 2

From now on, when we mention an outer-1-planar graph G, we always refer to its *outer-1-planar diagram*, i.e, a drawing of G in the plane so that the outer-1-planarity of G is preserved and this drawing has the minimum number of crossings among all such outer-1-planar drawings.

To begin with, we define *base graphs* Π_i^1 and Π_i^2 with $1 \leq i \leq 3$ by Fig. 1. In each picture of this figure besides Π_1^1, all vertices are lying consecutively in an outer-1-planar diagram G as where they are drawn in that picture (i.e., the boundary edges incident with the black vertices in that picture form a sub-drawing of the outer-face of G). The two white vertices in each picture of Fig. 1 are called the *handles*.

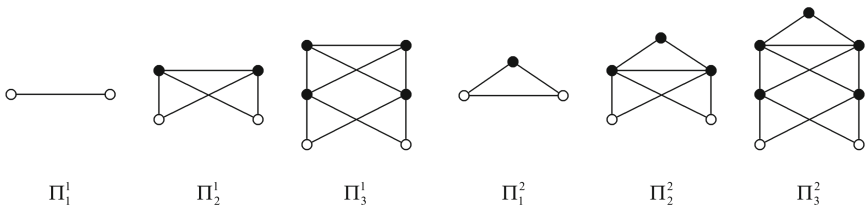

Fig. 1. Base graphs

Given two base graphs, say Π_i^j with handles u, v, and $\Pi_{i'}^{j'}$ with handles u', v', we have two operations:

- $\mathbf{\Pi_i^j \circ \Pi_{i'}^{j'}}$ Identifying v with v', see Fig. 2, and in the resulting graph let the degree of the vertex w corresponding v and v' be the number of edges incident with it in this partial drawing. The vertices u and u' in the resulting graph are called *linking handles*;
- $\mathbf{\Pi_i^j \otimes \Pi_{i'}^{j'}}$ Adding edges vv', uv' and $u'v$ so that uv' crosses $u'v$, see Fig. 2, and in the resulting graph let the degree of the vertex v or v' be the number of edges incident with it in this partial drawing. The vertices u and u' in the resulting graph are called *crossed-linking handles*.

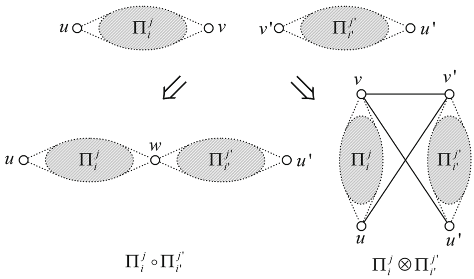

Fig. 2. Two operations generated by Π_i^j and $\Pi_{i'}^{j'}$

Note that $\Pi_i^j \circ \Pi_{i'}^{j'}$ or $\Pi_i^j \otimes \Pi_{i'}^{j'}$ is still an outer-1-planar diagram. We prove the following

Theorem 4. *Every* 2*-connected Anicop graph with maximum degree at most* 4 *contains one of the configurations among*

(C1) *a vertex* u *of degree* 2 *adjacent to a vertex* v *of degree at most* 3*;*
(C2) *a cycle of length* 4 *with two nonadjacent vertices of degree* 2*;*
(C3) *a triangle* uvw *with* $d(v) = 2$ *and* u *adjacent to a vertex* x *of degree* 2*;*
(C4) $\Pi_1^1 \otimes \Pi_1^2$*;*
(C5) $\Pi_1^2 \otimes \Pi_1^2$*;*
(C6) Π_3^1*;*
(C7) Π_2^1 *or* Π_2^2 *or* Π_3^2*, with a handle of degree at most* 3*;*
(C8) Π_2^1 *or* Π_2^2 *or* Π_3^2*, with a handle adjacent to a vertex of degree* 2*;*
(C9) Π_2^1 *or* Π_2^2 *or* Π_3^2*, with the two handles being adjacent;*
(C10) $\Pi_2^1 \circ \Pi_2^1$*, or* $\Pi_2^1 \circ \Pi_2^2$*, or* $\Pi_2^2 \circ \Pi_2^2$*;*
(C11) $\Pi_1^1 \otimes \Pi_2^1$*, or* $\Pi_1^1 \otimes \Pi_2^2$*, or* $\Pi_1^2 \otimes \Pi_2^1$*, or* $\Pi_1^2 \otimes \Pi_2^2$*.*

In this section, we apply Theorem 4 to prove the following theorem, which is slightly stronger than Theorem 2.

Theorem 5. *If* G *is an Anicop graph with maximum degree at most* 4*, then* $\chi''(G) \leq 5$*.*

Proof. (sketch). Let G be a counterexample with the minimum number of vertices. Clearly, G is 2-connected. It is sufficient to prove that G does not contain the configuration (Ci) for each $1 \leq i \leq 11$, contradicting Theorem 4. The proof of each item proceeds as follows. First, we construct a graph G' with $\Delta(G') \leq 4$ and $|G'| < |G|$ via removing some vertices appearing in (Ci) from G (we suppose, to the contrary, that (Ci) occurs), and after that, adding non-crossed edges inside the outer boundary (this operation applies sometimes, not always). Next, we prove that a total 5-coloring of G' can be extended to a total 5-coloring of G (sometimes the recoloring shall be involved). Note that if we remove vertices from an Anicop graph, or add non-crossed edges inside the outer boundary of an Anicop graph, the resulting graph is still an Anicop graph. So, by the minimality of G, G' is total-5-colorable, which implies $\chi''(G) \leq 5$, a contradiction.

3 Structures: The Proof of Theorem 4

3.1 Preliminaries

We first review some useful notations that were often used in many papers including [13–19].

Given a 2-connected Anicop graph G, by $v_1, v_2, \ldots, v_{|G|}$ we denote the vertices of G that lie in a clockwise sequence on the outer boundary. Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \ldots, v_j\}$ and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j] \backslash \{v_i, v_j\}$, where the subscripts are taken modulo $|G|$. Set $\mathcal{V}[v_i, v_i] = V(G)$ and $\mathcal{V}(v_i, v_i) = V(G) \setminus \{v_i\}$.

A vertex set $\mathcal{V}[v_i, v_j]$ is a *non-edge* if $j = i + 1 \pmod{|G|}$ and $v_i v_j \notin E(G)$, and is a *path* if $v_i v_{i+1} \cdots v_j$ (the subscripts are taken modulo $|G|$) forms a path. An edge $v_i v_j$ is a *chord* if $j = i + 1 \pmod{|G|}$. By $\mathcal{C}[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$.

Let $v_i v_j$ and $v_k v_l$ be two chords in an Anicop graph G so that $v_i v_j$ crosses $v_k v_l$ and v_i, v_k, v_j and v_l lie in a clockwise sequence on the outer boundary of G. We say that $v_i v_j$ *co-crosses* $v_k v_l$, and $v_i v_j, v_k v_l$ are *co-crossed chords*, if $v_i v_k, v_k v_j, v_j v_l \in E(G)$, $l - j = k - i = 1 \pmod{|G|}$, and $j - k = 1$ and $d(v_k) = d(v_j) = 3$ (see the 1^{st} picture of Fig. 3), or $j - k = 2$, $v_k v_{k+1}, v_{k+1} v_j \in E(G)$, $d(v_k) = d(v_j) = 4$ and $d(v_{k+1}) = 2$ (see the 2^{nd} picture of Fig. 3).

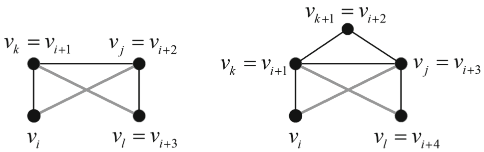

Fig. 3. $v_i v_j$ co-crosses $v_k v_l$

By the partial drawings of G as showed in Fig. 4, we define different types of *clusters* that will be frequently used in the following arguments. In any picture of this figure, vertices are all distinct, the edges drawn as crossed have to be crossed in G, and the curving edges are chords. Note that any graph in Fig. 4 contains a base graph as a subgraph.

We call H an *I-cluster* in G if H is either a left I^1-cluster, or a right I^1-cluster, or a left I^2-cluster, or a right I^2-cluster. The *II-cluster, III-cluster* and *IV-cluster* are defined similarly. The *width* of a cluster is the value of $|\mathcal{V}[v_L, v_R]|$, where L and R are the subscripts of the far left vertex and the far right vertex on the outer boundary (see in a clockwise direction from left to right). For convenience, we use $\{v_L, v_R\}_1$, $\{v_L, v_R\}_2$, $\{v_L, v_R\}_3$, and $\{v_L, v_R\}_4$ to represent a I-cluster, II-cluster, III-cluster, and IV-cluster, respectively. For example, the width of the left I^1-cluster $\{v_j, v_{i+3}\}_1$ is $(i + 3) - j + 1 = i - j + 4 \pmod{|G|}$, and the width of the right I^1-cluster $\{v_i, v_j\}_1$ is $j - i + 1 \pmod{|G|}$. Note that for a cluster, say a III-cluster for example, the left-type can be transferred to the right-type just by taking inversion.

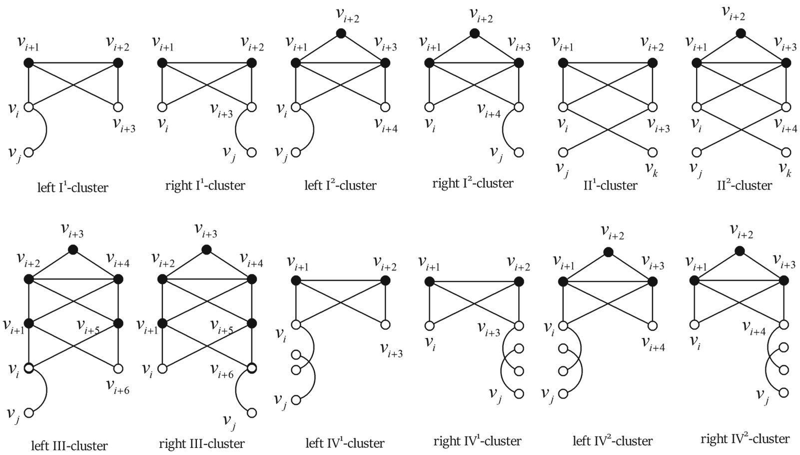

Fig. 4. The definitions of different types of clusters

The following three lemmas were originally proved for outer-1-plane graphs or Nicop graphs, and there is no double that their proofs also work for Anicop graphs.

Lemma 6 [19, Claim 1]. *Let v_i and v_j be vertices of a* 2*-connected outer*-1*-plane graph (or Anicop graph) G. If there is no crossed chord in $\mathcal{C}[v_i, v_j]$ and no edge between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then $\mathcal{V}[v_i, v_j]$ is either a non-edge or a path.*

Lemma 7. *Let v_iv_j and v_kv_l with $i < k < j < l$ be two crossed chords in a* 2*-connected outer*-1*-plane graph (or Anicop graph) G with $\Delta(G) \leq 4$ so that v_iv_j crosses v_kv_l and there is no other pair of crossed chords contained in the drawing induced by $\mathcal{V}[v_i, v_l]$. We have*

(1) *at most one of $\mathcal{V}[v_i, v_k], \mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is a non-edge* [19, Claim 3]*;*
(2) *if one of $\mathcal{V}[v_i, v_k], \mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is a non-edge, then G has a subgraph isomorphic to one of the configurations among* (C1)*,* (C2)*, and* (C3) [19, Claims 2 and 4]*;*
(3) *if all of $\mathcal{V}[v_i, v_k], \mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are paths, then either v_iv_j co-crosses v_kv_l in G, or G has a subgraph isomorphic to one of the configurations among* (C1)*,* (C2)*,* (C3)*,* (C4)*, and* (C5) [19, Claims 2 and 5].

Lemma 8 [15, Lemma 2.2]. *Let $\mathcal{V}[v_i, v_j]$ with $j - i \geq 3$ be a path in a* 2*-connected Nicop graph (or Anicop graph) G with $\Delta(G) \leq 4$. If there is no crossed chord in $\mathcal{C}[v_i, v_j]$ and no edges between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then G contains* (C1) *or* (C2)*.*

3.2 Proofs by Combinatorial Analyses

Let G be a 2-connected Anicop graph with $\Delta(G) \leq 4$. If G does not contain a crossing, then G is an outerplane graph, and the following is immediate.

Lemma 9 [11, Corollary 2.5]. *If G does not contain a crossing, then it contains* (C1) *or* (C3).

If G contains a crossing, then choose one pair of crossed chords v_iv_j and v_kv_l such that v_iv_j crosses v_kv_l, and $\mathcal{C}[v_i, v_l]$ contains no other crossed chord besides v_iv_j and v_kv_l. Applying Lemmas 6 and 7, one can conclude that v_iv_j co-crosses v_kv_l unless G contains one of the configurations among (C1), (C2), (C3), (C4), and (C5).

Hence in the following we assume that v_iv_j co-crosses v_kv_l with $1 = i < k < j < l$, and G does not contain any configurations among (C6)—(C11) (otherwise we win).

Since (C7) and (C7) are absent, $d(v_l) \geq 4$ and thus there is a chord v_lv_s with $l < s \leq n$. In this case the drawing induced by $\mathcal{V}[v_i, v_l]$ and v_lv_s is a I-cluster $\{v_i, v_s\}_1$. We make the following assumption, otherwise we can choose the shorter one I-cluster to replace $\{v_i, v_s\}_1$.

Assumption 1. *$\{v_i, v_s\}_1$ is the shortest I-cluster contained in the drawing induced by $\mathcal{V}[v_i, v_s]$.*

Lemma 10. *Suppose that v_a and v_b are two vertices with $l \leq a < b \leq s$. If there is no edge between $\mathcal{V}(v_a, v_b)$ and $\mathcal{V}(v_b, v_a)$, and there is a pair of chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $a \leq i' < k' < j' < l' \leq b$, then there is a II-cluster contained in the drawing induced by $\mathcal{V}[v_a, v_b]$ unless $\{i', l'\} = \{a, b\}$ and $v_{i'}v_{j'}$ co-crosses $v_{k'}v_{l'}$.*

Proof. Suppose that $v_{i'}v_{j'}$ does not co-cross $v_{k'}v_{l'}$. By Lemmas 6 and 7, there is another pair of crossed chords besides $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$, say $v_{i''}v_{j''}$ and $v_{k''}v_{l''}$ with $i' \leq i'' < k'' < j'' < l' \leq l'$, in $\mathcal{C}[v_{i'}, v_{l'}]$. We choose $v_{i''}v_{j''}$ and $v_{k''}v_{l''}$ carefully so that there is no other pair of crossed chords in $\mathcal{C}[v_{i''}, v_{j''}]$ besides them. This implies that $v_{i''}v_{j''}$ co-crosses $v_{k''}v_{l''}$, because otherwise one of the configurations among (C1), (C2), (C3), (C4), and (C5) would appear by Lemmas 6 and 7. Since $\{i'', l''\} \neq \{i', l'\}$, $\{i'', l''\} \neq \{a, b\}$. By the absences of (C7) and (C9), and by Assumption 1, there are chords $v_{i''}v_{t''}$ and $v_{l''}v_{s''}$ with $l'' < t'' \leq b$ and $a \leq s'' < i''$. Therefore, a II-cluster $\{v_{s''}, v_{t''}\}_2$ is found in the drawing induced by $\mathcal{V}[v_a, v_b]$.

On the other hand, we assume that $v_{i'}v_{j'}$ co-crosses $v_{k'}v_{l'}$ but $\{i', l'\} \neq \{a, b\}$. Actually, one can see that $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ play the same role as $v_{i''}v_{j''}$ and $v_{k''}v_{l''}$ in the previous paragraph. Therefore, we can again find a II-cluster in the drawing induced by $\mathcal{V}[v_a, v_b]$.

In the following proofs, we distinguish two major cases.

The First Case: v_lv_s is Non-crossed

Lemma 11. *There exists a II-cluster contained in the drawing induced by $\mathcal{V}[v_l, v_s]$.*

Proof. If there is no crossed chord in $\mathcal{C}[v_l, v_s]$, then $\mathcal{V}[v_l, v_s]$ is a path by Lemma 6. If $s - l = 2$, then $d(v_{l+1}) = 2$ and (C8) appears. If $s - l \geq 3$, then (C1) or (C2) appears by Lemma 8. Hence there is a pair of crossed chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $l \leq i' < k' < j' < l' \leq s$, and by Lemma 10, there is a II-cluster contained in the drawing induced by $\mathcal{V}[v_l, v_s]$ unless $\{i', l'\} = \{l, s\}$ and $v_{i'}v_{j'}$ co-crosses $v_{k'}v_{l'}$, which case would not occur because otherwise $d(v_l) \geq 5$.

By Lemma 11, there are chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $l < i' < k' < j' < l' < s$ so that $v_{i'}v_{j'}$ co-crosses $v_{k'}v_{l'}$, and moreover, there are chords $v_{i'}v_{t'}$ with $l' < t' \leq s$ and $v_{l'}v_{s'}$ with $l \leq s' < i'$. In other words, this structure is indeed a II-cluster $\{v_{s'}, v_{t'}\}_2$. Typically, the following assumption is natural.

Assumption 2. *$\{v_{s'}, v_{t'}\}_2$ is the shortest II-cluster contained in the drawing induced by $\mathcal{V}[v_{s'}, v_{t'}]$.*

Lemma 12. *The drawing induced by $\mathcal{V}[v_{s'}, v_{t'}]$ has a copy of Π_3^2 with handles $v_{s'}$ and $v_{t'}$.*

Proof. By Lemma 10 and by the fact that $\Delta(G) \leq 4$, there is no crossed chord in $\mathcal{C}[v_{l'}, v_{t'}]$, because otherwise we would find in the drawing induced by $\mathcal{V}[v_{s'}, v_{t'}]$ a shorter II-cluster than $\mathcal{V}[v_{s'}, v_{t'}]$, contradicting Assumption 2. By Lemma 6, $\mathcal{V}[v_{l'}, v_{t'}]$ is non-edge or path. If $\mathcal{V}[v_{l'}, v_{t'}]$ is a non-edge, then $d(v_{l'}) = 3$ and (C7) appears. Hence $\mathcal{V}[v_{l'}, v_{t'}]$ is a path. If $t' - l' \geq 3$, then by Lemma 8, G contains (C1) or (C2). If $t' - l' = 2$, then $d(v_{l'+1}) = 2$ and (C8) appears. Hence $t' - l' = 1$ and $v_{l'}v_{t'} \in E(G)$. By symmetry, $i' - s' = 1$ and $v_{s'}v_{i'} \in E(G)$. This implies that the drawing induced by $\mathcal{V}[v_{s'}, v_{t'}]$ contains either Π_3^1 or Π_3^2 with handles $v_{s'}$ and $v_{t'}$. However, Π_3^1 is forbidden in G, so it must be a copy of Π_3^2 with handles $v_{s'}$ and $v_{t'}$.

Since (C7) and (C9) are absent from G, there are chords $v_{t'}v_p$ and $v_{s'}v_q$ with $p \neq s', i'$ and $q \neq t', l'$. Since $s' \neq l$ and $v_{t'}v_p$ cannot cross $v_{s'}v_q$ by the definition of the Anicop graphs, either $t' < p \leq s$ or $l \leq r < s'$. We assume, without loss of the generality, the former, and in this case there is a III-cluster, say $\{v_{s'}, v_p\}_3$, contained in the drawing induced by $\mathcal{V}[v_{s'}, v_p]$. Again, we do the following natural assumption.

Assumption 3. *$\{v_{s'}, v_p\}_3$ is the shortest III-cluster contained in the drawing induced by $\mathcal{V}[v_{s'}, v_p]$.*

Lemma 13. *There is no crossed chord in $\mathcal{C}[v_{t'}, v_p]$.*

Proof. Suppose, to the contrary, that there is a pair of crossed chords there. By Lemma 11, there exists a II-cluster contained in the drawing induced by $\mathcal{V}[v_{t'}, v_p]$. Here one shall note that t' would not be incident with any crossed edge in the drawing induced by $\mathcal{V}[v_{t'}, v_p]$ by the definition of the Anicop graphs. Assume that $\{v_{s''}, v_{t''}\}$ with $t' < s'' < t''$ is the shortest II-cluster contained in the drawing induced by $\mathcal{V}[v_{t'}, v_p]$. By similar arguments as in the proof of Lemma 12, the drawing induced by $\mathcal{V}[v_{s''}, v_{t''}]$ contains a copy of Π_3^2 with handles

$v_{s''}$ and $v_{t''}$. Again, by the absences of (C7) and (C9) and by the definition of the Anicop graphs, there is a chord $v_{t''}v_{p'}$ with $t'' < p' \leq p$ or a chord $v_{s''}v_{q'}$ with $t' \leq q' < s''$. In each case we find in the drawing induced by $\mathcal{V}[v_{t'}, v_p] \subset \mathcal{V}[v_{s'}, v_p]$ a shorter III-cluster than $\{v_{s'}, v_p\}_3$, contradicting Assumption 3.

By Lemmas 6 and 13, $\mathcal{V}[v_{t'}, v_p]$ is a path. If $p - t' = 2$, then $d(v_{t'+1}) = 2$ and (C8) appears. If $p - t' \geq 3$, then (C1) or (C2) appears by Lemma 8. This is the end of the discussions for the first case.

The Second Case: $v_l v_s$ is Crossed. Suppose that $v_l v_s$ is crossed by a chord $v_r v_t$ with $l < r < s$, where $t = i$ is possible. Recall that when Assumption 1 is applied (in the proof of Lemma 10, for example), we actually only use the fact that there is no I-cluster in the drawing induced by $\mathcal{V}[v_l, v_s]$ with width at most $s - l$. Therefore, if we assume that there is no I-cluster in the drawing induced by $\mathcal{V}[v_s, v_t]$ with width at most $t - s$, then Lemma 10 still holds while l is replaced by s and s is replaced by t.

Lemma 14.

(1) *There is no crossed chord in* $\mathcal{C}[v_l, v_r]$*;*
(2) *There is no crossed chord in* $\mathcal{C}[v_r, v_s]$*;*
(3) *If there is no I-cluster in the drawing induced by* $\mathcal{V}[v_s, v_t]$ *with width at most* $t - s$*, then there is no crossed chord in* $\mathcal{C}[v_s, v_t]$*.*

Proof. The proof can be completed by similar arguments as we had presented in Sect. 3.2. We summary the idea for the readers.

Suppose that there is a pair of crossed chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $i' < k' < j' < l'$ in $\mathcal{C}[v_l, v_r]$ (or $\mathcal{C}[v_r, v_s]$, or $\mathcal{C}[v_s, v_t]$). If there is a II-cluster contained in the drawing induced by $\mathcal{V}[v_l, v_r]$ (or $\mathcal{V}[v_r, v_s]$, or $\mathcal{V}[v_s, v_t]$), then we choose one, say $\{v_{s'}, v_{t'}\}_2$, with the shortest width. Next, we prove that the drawing induced by $\mathcal{V}[v_{s'}, v_{t'}]$ has a copy of Π_3^2 with handles $v_{s'}$ and $v_{t'}$ (note that by the definition of the Anicop graphs, $s' \neq l, r, s$), based on which we can find a III-cluster in the drawing induced by $\mathcal{V}[v_l, v_r]$ (or $\mathcal{V}[v_r, v_s]$, or $\mathcal{V}[v_s, v_t]$). Again, choose the shortest III-cluster, say $\{v_{s'}, v_p\}_3$, and we can finally find some configuration that is forbidden in the graph induced by $\mathcal{V}[v_{s'}, v_p]$.

On the other hand, if no II-cluster is contained in the drawing induced by $\mathcal{V}[v_l, v_r]$ (or $\mathcal{V}[v_r, v_s]$, or $\mathcal{V}[v_s, v_t]$), then by Lemma 10, we conclude that $\{i', l'\} = \{l, r\}$ (or $\{i', l'\} = \{r, s\}$, or $\{i', l'\} = \{s, t\}$) and $v_{i'}v_{j'}$ co-crosses $v_{k'}v_{l'}$, which is impossible by the definition of the Anicop graphs.

Lemma 15. $r - l = 1$.

Proof. By Lemmas 6 and 14(1), $\mathcal{V}[v_l, v_r]$ is a non-edge or a path. If $\mathcal{V}[v_l, v_r]$ is a non-edge, then it is trivial that $r - l = 1$. If $\mathcal{V}[v_l, v_r]$ is a path, then by Lemma 8 and the absence of (C8), we also have $r - l = 1$.

Lemma 16. $\mathcal{V}[v_r, v_s]$ *is a path such that* $s - r \leq 2$ *and* $v_r v_s \in E(G)$*. Moreover, if* $s - r = 2$*, then* $v_l v_r \in E(G)$*.*

Proof. By Lemmas 6 and 14(2), $\mathcal{V}[v_r, v_s]$ is a non-edge or a path. If it is a non-edge, then $v_l v_r \in E(G)$ by the 2-connectedness of G. Hence $d(v_r) = 2$ by Lemma 15, and thus (C8) occurs. If $\mathcal{V}[v_r, v_s]$ is a path, then $s-r \leq 2$ by Lemma 8. If $s - r = 1$, then $v_r v_s \in E(G)$, because otherwise $v_l v_r \in E(G)$ and $d(v_r) = 2$ by the 2-connectedness of G and by Lemma 15, which implies that (C8) occurs. If $s - r = 2$, then $d(v_{r-1}) = 2$. Since (C1) is forbidden, $d(v_r) \geq 4$, which implies that $v_l v_r, v_r v_s \in E(G)$.

Lemma 17. $t \neq i = 1$.

Proof. Suppose, to the contrary, that $t = i$. If $s - r = 2$, then by Lemma 16, one can see that the drawing induced by $\mathcal{V}[v_i, v_s]$ contains a copy of $\Pi_1^2 \otimes \Pi_2^1$ or $\Pi_1^2 \otimes \Pi_2^2$ with crossed-linking handles v_i and v_s. If $s - r = 1$, then $v_l v_r \in E(G)$, because otherwise $d(v_r) = 2$. Since $v_i v_r \in E(G)$, (C8) appears. In this case, the drawing induced by $\mathcal{V}[v_i, v_s]$ has a copy of $\Pi_1^1 \otimes \Pi_2^1$ or $\Pi_1^1 \otimes \Pi_2^2$ with crossed-linking handles v_i and v_s. So we say that (C11) occurs.

Until now, we have actually proved the following result, which will be frequently used during the remaining arguments.

Lemma 18. *If $v_i v_j$ co-crosses $v_k v_j$ and $v_l v_s$ is a chord with $i < k < j < l < s$ such that $\{v_i, v_s\}_1$ is the shortest I-cluster contained in the drawing induced by $\mathcal{V}[v_i, v_s]$, then $v_l v_s$ is crossed by a chord $v_r v_t$ so that*

(1) *$s < t \neq i$;*
(2) *$r - l = 1$;*
(3) *$\mathcal{V}[v_r, v_s]$ is a path with $s - r \leq 2$ and $v_r v_s \in E(G)$, and if $s - r = 2$, then $v_l v_r \in E(G)$.*

Lemma 19. *There is a I-cluster in the drawing induced by $\mathcal{V}[v_s, v_t]$ with width at most $t - s$.*

Proof. If the opposite holds, then by Lemmas 6 and 14(3), there is no crossed chord in $\mathcal{C}[v_s, v_t]$, and thus $\mathcal{V}[v_s, v_t]$ is a non-edge or a path. If it is a non-edge, then $s - r = 1$ and $v_r v_s \in E(G)$, because otherwise $d(v_{s-1}) = 2$ and $d(v_s) = 3$, which implies that (C1) occurs. However, if $s - r = 1$ and $v_r v_s \in E(G)$, then $d(v_r) \leq 3$ and $d(v_s) = 2$ by Lemma 15, again implying the appearance of (C1). Hence $\mathcal{V}[v_s, v_t]$ is a path, and by Lemma 8, $t - s \leq 2$.

Suppose that $t - s = 2$. It follows that $d(v_{s+1}) = 2$. If $v_s v_t \in E(G)$, then $s - r = 1$ and $v_r v_s \in E(G)$, because otherwise $v_{s-1} v_s \in E(G)$ and $d(v_{s-1}) = 2$ by Lemma 16, which implies that (C3) appears. Similarly, $v_l v_r \in E(G)$, because otherwise $d(v_r) = 2$ and (C3) occurs again. In this case, the drawing induced by $\mathcal{V}[v_l, v_t]$ has a copy of $\Pi_1^1 \otimes \Pi_1^2$ with crossed-linking handles v_l and v_t, and thus (C11) occurs. On the other hand, if $v_s v_t \notin E(G)$, then $s - r = 2$ because otherwise $d(v_s) = 3$ and (C1) occurs. However, if $s - r = 2$, then $v_r v_s \in E(G)$ and $d(v_{s-1}) = 2$, which implies the appearance of (C3).

Hence $t - s = 1$ and $v_s v_t \in E(G)$. If $s - r = 2$, then by Lemma 16, the drawing induced by $\mathcal{V}[v_l, v_t]$ is a copy of Π_2^2 with handles v_l and v_t, and thus the drawing induced by $\mathcal{V}[v_i, v_t]$ has a copy of $\Pi_2^1 \circ \Pi_2^2$ or $\Pi_2^2 \circ \Pi_2^2$ with linking handles v_i and v_t. If $s-r=1$, then $v_l v_r \in E(G)$ because otherwise $d(v_r)=2$ and $d(v_s)=3$, which implies that (C1) appears. In this case, the drawing induced by $\mathcal{V}[v_l, v_t]$ is a copy of Π_2^1 with handles v_l and v_t, and thus the drawing induced by $\mathcal{V}[v_i, v_t]$ has a copy of $\Pi_2^1 \circ \Pi_2^1$ or $\Pi_2^1 \circ \Pi_2^2$ with linking handles v_i and v_t. So we say that (C10) occurs.

Note that the drawing induced by $\mathcal{V}[v_i, v_l]$ and chords $v_l v_s, v_r v_t$ is a IV-cluster, say $\{v_i, v_t\}_4$, such that $t \neq i$, and the drawing induced by $\mathcal{V}[v_l, v_s]$ has the properties described by Lemmas 15 and 16. We call such a IV-cluster a *d*etermined IV-cluster. We do the following assumption.

Assumption 4. *$\{v_i, v_t\}_4$ is the shortest determined IV-cluster contained in the drawing induced by $\mathcal{V}[v_i, v_t]$.*

According to Lemma 19, we assume, without loss of generality, that $v_{i'} v_{j'}$ co-crosses $v_{k'} v_{l'}$ and $v_{l'} v_{s'}$ is a chord such that $s \le i' < k' < j' < l' < s' \le t$ (i.e., there is a I-cluster $\{v_{i'}, v_{s'}\}_1$ in the drawing induced by $\mathcal{V}[v_s, v_t]$). Clearly, we can carefully choose, in advance, i', k', j', l' and s' so that

Assumption 5. *$\{v_{i'}, v_{s'}\}_1$ is the shortest I-cluster contained in the drawing induced by $\mathcal{V}[v_{i'}, v_{s'}]$.*

By Lemma 18, $v_{l'} v_{s'}$ is crosses by a chord $v_{r'} v_{t'}$ with $l' < r' < s'$. If $s' < t' \le t$, then there is a determined IV-cluster with width $t'-i'+1 < t-i+1$, say $\{v_{i'}, v_{t'}\}_4$, contained in the drawing induced by $\mathcal{V}[v_{i'}, v_{v_{t'}}]$, contradicting Assumption 4. Hence $s \le t' < i'$.

By the absences of (C7) and (C9), there is a chord $v_{q'} v_{i'}$ with $t' \le q' < i'$. If the I-cluster $\{v_{q'}, v_{l'}\}_1$ is the shortest one contained in the drawing induced by $\mathcal{V}[v_{q'}, v_{l'}]$, then by Lemma 18, $v_{q'} v_{i'}$ is crossed by a chord $v_{y'} v_{p'}$ with $t' \le y' < q' < p' < i'$, and furthermore, there is a determined (left) IV-cluster with width $l' - y' + 1 < t - i + 1$, say $\{v_{y'}, v_{l'}\}_4$, contained in the drawing induced by $\mathcal{V}[v_{y'}, v_{v_{l'}}]$, contradicting Assumption 4. Hence there is a shorter I-cluster contained in the drawing induced by $\mathcal{V}[v_{q'}, v_{l'}]$. Among those I-clusters contained in the drawing induced by $\mathcal{V}[v_{q'}, v_{l'}]$, we choose the shortest one, say $\{v_{i''}, v_{s''}\}_1$ for example. Precisely, $v_{i''} v_{j''}$ co-crosses $v_{k''} v_{l''}$ and $v_{l''} v_{s''}$ is a chord with $q' \le i'' < k'' < j'' < l'' < s'' \le i'$. By Lemma 18, $v_{l''} v_{s''}$ is crossed by a chord $v_{r''} v_{t''}$ with $l'' < r'' < s''$. If $s'' < t'' \le i'$, then there is a determined IV-cluster with width $t'' - i'' + 1 < t - i + 1$, say $\{v_{i''}, v_{t''}\}_4$, contained in the drawing induced by $\mathcal{V}[v_{i''}, v_{v_{t''}}]$, contradicting Assumption 4. Hence $q' \le t'' < i''$. We reset $\{t', i', k', j', l', r', s'\} := \{t'', i'', k'', j'', l'', r'', s''\}$ and come back to the beginning of this paragraph. Since $s'' - t'' < s' - t'$ and the graph is finite, this iteration can stop somewhere.

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