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Light edges in 1-planar graphs of minimum degree 3 ☆

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ABSTRACT

A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one another edge. In this work we prove that each 1-planar graph of minimum degree at least 3 contains an edge with degrees of its endvertices of type $(3, \leq 23)$ or $(4, \leq 11)$ or $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$. Moreover, the upper bounds 9, 8 and 7 here are sharp and the upper bounds 23 and 11 are very close to the possible sharp ones, which may be 20 and 10, respectively. This generalizes a result of Fabrici and Madaras (2007) which says that each 3-connected 1-planar graph contains a light edge, and improves a result of Hudák and Šugerek (2012), which states that each 1-planar graph of minimum degree at least 4 contains an edge with degrees of its endvertices of type $(4, \leq 13)$ or $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Notations are standard (cf. [1]) unless we state otherwise.

A *planar graph* is a graph that can be drawn in the plane in such a way that no edges cross each other. Such a drawing is called a *plane graph*. For a plane graph G, $V(G)$, $E(G)$ and $F(G)$ denote the set of vertices, edges, and faces of G, respectively. A k-, k^+- and k^--*vertex* (resp. *face*) is a vertex (resp. face) of degree k, at least k and at most k, respectively. An edge uv is *of type* $(a, \leq b)$ if $d(u) = a$ and $d(v) \leq b$. Similarly we can define edges of type $(\leq a, \leq b)$ or $(a, \geq b)$ or $(\geq a, \geq b)$. A graph is *1-planar* if it can be drawn in the plane so that each edge is crossed by at most one another edge. Such a drawing so that the number of crossings is as small as possible is called a *1-plane graph*. The notion of 1-planarity was introduced by Ringel [6] while trying to simultaneously color the vertices and faces of a plane graph such that any pair of adjacent or incident elements receive different colors.

A well-known consequence of the Euler's Polyhedron Formula says that each planar graph has a vertex of degree at most 5. The beautiful Kotzig's Theorem [5] states that each 3-connected planar graph contains an edge whose sum of degrees of its endvertices is at most 13, and at most 11 if 3-vertices are absent. In addition, the bounds 13 and 11 are sharp. For other relative results on the light subgraphs of graphs embedded in the plane, we refer the readers to a recent survey contributed by Jendrol' and Voss [4].

For 1-planar graphs, there are analogical results. For example, Fabrici and Madaras [2] showed that each 1-planar graph contains a vertex of degree at most 7, and proved that each 3-connected 1-planar graph contains an edge with both endvertices of degrees at most 20. Here the bound 20 is also sharp.

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As we know, every 3-connected graph has minimum degree at least 3. Hence a natural question is to ask whether each 1-planar graph of minimum degree at least 3 contains a light edge (i.e., an edge such that the sum, or the maximum, of degrees of its endvertices is bounded by a constant that is independent of the given graph). Actually, the answer to the above question is positive for 1-planar graph of minimum degree at least 4. Precisely, Hudák and Šugerek [3] proved

Theorem 1.1 (*[3]*)**.** *Each 1-planar graph of minimum degree at least* 4 *contains an edge of type* $(4, \leq 13)$ *or* $(5, \leq 9)$ *or* $(6, \leq 8)$ *or* $(7, 7)$.

Moreover, they also claimed that for 1-planar graphs of minimum degree at least 5, these bounds in Theorem 1.1 are best possible and the list of edges is minimal (in the sense that, for each of the considered edge types there are 1-planar graphs whose set of types of edges contains just the selected edge type). Actually, there exists 1-planar graph with only edges of type $(5, 9)$, $(5, 10)$ and $(9, 10)$, or with only edges of type $(6, 8)$ and $(8, 8)$, or with only edges of type $(7, 7)$. The first two graphs were constructed by Hudák and Šugerek [3], and the last graph (i.e., 7-regular 1-planar graph) was introduced by Fabrici and Madaras [2].

Motivated by Theorem 1.1 of Hudák and Šugerek, and also by the above mentioned result of Fabrici and Madaras [2], we investigate light edges in 1-planar graphs by proving that each 1-planar graph of minimum degree at least 3 contains a light edge. More precisely, we are able to prove the following main theorem of this paper.

Theorem 1.2. *Each 1-planar graph of minimum degree at least* 3 *contains an edge of type* $(3, \leq 23)$ *or* $(4, \leq 11)$ *or* $(5, \leq 9)$ *or* $(6, \leq 8)$ *or* $(7, 7)$.

Clearly, Theorem 1.2 can be seen as an improvement and also a generalization of Theorem 1.1. Although we improve 13 in Theorem 1.1 to 11, we still do not know whether 11 is sharp. If it can be improved, then it shall be 10, since Hudák and Šugerek [3] constructed a 1-planar graph with only edges of type $(4, 10)$ and $(10, 10)$. On the other hand, the sharpness of the upper bound 23 in Theorem 1.2 is unclear. Since Fabrici and Madaras [2] constructed a 1-planar graph with only edges of type $(3, 20)$ and $(20, 20)$, we want to know whether the upper bound 23 in Theorem 1.2 can be replaced by 20. In conclusion, we raise the following problem.

Problem 1.3. Does each 1-planar graph of minimum degree at least 3 contain an edge of type $(3, \leq 20)$ or $(4, \leq 10)$ or $(5, \leq 9)$ or $(6, \leq 8)$ or $(7, 7)$?

2. The existence of a light edge

The *associated plane graph* $G^\times$ of a 1-plane graph G is the plane graph that is obtained from G by turning all crossings of G into new vertices of degree four. Those new 4-vertices are called *false vertices* of $G^\times$, and the original vertices of G are called *true vertices* of $G^\times$. A face of $G^\times$ is *false* if it is incident with at least one false vertex, and *true* otherwise.

Lemma 2.1. *If G is a 1-plane graph, then*
(a) false vertices in $G^\times$ are not adjacent;
(b) if a 3-vertex v is incident with two 3-faces and adjacent to two false vertices in $G^\times$, then v is incident with a 5^+-face;
(c) there exists no edge uv in $G^\times$ such that $d_{G^\times}(v) = 3$, u is a false vertex, and uv is incident with two 3-faces;
(d) if v is a true 4-vertex in $G^\times$, then v is incident with at most three false 3-faces.

Proof. The conclusions (a), (b) and (c) come from [7, Lemma 1]. For (d), suppose that v is a true 4-vertex in $G^\times$ incident with four false 3-faces $\{vv_1v_2\}$, $\{vv_2v_3\}$, $\{vv_3v_4\}$ and $\{vv_4v_1\}$. By (a), we assume, without loss of generality, that v_1 and v_3 are false. Now, there are two edges in G connecting v_2 to v_4, one of which passes through v_1 and the other passes through v_3. This contradicts the fact that G is simple. □

The Proof of Theorem 1.2. Suppose that there is a 1-plane graph G of minimum degree at least 3 contradicting Theorem 1.2. So G contains only edges of type $(3, \geq 24)$ or $(4, \geq 12)$ or $(5, \geq 10)$ or $(6, \geq 9)$ or $(\geq 7, \geq 8)$. We apply the discharging method to the associated plane graph $G^\times$ of G. Formally, for each vertex $v \in V(G^\times)$, let $c(v) := d_{G^\times}(v) - 4$ be its initial charge, and for each face $f \in F(G^\times)$, let $c(f) := d_{G^\times}(f) - 4$ be its initial charge. Clearly,

$$\sum_{x \in V(G^\times) \bigcup F(G^\times)} c(x) = -8 < 0$$

by the well-known Euler's formula.

Let $f = \{vv_1v_2\}$ be a false 3-face of $G^\times$ such that v is a false vertex generating by v_1v_3 crossing v_2v_4 in G. If $d_{G^\times}(v_1) = k$, $v_2v_3 \in E(G)$ and

$$d_{G^\times}(v_4) \leq \begin{cases} 11, & \text{if } k = 4 \\ 9, & \text{if } k = 5 \\ 8, & \text{if } k = 6 \end{cases}$$

then f is said to be *k-special*, where $4 \leq k \leq 6$.

We define discharging rules as follows.

R1 Every true 4-vertex of $G^\times$ sends $\frac{1}{6}$ to each of its incident 4-special faces.

R2 Every 5-vertex of $G^\times$ sends $\frac{3}{10}$ to each of its incident 5-special faces, and $\frac{1}{5}$ to each of its incident 3-faces that are not 5-special.

R3 Every 6-vertex of $G^\times$ sends $\frac{7}{18}$ to each of its incident 6-special faces, and $\frac{1}{3}$ to each of its incident 3-faces that are not 6-special.

R4 Every 7-vertex of $G^\times$ sends $\frac{1}{2}$ to each of its incident false 3-faces.

R5 Every 8^+-vertex v of $G^\times$ sends $\frac{d_{G^\times}(v)-4}{d_{G^\times}(v)}$ to each of its incident faces.

R6 Let v be a false vertex of $G^\times$ such that v_1v_3 crossed v_2v_4 in G at v, and let f_i with $1 \le i \le 4$ be the face that is incident with vv_i and vv_{i+1} in $G^\times$ (here v_5 is recognized as v_1).

R6.1 Suppose that $\min\{d_{G^\times}(v_1), d_{G^\times}(v_2)\} \ge 24$ and $d_{G^\times}(v_3) = 3$.

- If $d_{G^\times}(v_4) = 3$, then f_1 sends $\frac{1}{6}$, through v, to each of the elements among f_2, f_4, v_3, v_4.
- If $d_{G^\times}(v_4) \ge 4$, then f_1 sends $\frac{1}{3}$ to both f_2 and v_3 through v.

R6.2 Suppose that $23 \ge \min\{d_{G^\times}(v_1), d_{G^\times}(v_2)\} \ge 12$ and $d_{G^\times}(v_3) \le 6$.

- If f_1 is a 3-face, then f_1 sends $\frac{1}{6}$ to both f_2 and f_4 through v while $d_{G^\times}(v_4) \le 6$, and $\frac{1}{3}$ to f_2 through v while $d_{G^\times}(v_4) \ge 7$.
- If f_1 is a 4^+-face, then f_1 sends $\frac{1}{3}$ to both f_2 and f_4 through v.

R6.3 Suppose that $11 \ge \min\{d_{G^\times}(v_1), d_{G^\times}(v_2)\} \ge 10$ and $d_{G^\times}(v_3) \le 6$.

- If f_1 is a 3-face, then f_1 sends $\frac{1}{10}$ to both f_2 and f_4 through v while $d_{G^\times}(v_4) \le 6$, and $\frac{1}{5}$ to f_2 through v while $d_{G^\times}(v_4) \ge 7$.
- If f_1 is a 4^+-face, then f_1 sends $\frac{3}{10}$ to both f_2 and f_4 through v.

R6.4 Suppose that $\min\{d_{G^\times}(v_1), d_{G^\times}(v_2)\} = 9$ and $d_{G^\times}(v_3) \le 6$.

- If f_1 is a 3-face, then f_1 sends $\frac{1}{18}$ to both f_2 and f_4 through v while $d_{G^\times}(v_4) \le 6$, and $\frac{1}{9}$ to f_2 through v while $d_{G^\times}(v_4) \ge 7$.
- If f_1 is a 4^+-face, then f_1 sends $\frac{5}{18}$ to both f_2 and f_4 through v.

R7 Every 4^--face of $G^\times$ redistributes its remaining charge after applying the previous rules equitably to each of its incident true 4^-vertices.

R8 Every 5^+-face of $G^\times$ sends $\frac{2}{3}$ to each of its incident 3-vertices, and then redistributes its remaining charge after applying the previous rules equitably to each of its incident true 4-vertices.

Let $c'(x)$ be the charge of $x \in V(G^\times) \cup F(G^\times)$ after applying the above rules. Since our rules only move charge around, and do not affect the sum, we have

$$\sum_{x\in V(G^\times)\cup F(G^\times)} c'(x) = \sum_{x\in V(G^\times)\cup F(G^\times)} c(x) < 0.$$

Next, we prove that $c'(x) \ge 0$ for each $x \in V(G^\times) \cup F(G^\times)$. This leads to

$$\sum_{x\in V(G^\times)\cup F(G^\times)} c'(x) \ge 0,$$

a contradiction.

Claim 1. *Every true 3-face incident with one 3-vertex v sends at least $\frac{2}{3}$ to v.*

Proof. Such a true 3-face sends to v at least $2 \times \frac{24-4}{24} - 1 = \frac{2}{3}$ by R5 and R7, since the neighbors of v on this face are 24^+-vertices. □

Claim 2. *Every true 3-face incident with one 4-vertex v sends at least $\frac{1}{3}$ to v.*

Proof. Such a true 3-face sends to v at least $2 \times \frac{12-4}{12} - 1 = \frac{1}{3}$ by R5 and R7, since the neighbors of v on this face are 12^+-vertices. □

A *transitive false vertex* v on $f \in F(G^\times)$ is a false vertex such that its two neighbors u, w on f have degrees both at least 9. If f sends out charges via a false vertex, then this false vertex must be transitive by R6.

Claim 3. *Let f be a face in $G^{\times}$ and let $\rho^{+}(f)$, $\rho^{-}(f)$ respectively be the total charges that f receives from its incident 9^{+}-vertices, and that f sends out via its incident transitive false vertices. If $d_{G^{\times}}(f) \geq 4$, then $\rho^{+}(f) \geq \rho^{-}(f)$, and if $d_{G^{\times}}(f) = 3$, then $\rho^{+}(f) \geq \rho^{-}(f) + 1$.*

Proof. If f is true, or not incident with a transitive false vertex, then there is nothing to prove. Hence we assume that there are some transitive false vertices $v_1, v_2, \ldots, v_k$ on f. For each v_i with $1 \leq i \leq k$, let u_i and w_i be two neighbors of v_i on the face f. Since false vertices are not adjacent in $G^{\times}$, u_i and w_i are 9^{+}-vertices by the definition of the transitive false vertex. The *contribution* of v_i $(1 \leq i \leq k)$ is denoted by $\pi^{+}(v_i) = \frac{d_{G^{\times}}(u_i)-4}{d_{G^{\times}}(u_i)} + \frac{d_{G^{\times}}(w_i)-4}{d_{G^{\times}}(w_i)}$, and the *demand* $\pi^{-}(v_i)$ of v_i is the amount of charges that f sends out via v_i. By R6, one can check that $\pi^{+}(v_i) \geq 2\pi^{-}(v_i)$ for each $1 \leq i \leq k$. Therefore,

$$\rho^{+}(f) \geq \frac{1}{2}\sum_{i=1}^{k}\pi^{+}(v_i) \geq \frac{1}{2}\sum_{i=1}^{k}2\pi^{-}(v_i) = \sum_{i=1}^{k}\pi^{-}(v_i) = \rho^{-}(f)$$

if $d_{G^{\times}}(f) \geq 4$. On the other hand, if $d_{G^{\times}}(f) = 3$, then it is easy to see from R6 that $\pi^{+}(v_1) - \pi^{-}(v_1) \geq \min\{2 \times \frac{5}{6} - 4 \times \frac{1}{6}, 2 \times \frac{2}{3} - 2 \times \frac{1}{6}, 2 \times \frac{3}{5} - 2 \times \frac{1}{10}, 2 \times \frac{5}{9} - 2 \times \frac{1}{18}\} = 1$, which implies that $\rho^{+}(f) = \pi^{+}(v_1) \geq \pi^{-}(v_1) + 1 = \rho^{-}(f) + 1$. □

Claim 4. *Suppose that f is a 4-face that is not incident with two false vertices.*

(1) If f is incident with at least one 3-vertex, then f sends at least $\frac{5}{12}$ to each of its incident true 4^{-}-vertices;

(2) If f is not incident with any 3-vertex and f is incident with at least one true 4-vertex, then f sends at least $\frac{1}{3}$ to each of its incident true 4-vertices.

Proof. (1) Let $f = \{v_1v_2v_3v_4\}$ be such a 4-face so that $d_{G^{\times}}(v_1) = 3$. Since v_2 and v_4 cannot be both false, at least one of them is a 24^{+}-vertex, and moreover, neither v_2 nor v_4 can be a transitive false vertex. If v_3 is true (so may be a true 4^{-}-vertex), then f sends to each of its incident true 4^{-}-vertices at least $\frac{1}{2} \times (4 - 4 + \frac{5}{6}) = \frac{5}{12}$ by R5 and R7. If v_3 is false, then it is a transitive false vertex because v_2 and v_4 are 24^{+}-vertices. So f sends to each of its incident true 4^{-}-vertices at least $4 - 4 + 2 \times \frac{5}{6} - 2 \times \frac{1}{3} = 1$ by R5, R6.1 and R7.

(2) Let $f = \{v_1v_2v_3v_4\}$ be such a 4-face so that v_1 is a true 4-vertex. Since v_2 and v_4 cannot be both false, at least one of them is a 12^{+}-vertex, and moreover, neither v_2 nor v_4 can be a transitive false vertex.

If v_3 is true (so may be a true 4-vertex), then f sends to each of its incident true 4-vertices at least $\frac{1}{2} \times (4-4+2 \times \frac{1}{3}) = \frac{1}{3}$ by R5 and R7. If v_3 is false, then it is a transitive false vertex because v_2 and v_4 are 12^{+}-vertices. So f sends to each of its incident true 4^{-}-vertices at least $4 - 4 + 2 \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{2}{3}$ by R5, R6.1, R6.2 and R7. □

Claim 5. *Every 5^{+}-face incident with true 4^{-}-vertices sends at least $\frac{1}{3}$ to each of its incident true 4-vertices (if exist).*

Proof. Suppose that f is incident with s 3-vertices, and t true 4-vertices. If $t = 0$, then there is noting to be proved, so we may assume that $t \geq 1$. Since true 4^{-}-vertices are not adjacent in $G^{\times}$, $s + t \leq \lfloor \frac{d_{G^{\times}}(f)}{2} \rfloor$. By R8 and Claim 3, f sends to each of its incident true 4-vertices at least

$$\begin{aligned} \frac{d_{G^{\times}}(f) - 4 + \rho^{+}(f) - \rho^{-}(f) - \frac{2}{3}s}{t} &\geq \frac{d_{G^{\times}}(f) - 4 - \frac{2}{3}s}{t} \\ &\geq \frac{d_{G^{\times}}(f) - 4 + \frac{2}{3}t - \frac{2}{3}\lfloor \frac{d_{G^{\times}}(f)}{2} \rfloor}{t} \\ &\geq \frac{d_{G^{\times}}(f) - 4 + \frac{2}{3}t - \frac{2}{3} \cdot \frac{d_{G^{\times}}(f)}{2}}{t} \\ &= \frac{\frac{2}{3}d_{G^{\times}}(f) - 4 + \frac{2}{3}t}{t}, \end{aligned}$$

which is at least $\frac{2}{3}$ provided that $d_{G^{\times}}(f) \geq 6$, and at least $\frac{1}{3}$ provided $d_{G^{\times}}(f) = 5$ and $t \geq 2$ (actually, if $d_{G^{\times}}(f) = 5$ then $t \leq 2$ since $s + t \leq \lfloor \frac{5}{2} \rfloor = 2$).

If $d_{G^{\times}}(f) = 5$ and $t = 1$, then by $s + t \leq 2$, we have $s \leq 1$. So f sends to its incident 4-vertex at least $5 - 4 - \frac{2}{3} = \frac{1}{3}$ by R8 and Claim 3. □

Proposition 1. *After the application of Rules, the charge of every face of $G^{\times}$ is non-negative.*

Proof. Claim 3 along with R7 and R8 deduces that $c'(f) \geq 0$ for each 4^{+}-face f and each 3-face that is incident with a transitive false vertex. Now we calculate the final charges of true 3-faces and false 3-faces incident with one non-transitive false vertex.

First of all, we assume that $f_1 = \{vv_1v_2\}$ is a true 3-face with $d_{G^{\times}}(v) \leq d_{G^{\times}}(v_1) \leq d_{G^{\times}}(v_2)$.

If v is a 3-vertex, then v_1 and v_2 are 24^+-vertices, thus the charge of f_1 is at least $3 - 4 + 2 \times \frac{5}{6} > 0$ after applying R1–R6. Therefore, $c'(f) = 0$ by R7.

If v is a 4-vertex, then v_1 and v_2 are 12^+-vertices, thus the charge of f_1 is at least $3 - 4 + 2 \times \frac{2}{3} > 0$ after applying R1–R6. Therefore, $c'(f) = 0$ by R7.

If v is a 5-vertex, then v_1 and v_2 are 10^+-vertices, thus $c'(f_1) \geq 3 - 4 + 2 \times \frac{3}{5} > 0$ by R5.

If v is a 6-vertex, then v_1 and v_2 are 9^+-vertices, thus $c'(f_1) \geq 3 - 4 + 2 \times \frac{5}{9} > 0$ by R5.

If v is a 7^+-vertex, then v_1 and v_2 are 8^+-vertices, thus $c'(f_1) \geq 3 - 4 + 2 \times \frac{1}{2} = 0$ by R5.

On the other hand, assume that $f_1 = \{vv_1v_2\}$ is a false face so that v is a non-transitive false vertex and $d_{G^\times}(v_1) \leq d_{G^\times}(v_2)$. Suppose that v_1v_3 crosses v_2v_4 in G at v, and f_i with $1 \leq i \leq 4$ is the face that is incident with vv_i and vv_{i+1} in $G^\times$ (here v_5 is recognized as v_1).

If $d_{G^\times}(v_1) = 3$, then v_2, v_3 are 24^+-vertices. By R5, v_2 sends at least $\frac{5}{6}$ to f_1, and by R6.1, f_2 sends at least $\frac{1}{6}$ to f_1. Therefore, $c'(f_1) \geq 3 - 4 + \frac{5}{6} + \frac{1}{6} = 0$.

If $d_{G^\times}(v_1) = 4$, then v_2, v_3 are 12^+-vertices. If f_1 is a 4-special face, then f receives $\frac{1}{6}$ from v_1 by R1, $\frac{2}{3}$ from v_2 by R5, at least $\frac{1}{6}$ from f_2 by R6.2, and thus $c'(f_1) \geq 3 - 4 + \frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 0$. If f_1 is not a 4-special face, then f_1 receives $\frac{2}{3}$ from v_2 by R5, $\frac{1}{3}$ from f_2 by R6.2, and thus $c'(f_1) \geq 3 - 4 + \frac{2}{3} + \frac{1}{3} = 0$.

If $d_{G^\times}(v_1) = 5$, then v_2, v_3 are 10^+-vertices. If f_1 is a 5-special face, then f_1 receives $\frac{3}{10}$ from v_1 by R2, $\frac{3}{5}$ from v_2 by R5, at least $\frac{1}{10}$ from f_2 by R6.3, and thus $c'(f_1) \geq 3 - 4 + \frac{3}{10} + \frac{3}{5} + \frac{1}{10} = 0$. If f_1 is not a 5-special face, then f_1 receives $\frac{1}{5}$ from v_1 by R2, $\frac{3}{5}$ from v_2 by R5, at least $\min\{\frac{1}{5}, \frac{3}{10}\} = \frac{1}{5}$ from f_2 by R6.3, and thus $c'(f_1) \geq 3 - 4 + \frac{1}{5} + \frac{3}{5} + \frac{1}{5} \geq 0$.

If $d_{G^\times}(v_1) = 6$, then v_2, v_3 are 9^+-vertices. If f_1 is a 6-special face, then f_1 receives $\frac{7}{18}$ from v_1 by R3, $\frac{5}{9}$ from v_2 by R5, at least $\frac{1}{18}$ from f_2 by R6.4, and thus $c'(f_1) \geq 3 - 4 + \frac{7}{18} + \frac{5}{9} + \frac{1}{18} = 0$. If f is not a 6-special face, then f_1 receives $\frac{1}{3}$ from v_1 by R3, $\frac{5}{9}$ from v_2 by R5, at least $\min\{\frac{1}{9}, \frac{5}{18}\} = \frac{1}{9}$ from f_2 by R6.4, and thus $c'(f_1) \geq 3 - 4 + \frac{1}{3} + \frac{5}{9} + \frac{1}{9} \geq 0$.

If $d_{G^\times}(v_1) \geq 7$, then v_2 is a 8^+-vertex. By R4 and R5, each of v_1 and v_2 sends at least $\frac{1}{2}$ to f_1, which implies $c'(f) \geq 3 - 4 + \frac{1}{2} + \frac{1}{2} = 0$. □

For a true k-vertex v of $G^\times$, denote by $v_1, v_2, \ldots, v_k$ the neighbors of v in $G^\times$ that lie consecutively around v, and by f_i the face that is incident with vv_i and vv_{i+1} in $G^\times$ (the subscript is taken by modular k). These notations will be used in the proof of the next propositions without explaining their meanings again.

Proposition 2. *After the application of Rules, the charge of every 3-vertex of $G^\times$ is non-negative.*

Proof. Suppose that v is a 3-vertex of $G^\times$.

(1) If v is incident only with 3-faces, then they are all true for otherwise $G^\times$ has two adjacent false vertices or G has a multi-edge. By Claim 1, $c'(v) \geq 3 - 4 + 3 \times \frac{2}{3} > 0$.

(2) If v is incident with one 4^+-face, say f_3, and two 3-faces f_1 and f_2, then we consider the following subcases.

First, if f_1 and f_2 are true, then $c'(v) \geq 3 - 4 + 2 \times \frac{2}{3} > 0$ by Claim 1.

Second, if only one of f_1 and f_2 is true, then by the symmetry, assume that f_1 is true. Therefore, f_2 is false and thus v_3 is a false vertex. If f_3 is a 5^+-face, then it sends $\frac{2}{3}$ to v by R8. By Claim 1, f_1 sends $\frac{2}{3}$ to v. Hence $c'(v) \geq 3 - 4 + \frac{2}{3} + \frac{2}{3} > 0$. On the other hand, if f_3 is a 4-face, then it is incident with only one false vertex v_3, and moreover, v_3 is not a transitive false vertex. Since v_1 is a 24^+-vertex, f_3 send by R7 to v at least $\frac{1}{2} \times (4 - 4 + \frac{5}{6}) = \frac{5}{12}$. Counting together the charge $\frac{2}{3}$ that f_1 sends to v by Claim 1, we conclude $c'(v) \geq 3 - 4 + \frac{5}{12} + \frac{2}{3} > 0$.

Third, if f_1 and f_2 are both false, then both v_1 and v_3 are false by Lemma 2.1(c), and furthermore, f_3 is a 5^+-face by Lemma 2.1(b), which sends $\frac{2}{3}$ to v by R8. The face adjacent to f_1 in $G^\times$ that is different from f_2, f_3 is denoted by h_1, and the face adjacent to f_2 in $G^\times$ that is different from f_1, f_3 is denoted by h_2. By R6.1, each of h_1 and h_2 sends at least $\frac{1}{6}$ to v. Therefore, $c'(v) \geq 3 - 4 + \frac{2}{3} + 2 \times \frac{1}{6} \geq 0$.

(3) If v is incident with one 3-face, say f_1, and two 4^+-faces f_2 and f_3, then we consider the following subcases.

First, if f_1 is true, then it sends $\frac{2}{3}$ to v by Claim 1. If f_2 or f_3, say f_2, is a 5^+-face, then v receives $\frac{2}{3}$ from f_2 by R8, which implies $c'(v) \geq 3 - 4 + \frac{2}{3} + \frac{2}{3} > 0$. If f_2 and f_3 are both 4-faces, then each of them is incident with at most one false vertex. Hence by Claim 4(1), each of f_2 and f_3 sends at least $\frac{5}{12}$ to v, which implies $c'(v) \geq 3 - 4 + \frac{2}{3} + 2 \times \frac{5}{12} > 0$.

Second, if f_1 is false, then assume by the symmetry that v_1 is false. The face adjacent to f_1 in $G^\times$ that is different from f_2, f_3 is denoted by h_1. By R6.1, h_1 sends at least $\frac{1}{6}$ to v.

If v_3 is true, then f_3 is either a 4-face that is not incident with two false vertices or a 5^+-face, and so does f_2. By Claim 4(1) and R8, each of f_2 and f_3 sends at least $\min\{\frac{5}{12}, \frac{2}{3}\} = \frac{5}{12}$ to v, which implies $c'(v) \geq 3 - 4 + \frac{1}{6} + 2 \times \frac{5}{12} = 0$.

If v_3 is false, then f_2 still sends at least $\frac{5}{12}$ to v by the same reason as above. At this stage, if f_3 is a 5^+-face, then it sends $\frac{2}{3}$ to v by R8, and thus $c'(v) \geq 3 - 4 + \frac{1}{6} + \frac{5}{12} + \frac{2}{3} > 0$. Hence we assume that f_3 is a 4-face and let $f_3 = \{vv_1u_3v_3\}$.

If $d_{G^\times}(u_3) \geq 4$, then h_1 sends $\frac{1}{3}$ to v by R6.1. If f_2 is a 5^+-face now, then it sends $\frac{2}{3}$ to v by R8, which implies $c'(v) \geq 3 - 4 + \frac{1}{3} + \frac{2}{3} = 0$. If f_2 is a 4-face, then let $f_2 = \{vv_2u_2v_3\}$. Since $u_2u_3 \in E(G)$, u_2 or u_3 is a 8^+-vertex. If u_2 is a 8^+-vertex, then f_2 sends to v at least $\frac{1}{2} + \frac{5}{6} = \frac{4}{3}$ by R5 and R7 and thus $c'(v) \geq 3 - 4 + \frac{1}{3} + \frac{4}{3} > 0$. If u_3 is a 8^+-vertex, then f_3 sends to v at least $\frac{1}{2}$ by R5. By Claim 4(1), f_2 sends to v at least $\frac{5}{12}$. Therefore, $c'(v) \geq 3-4+\frac{1}{3}+\frac{1}{2}+\frac{5}{12} > 0$. Note that v_3 is not a transitive false vertex on f_2, and neither v_1 nor v_3 is a transitive false vertex on f_3.

If $d_{G^\times}(u_3)=3$, then we look at the degree of f_2. If f_2 is a 4-face, then let $f_2=\{vv_2u_2v_3\}$. Since u_2 is adjacent to u_3 in G, u_2 is a 24^+-vertex, which implies that f_2 sends to v at least $2\times\frac{5}{6}=\frac{5}{3}$ by R5 and R7. Note that v_3 is not a transitive false vertex on f_2. Hence $c'(v)\geq 3-4+\frac{5}{3}>0$. If f_2 is a 5^+-face, then it gives $\frac{2}{3}$ to v by R8. Suppose that the crossing v_3 is produced by vw crossing u_2u_3 in G. Clearly, w and u_2 are both 24^+-vertices. Let h_2 be the face in $G^\times$ that is incident with wv_3 and u_2v_3. By R6.1, h_2 sends $\frac{1}{6}$ to v. Recall that h_1 sends at least $\frac{1}{6}$ to v. We then have $c'(v)\geq 3-4+\frac{2}{3}+\frac{1}{6}+\frac{1}{6}=0$.

(4) If v is incident only with 4^+-faces, then we consider the following subcases.

If v is incident with at least two 5^+-faces, then it is clear that $c'(v)\geq 3-4+2\times\frac{2}{3}>0$ by R8.

If v is incident with one 5^+-face f_3 and two 4-faces f_1 and f_2, then f_3 sends $\frac{2}{3}$ to v by R8. If f_1 or f_2, say f_1, is incident with at most one false vertex, then by Claim 4(1), f_1 sends at least $\frac{5}{12}$ to v, which implies $c'(v)\geq 3-4+\frac{2}{3}+\frac{5}{12}>0$. Hence we assume that both f_1 and f_2 are incident with two false vertices. Let $f_1=\{vv_1u_1v_2\}$ and $f_2=\{vv_2u_2v_3\}$. We then conclude that v_1, v_2, v_3 are all false and $u_1u_2\in E(G)$. Therefore, u_1 or u_2 is a 8^+-vertex, since any two 7^--vertices are not adjacent in G. By the symmetry, assume that u_1 is a 8^+-vertex. By R5 and R7, f_1 sends at least $\frac{1}{2}$ to v, since neither v_1 nor v_2 is a transitive false vertex on f_1. Hence $c'(v)\geq 3-4+\frac{2}{3}+\frac{1}{2}>0$.

If v is incident only with 4-faces, then let $f_1=\{vv_1u_1v_2\}$, $f_2=\{vv_2u_2v_3\}$ and $f_3=\{vv_3u_3v_1\}$. If there is at most one false vertex among v_1, v_2 and v_3, then each of f_1, f_2 and f_3 is incident with at most one false vertex. By Claim 4(1), each of them sends at least $\frac{5}{12}$ to v, which implies that $c'(v)\geq 3-4+3\times\frac{5}{12}>0$. If v_1, v_2 are false and v_3 is true, then v_3 is a 24^+-vertex and $u_1u_2, u_1u_3\in E(G)$. If $d_{G^\times}(u_1)\geq 8$, then f_1 sends at least $\frac{1}{2}$ to v by R5 and R7, since neither v_1 nor v_2 is a transitive false vertex on f_1. Counting together the charge $2\times\frac{5}{12}=\frac{5}{6}$ receiving from f_2 and f_3 by Claim 4(1), we conclude $c'(v)\geq 3-4+\frac{1}{2}+\frac{5}{6}>0$. On the other hand, if $d_{G^\times}(u_1)\leq 7$, then $d_{G^\times}(u_2)\geq 8$. Since v_2 is not a transitive false vertex on f_2, f_2 sends at least $\frac{1}{2}+\frac{5}{6}=\frac{4}{3}$ to v by R5 and R7, which immediately implies that $c'(v)\geq 3-4+\frac{4}{3}>0$. At last, we look at the case that v_1, v_2, v_3 are all false. This implies that $u_1u_2u_3$ is a triangle in G, and then at least two vertices among u_1, u_2 and u_3, say u_1 and u_2, are 8^+-vertices. Since neither v_1 nor v_2 is a transitive false vertex on f_1, f_1 sends at least $\frac{1}{2}$ to v by R5 and R7. Same result also holds for f_2. Therefore, $c'(f)\geq 3-4+\frac{1}{2}+\frac{1}{2}=0$. □

Proposition 3. *After the application of Rules, the charge of every 4-vertex of $G^\times$ is non-negative.*

Proof. If v is a false vertex, then it is clear that $c'(v)=c(v)=4-4=0$. Hence we assume in the following that v is a true 4-vertex. By Lemma 2.1(d), v is incident with at most three false 3-faces.

If v is not incident with any 4-special face, then v sends out nothing and thus $c'(v)\geq c(v)=4-4=0$. So, we suppose that f_1 is a 4-special face so that v_1 is a false vertex. Let w, u_4 be vertices such that vw crosses v_2u_4 in G at v_1. Since f_1 is a 4-special face, $v_2u_4\in E(G)$ and $d_{G^\times}(u_4)\leq 11$. This implies that $u_4\neq v_4$ (for otherwise vv_4 is an edge of type $(4, \leq 11)$), and thus f_4 is a 4^+-face. Similarly, if f_2 is a 4-special face, then f_3 is a 4^+-face. This implies that v is incident with at most two 4-special faces, to which v sends at most $2\times\frac{1}{6}=\frac{1}{3}$ by R1.

If f_4 is a 5^+-face, or a 4-face incident with at most one false vertex, then by Claim 5, or Claim 4, f_4 sends at least $\frac{1}{3}$ to v. This implies $c'(v)\geq 4-4-\frac{1}{3}+\frac{1}{3}=0$. If f_4 is a 4-face incident with exactly two false vertices, then v_4 is false. We now look at the face f_2.

If f_2 is not a false 3-face, then it is either a true 3-face, or a 4-face that is incident with at most one false vertex, or a 5^+-face. In any case, f_2 sends at least $\frac{1}{3}$ to v by Claim 2, or Claim 5, or Claim 4. This concludes that $c'(v)\geq 4-4-\frac{1}{3}+\frac{1}{3}=0$. Hence we are left the case that f_2 is a false 3-face, that is, $v_2v_3\in E(G^\times)$ and v_3 is false.

If f_3 is a 5^+-face, then it sends at least $\frac{1}{3}$ to v by Claim 5, which implies that $c'(v)\geq 4-4-\frac{1}{3}+\frac{1}{3}=0$. If f_3 is a 4-face, then let $f_3=\{vv_3u_3v_4\}$. Since $u_3u_4\in E(G)$, either u_3 or u_4 is a 8^+-vertex. Without loss of generality, assume that $d_{G^\times}(u_3)\geq 8$. Since neither v_3 nor v_4 is a transitive false vertex on f_3, f_3 sends at least $\frac{1}{2}$ to v by R5 and R7. Therefore, $c'(v)\geq 4-4-\frac{1}{3}+\frac{1}{2}>0$. □

Proposition 4. *After the application of Rules, the charge of every 5-vertex of $G^\times$ is non-negative.*

Proof. If a 5-vertex v is not incident with any 5-special face, then $c'(v)\geq 5-4-5\times\frac{1}{5}=0$ by R2. Hence we assume that v is incident with at least one 5-special face.

Suppose that f_1 is a special 5-face so that v_1 is a false vertex. Let w, u_5 be vertices such that vw crosses v_2u_5 in G at v_1. Since f_1 is a 5-special face, $wv_2\in E(G)$ and $d_{G^\times}(u_5)\leq 9$. This implies that $u_5\neq v_5$ (for otherwise vv_5 is an edge of type $(5, \leq 9)$), and thus f_5 is a 4^+-face. This fact tells us that if v is incident with a 5-special face, then it must be incident with one 4^+-face. Hence v is incident with at most four 3-faces, among which at most three are 5-special.

If v is incident with three 5-special faces, then it is incident with two 4^+-faces, and thus $c'(v)\geq 5-4-3\times\frac{3}{10}>0$ by R2. If v is incident with at most two 5-special, then $c'(v)\geq 5-4-2\times\frac{3}{10}-2\times\frac{1}{5}=0$ by R2. □

Proposition 5. *After the application of Rules, the charge of every 6-vertex of $G^\times$ is non-negative.*

Proof. The proof is highly similar to the previous one. For the completeness of the paper, we add this proof here.

If a 6-vertex v is not incident with any 6-special face, then $c'(v) \geq 6 - 4 - 6 \times \frac{1}{3} = 0$. Hence we assume that v is incident with at least one 6-special face.

Suppose that f_1 is a special 6-face so that v_1 is a false vertex. Let w, u_6 be vertices such that vw crosses v_2u_6 in G at v_1. Since f_1 is a 6-special face, $v_2u_6 \in E(G)$ and $d_{G^\times}(u_6) \leq 8$. This implies that $u_6 \neq v_6$ (for otherwise vv_6 is an edge of type $(6, \leq 8)$), and thus f_6 is a 4^+-face. This fact tells us that if v is incident with a 6-special face, then it must be incident with one 4^+-face. Hence v is incident with at most five 3-faces, among which at most four are 6-special.

If v is incident with four 6-special faces, then it is incident with two 4^+-faces, and thus $c'(v) \geq 6 - 4 - 4 \times \frac{7}{18} > 0$ by R3. If v is incident with at most three 6-special, then $c'(v) \geq 6 - 4 - 3 \times \frac{7}{18} - 2 \times \frac{1}{3} > 0$ by R3. □

Proposition 6. *After the application of Rules, the charge of every 7^+-vertex of $G^\times$ is non-negative.*

Proof. If v is a 7-vertex, then v is incident with at most six false 3-faces, for otherwise two false vertices are adjacent in $G^\times$. Hence we have, by R4, that $c'(v) \geq 7-4-6 \times \frac{1}{2} = 0$. If v is a 8^+-vertex, then $c'(v) \geq d_{G^\times}(v) - 4 - \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} \cdot d_{G^\times}(v) = 0$ by R5. □

This is the end of the whole proof. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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