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Linear Arboricity of NIC-Planar Graphs

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Abstract A graph is NIC-planar if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing edges share at most one common end vertex. It is proved that every NIC-planar graph with minimum degree at least 2 (resp. 3) contains either an edge with degree sum at most 23 (resp. 17) or a 2-alternating cycle (resp. 3-alternating quadrilateral). By applying those structural theorems, we confirm the Linear Arboricity Conjecture for NIC-planar graphs with maximum degree at least 14 and determine the linear arboricity of NIC-planar graphs with maximum degree at least 21.

Keywords NIC-planar graph; linear arboricity; light edge
2000 MR Subject Classification 05C15; 05C10

1 Introduction

All graphs considered in this paper are simple and undirected. By $V(G), E(G), \Delta(G)$ and $\delta(G)$, we denote the vertex set, the edge set, the maximum degree and the minimum degree of a graph G, respectively. A *planar graph* is a graph that can be drawn in the plane so that no edge is crossed, and such a drawing is a *plane graph*. For a plane graph G, we use $F(G)$ to denote its face set. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges that are incident with v in G. By k-, k^+-, and k^--*vertex* (resp. face), we denote a vertex (resp. face) of degree k, at least k, and at most k, respectively. For other undefined concepts we refer the readers to [4].

A graph is *1-planar* if it can be drawn in a plane so that each edge is crossed by at most one other edge. The notion of the 1-planarity was introduced by Ringel[7] in 1965 when he considered the vertex-face coloring of plane graphs, which can be translated to the vertex coloring of 1-planar graphs. A graph is *IC-planar* (independent-crossing-planar) if it has a 1-planar drawing so that each vertex is incident with at most one crossing edge. A graph is *NIC-planar* (near-independent-crossing-planar) if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing edges share at most one common end vertex. The IC-planarity was introduced by Albertson[2] in 2008 and the NIC-planarity was introduced by Zhang[11] in 2014. Both of them specialize 1-planarity, but generalize planarity. Recently, Bachmaier et al.[3] investigated the structure of the NIC-planar graphs and IC-planar graphs.

A *linear forest* is a forest (i.e., an acyclic graph) in which every component is a path. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests needed to partition the edge set of G.

The following Conjecture 1.1 is known as the Linear Arboricity Conjecture (LAC), which was raised by [1].

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Conjecture 1.1. *If G is a simple graph, then* $\lceil \frac{\Delta(G)}{2} \rceil \le \mathrm{la}(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$.

Note that LAC is still quite open and it was verified for planar graphs[9,10]. Moreover, Cygan et al.[5] proved that if G is a planar graph with $\Delta(G) \ge 9$ then $\mathrm{la}(G) = \lceil \frac{\Delta(G)}{2} \rceil$. For 1-planar graphs G, Zhang, Liu and Wu[12] showed that if $\Delta(G) \ge 33$ then $\mathrm{la}(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

In this paper, we aim to partially solve LAC for NIC-planar graphs with large maximum degree by proving the following

Theorem 1.2. *If G is an NIC-planar graph with $\Delta(G) \ge 14$, then* $\mathrm{la}(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$.

In addition, we prove the second theorem on the linear arboricity of NIC-planar graphs.

Theorem 1.3. *If G is an NIC-planar graph with $\Delta(G) \ge 21$, then* $\mathrm{la}(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

2 Structural Theorems

A *good drawing* of an NIC-planar graph is a drawing so that its NIC-planarity is preserved, and moreover, the number of crossings is as small as possible. The *associated plane graph* of an NIC-planar graph G, denoted by $G^\times$, is a plane graph derived from the good drawing of G by turning all its crossings into new vertices of degree four, which are called *false* vertices of $G^\times$ while vertices in $V(G)$ are called *true* vertices of $G^\times$. A face of $G^\times$ is *false* if it is incident with at least one false vertex, and is *true* otherwise. The following one lemma is straightforward.

Lemma 2.1. *If G is a good drawing of an NIC-planar graph, then*

(1) any 2-vertex is not incident with a false 3-face in $G^\times$;

(2) if a 3-vertex is incident with three 3-faces in $G^\times$, then those faces are true;

(3) if a 3-vertex is incident with two 3-faces in $G^\times$, then at least one of them is true.

Proof. (1) Let $uvwu$ be a false 3-face in $G^\times$ such that u is a 2-vertex and v is false. Assume that uy crosses wx in G. We now adjust the drawing of G by pulling u into the area forming by the face of $G^\times$ incident with vw and vy. This helps us avoid the crossing v from G. Therefore, G is not a good drawing, a contradiction.

(2) Suppose that a 3-vertex u is incident with three 3-faces $uxyu, uyzu$ and $uzxu$ in $G^\times$. If one of them is false, then assume, without loss of generality, that x is a false vertex. In this case, we can find two different edges in G that connect y to z. Therefore, G is not a simple graph, a contradiction.

(3) Suppose that a 3-vertex u is incident with two 3-faces $uxyu$ and $uyzu$ in $G^\times$. If they are both false, then there are two probabilities. First, if y is a false vertex, then the path xyz in $G^\times$ is actually an edge in G that connects x to z. We pull the edge $xz \in E(G)$ into the area forming by the face of $G^\times$ incident with ux and uz. This operation erases the crossing y from G and implies that G is not a good drawing. Second, if x and z are false vertices, then u and y are two end vertices of the pair of crossing edges producing the crossings x (or z). Therefore, the NIC-planarity of G is destroyed, a contradiction. □

In the remaining of this section, we prove two structural theorems for NIC-planar graphs that are applied to prove the main theorems (Theorems 1.2 and 1.3) of this paper.

Theorem 2.2. *If G is a NIC-planar graph with minimal degree $\delta(G) \ge 2$, then G contains*

(a) an edge uv with $d_G(u) + d_G(v) \le 23$, or

(b) there is a 2-alternating cycle $v_0v_1 \cdots v_{2n-1}v_0$ such that $d_G(v_0) = d_G(v_2) = \cdots = d_G(v_{2n-2}) = 2$ and $\max\limits_{1\le i\le n} \mid N_2(v_{2i-1}) \mid \ge 3$.

Notation. Here and below, $N_2(v_{2i-1})$ denotes the number of 2-vertices that are adjacent to v_{2i-1} in G.

Proof. Suppose, to the contrary, that G is a counterexample. If $\Delta(G) \le 11$, then each edge uv of G satisfies $d_G(u) + d_G(v) \le 2\Delta(G) \le 22$, which implies (a), a contradiction. Hence we assume that $\Delta(G) \ge 12$. By the absence of (a), the neighbors of a 2-vertex in G are all 22^+-vertices.

Let H be the subgraph of G such that $E(H)$ consists of all edges incident with the 2-vertices of G. Since (b) is forbidden in G, every component of H is either a path or a cycle. This implies that $|E(H)| \le |V(H)|$. By the definition of H, $|E(H)| = 2|V_2|$ and $|V(H)| \le |V_2| + |V_{22^+}|$. Hence it is easy to conclude that $|V_2| \le |V_{22^+}|$. Here $|V_2|$ or $|V_{22^+}|$ is the number of 2-vertices or 22^+-vertices, respectively.

In what follows, we call a true vertex of $G^\times$ *big* if $d_{G^\times}(v) \ge 18$, *middle* if $5 \le d_{G^\times}(v) \le 17$, and *small* if $d_{G^\times}(v) \le 4$. A middle vertex is an M^{11^-}*-vertex* if $5 \le d_{G^\times}(v) \le 11$, and M^{12+}*-vertex* if $12 \le d_{G^\times}(v) \le 17$. Since (a) is forbidden in G, any two 11^--vertices are not adjacent in G. We use F, B, M^{11-}, M^{12+} and S to represent false vertex, big vertex, M^{11-}-vertex, M^{12+}-vertex and small vertex, respectively, and then use these notations to represent the structure of a face of $G^\times$. For example, we say that a face is an (F, S, B, S)-face if it is a 4-face with vertices u_1, u_2, u_3 and u_4 lying clockwise on the boundary of f such that u_1 is false, u_2 is small, u_3 is big and u_4 is small. A face in $G^\times$ is *burdened* if it is incident with at least one small vertex.

We now apply the discharging method to the associated plane graph $G^\times$ of G. Formally, for each vertex $v \in V(G^\times)$, let $c(v) := d_{G^\times}(v) - 4$ be its initial charge, and for each face $f \in F(G^\times)$, let $c(f) := d_{G^\times}(f) - 4$ be its initial charge. Clearly, $\sum_{x \in V(G^\times) \cup F(G^\times)} c(x) = -8 < 0$ by the well-known Euler's formula. The discharging rules are defined as follows.

R1 every middle vertex v sends $\frac{d_{G^\times}(v)-4}{d_{G^\times}(v)}$ to each of its incident faces.

R2 every big vertex sends $\frac{7}{9}$ to each of its incident faces.

R3 let f be a face in $G^\times$ incident with an edge $uv \in E(G^\times)$.

- **R3.1** if u is a M^{11-}-vertex, and uv is incident with a 3-face f' such that w is a false vertex (in this case v must be an M^{12+}- or B-vertex), then f sends $\frac{1}{45}$ to f' through uv (see the left picture of Figure 1).
- **R3.2** if u is a big vertex, v is a false vertex, and uv is incident with a 3-face f' such that w is a small vertex, then f sends $\frac{2}{9}$ to f', and $\frac{1}{18}$ to w, both through uv (see the right picture of Figure 1).

R4 every burdened true 3-face of $G^\times$ sends $\frac{5}{9}$ to each of its incident small vertices (if exists).

R5 every burdened 4-face f of $G^\times$ sends to each of its incident small vertices $\frac{7}{18}$ if f is an (B, S, F, S)-face, $\frac{34}{45}$ if f is an (B, S, F, M^{11-})-face, and $\frac{7}{9}$ otherwise.

R6 every burdened 5^+-face f of $G^\times$ sends to each of its incident small vertices $\frac{2}{3}$ if f is an (S, F, S, F, S, F)-face, and $\frac{13}{18}$ otherwise.

R7 every 22^+-vertex sends $\frac{8}{9}$ to a virtual box, from which every 2-vertex receives the same amount.

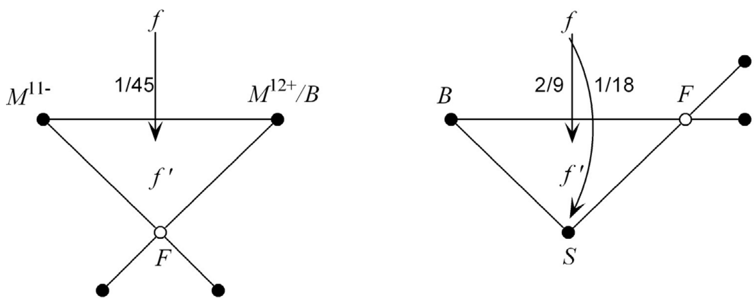

Figure 1. R3.1 and R3.2

Note that if f is a 3-face in R3.1, then f is true and not burdened, and if f is a 3-face in R3.2, then f is of type (B, F, B).

These can be easily seen from the definition of the NIC-planarity and the absence of (a). Therefore, the face f' as described in R3.1 or R3.2 will not lose charge through uv.

Let $c'(x)$ be the charge of $x \in V(G^\times) \cup F(G^\times)$ after applying the above rules. Since our rules only move charge around, and do not affect the sum, we have

$$\sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} c(x) < 0.$$

Since $|V_2| \le |V_{22^+}|$, the virtual box in R7 has no deficiency finally. Next, we prove that $c'(x) \ge 0$ for each $x \in V(G^\times) \cup F(G^\times)$. This leads to

$$\sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) \ge 0,$$

a contradiction.

We first calculate the final charge of each face $f \in F(G^\times)$.

Case 1. $f = uvwu$ is a true 3-face such that $d_{G^\times}(u) \le d_{G^\times}(v) \le d_{G^\times}(w)$.

If $d_{G^\times}(u) \le 4$, then v and w are of degree at least 20, and they are big vertices. Note that the sum of the degrees of the two end-vertices of an edge in G is at least 24 by the absence of (a). By R2 and R4, $c'(f) \ge 3 - 4 + 2 \times \frac{7}{9} - \frac{5}{9} = 0$.

If $5 \le d_{G^\times}(u) \le 11$, then $d_{G^\times}(v), d_{G^\times}(w) \ge 13$, and thus $c'(f) \ge 3 - 4 + 2 \times \min\{\frac{13-4}{13}, \frac{7}{9}\} - 2 \times \frac{1}{45} > 0$ by R1, R2 and R3.

If $d_{G^\times}(u) \ge 12$, then $c'(f) \ge 3 - 4 + 3 \times \min\left\{\frac{12-4}{12}, \frac{7}{9}\right\} > 0$ by R1 and R2.

Case 2. $f = uvwu$ is a false 3-face such that u is a false vertex, and $d_{G^\times}(v) \le d_{G^\times}(w)$.

If $d_{G^\times}(v) \le 4$, then w is a big vertex, from which f receives $\frac{7}{9}$ by R2. In addition, f would receive another $\frac{2}{9}$ from the other face, besides f, incident with uw by R3.2. Therefore, $c'(f) \ge 3 - 4 + \frac{7}{9} + \frac{2}{9} = 0$.

If $5 \le d_{G^\times}(v) \le 11$, then v is an M^{11^-}-vertex and thus by R3.1 f receives $\frac{1}{45}$ from the other face, besides f, incident with vw. Since $d_{G^\times}(v) + d_{G^\times}(w) = d_G(v) + d_G(w) \ge 24$, $c'(f) \ge 3 - 4 + \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} + \min\{\frac{d_{G^\times}(w)-4}{d_{G^\times}(w)}, \frac{7}{9}\} + \frac{1}{45} \ge 3 - 4 + \frac{5-4}{5} + \frac{7}{9} + \frac{1}{45} = 0$ by R1 and R2.

If $12 \le d_{G^\times}(v) \le 17$, then $c'(f) \ge 3 - 4 + \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} + \min\left\{\frac{d_{G^\times}(w)-4}{d_{G^\times}(w)}, \frac{7}{9}\right\} \ge 3 - 4 + \frac{12-4}{12} + \frac{12-4}{12} > 0$ by R1 and R2.

If $d_{G^\times}(v) \ge 18$, then f is of type (B, F, B), and thus $c'(f) \ge 3 - 4 + 2 \times \frac{7}{9} - 2 \times (\frac{2}{9} + \frac{1}{18}) = 0$ by R2 and R3.

Case 3. f is a true 4-face.

If f is not incident with small vertex, then f is incident with at least two 12^+-vertices, and thus $c'(f) \ge 4 - 4 + 2 \times \min\left\{\frac{12-4}{12}, \frac{7}{9}\right\} - 4 \times \frac{1}{45} > 0$ by R1, R2 and R3.1.

If f is incident with exactly one small vertex, then f is adjacent to at least two big vertices, thus $c'(f) \ge 4 - 4 + 2 \times \frac{7}{9} - 2 \times \frac{1}{45} - \frac{7}{9} > 0$ by R2, R3.1 and R5.

If f is incident with exactly two small vertices, then f is adjacent to two big vertices, and R3.1 will not be applied. Therefore, $c'(f) \ge 4 - 4 + 2 \times \frac{7}{9} - 2 \times \frac{7}{9} = 0$ by R2 and R5.

Case 4. $f = uvwyu$ is a false 4-face such that u is a false vertex.

Note that f is incident with at exactly one false vertex by the definition of the NIC-planarity.

If f is not incident with small vertex, then f is incident with at least one 12^+-vertex, and thus $c'(f) \ge 4 - 4 + \min\left\{\frac{12-4}{12}, \frac{7}{9}\right\} - 2 \times \frac{1}{45} - 2 \times \left(\frac{2}{9} + \frac{1}{18}\right) > 0$ by R1, R2 and R3.

If v is a small vertex, then w is big. If y is an M^{11^-}-vertex, then $c'(f) \ge 4 - 4 + \frac{5-4}{5} + \frac{7}{9} - \frac{1}{45} - \frac{34}{45} > 0$ by R1, R2, R3.1 and R5. If y is a 12^+-vertex, then $c'(f) \ge 4 - 4 + \frac{12-4}{12} + \frac{7}{9} - \left(\frac{2}{9} + \frac{1}{18}\right) - \frac{7}{9} >$

0 by R1, R2, R3.2 and R5. If y is a small vertex, then $c'(f) \geq 4-4+\frac{7}{9}-2\times\frac{7}{18}=0$ by R2 and R5.

If w is a small vertex, then v and y are big vertices, and $c'(f) \geq 4-4+2\times\frac{7}{9}-2\times(\frac{2}{9}+\frac{1}{18})-\frac{7}{9} > 0$ by R2, R3 and R5.

Case 5. f is a 5^+-face.

Suppose that f is incident with t big vertices and s small vertices. Since two small vertices are not adjacent, $s \leq \lfloor \frac{d_{G^\times}(f)}{2} \rfloor$ and there are $2s$ edges on f that are incident with a small vertex.

Let l_1 be the number of edges uv on f such that u is a big vertex and v is an M^{11-}-vertex or a false vertex. Through each of those edges, f may sends out at most $\max\{\frac{1}{45},\frac{2}{9}+\frac{1}{18}\}=\frac{5}{18}$ by R3.

Let l_2 be the number of edges on f that is incident with neither a small vertex nor a big vertex. Through each of those edges, f may sends out at most $\frac{1}{45}$ by R3.1.

Since $l_1+l_2 \leq d_{G^\times}(f)-2s$ and $l_1 \leq 2t$, we conclude by R2 and R6 that

$$\begin{aligned}
c'(f) \geq & d_{G^\times}(f)-4+\frac{7}{9}t-\frac{13}{18}s-\frac{5}{18}l_1-\frac{1}{45}l_2\\
= & d_{G^\times}(f)-4+\frac{7}{9}t-\frac{13}{18}s-\frac{1}{45}(l_1+l_2)-\frac{23}{90}l_1\\
\geq & d_{G^\times}(f)-4+\frac{7}{9}t-\frac{13}{18}s-\frac{1}{45}(d_{G^\times}(f)-2s)-\frac{23}{90}\cdot 2t\\
= & \frac{44}{45}d_{G^\times}(f)+\frac{4}{15}t-\frac{61}{90}s-4\\
\geq & \frac{44}{45}d_{G^\times}(f)+\frac{4}{15}t-\frac{61}{90}\cdot\lfloor\frac{1}{2}d_{G^\times}(f)\rfloor-4.
\end{aligned}$$

Clearly, $c'(f) \geq 0$ provided that $d_{G^\times}(f) \geq 7$, or $d_{G^\times}(f)=6$ and $s \leq 2$, or $d_{G^\times}(f)=5$ and $t \geq 2$, or $d_{G^\times}(f)=5$, $t \leq 1$ and $s \leq 1$.

Hence in the following we just consider two remaining cases. Firstly, assume that f is a 6-face that is incident with exactly three small vertices. If f is not incident with big vertices, then f shall be of type (S,F,S,F,S,F), and thus $c'(f) \geq 6-4-3\times\frac{2}{3}=0$ by R6. If f is incident with at least one big vertex, then by R2 and R6, $c'(f) \geq 6-4+\frac{7}{9}-3\times\frac{13}{18} > 0$. Secondly, assume that f is a 5-face that is incident with exactly two small vertices. Now f is incident with at least one big vertex and R3 (actually, R3.2) will be applied to f at most once. Therefore, $c'(f) \geq 5-4+\frac{7}{9}-(\frac{2}{9}+\frac{1}{18})-2\times\frac{13}{18} > 0$ by R2, R3 and R6.

Now we calculate the final charge of each vertex $v \in V(G^\times)$.

Case 6. v is a 2-vertex.

Note that v is not incident with a false 3-face by Lemma 2.1(1).

If v is incident with a true 3-face, then v is adjacent to two big vertices in $G^\times$, and the other face f incident with v is a 4^+-face (moreover, if f is a 4-face, then it is not of type (B,F,S,F) or (B,S,F,M^{11-})), thus $c'(v) \geq 2-4+\frac{5}{9}+\min\{\frac{7}{9},\frac{2}{3}\}+\frac{8}{9} > 0$ by R4, R5, R6 and R7.

If v is incident with two 4-faces, then at least one 4-face incident with v is not of type (B,S,F,S) or (B,S,F,M^{11-}) (otherwise two 11^--vertices are adjacent in G). Therefore, $c'(v) \geq 2-4+\min\{\frac{7}{18},\frac{34}{45}\}+\frac{7}{9}+\frac{8}{9} > 0$ by R5 and R7.

If v is incident with a 4-face and a 5^+-face, then we consider two cases. If the 4-face incident with v is of type (B,S,F,S), then the 5^+-face incident with v is not of type (S,F,S,F,S,F). Therefore, $c'(v) \geq 2-4+\frac{7}{18}+\frac{13}{18}+\frac{8}{9}=0$ by R5, R6 and R7. If the 4-face incident with v is not of type (B,S,F,S), then $c'(v) \geq 2-4+\frac{34}{45}+\frac{2}{3}+\frac{8}{9} > 0$ by R5, R6 and R7.

If v is incident with two 5^+-faces, then $c'(v) \geq 2-4+2\times\frac{2}{3}+\frac{8}{9} > 0$ by R6 and R7.

Case 7. v is a 3-vertex.

If v is incident with three 3-faces, then all of those 3-faces are true by Lemma 2.1(2), thus $c'(v) \geq 3 - 4 + 3 \times \frac{5}{9} > 0$ by R4.

If v is incident with two 3-faces, then at least one of them is true by Lemma 2.1(3). If they are both true, then $c'(v) \geq 3 - 4 + 2 \times \frac{5}{9} + \frac{7}{18} > 0$ by R4, R5 and R6. If v is incident with a false 3-face $f = uvwu$, then the other face, besides f, incident with uw would sends $\frac{1}{18}$ to v through uw by R3.2, which implies that $c'(v) \geq 3 - 4 + \frac{1}{18} + \frac{5}{9} + \frac{7}{18} = 0$ by R4, R5 and R6.

Now we assume that v is incident with at most one 3-face.

If v is incident with a 5^+-face, then besides this face, v is incident with another 4^+-face, thus $c'(v) \geq 3 - 4 + \frac{7}{18} + \frac{2}{3} > 0$ by R5 and R6.

If v is not incident with any 5^+-face, then v is incident with at least two 4-faces. Since two small vertices are not adjacent in G, among the 4-faces incident with v, at least one is not of type (B, S, F, S). Therefore, $c'(v) \geq 3 - 4 + \frac{7}{18} + \frac{34}{45} > 0$ by R5.

Case 8. v is a 4^+-vertex.

If v is a 4-vertex, then v do not give out any charge by R1-R7, and thus $c'(v) = c(v) = d_{G^\times}(v) = 0$.

If v is a middle vertex, then by R1, $c'(v) \geq d_{G^\times}(v) - 4 - \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} \cdot d_{G^\times}(v) = 0$.

If v is a big (i.e, 18^+-) and 21^--vertex, then $c'(v) \geq d_{G^\times}(v) - 4 - \frac{7}{9}d_{G^\times}(v) = \frac{1}{9}(2d_{G^\times}(v) - 36) \geq 0$ by R2.

If v is a 22^+-vertex, then by R2 and R7, $c'(v) \geq d_{G^\times}(v) - 4 - \frac{7}{9}d_{G^\times}(v) - \frac{8}{9} = \frac{1}{9}(2d_{G^\times}(v) - 44) \geq 0$. □

Remark. A direct corollary from Lemma 2.2 says that

every NIC-planar graph with $\delta(G) \geq 3$ contains an edge uv so that $d_G(u) + d_G(v) \leq 23$.

Actually, we conjecture this result also holds for 1-planar graphs. If so, then the upper bound 23 for the degree sum of the existing edge would be sharp. To see this, consider the graph of the icosahedron. Into each its 3-face xyz insert three new vertices u, v, w and add new edges $ux, uy, uz, vx, vy, vz, wx, wy, wz$ such that vx, uy (vz, wy and wx, uz) cross exactly once (see Figure 2). The degree of any vertex of the resulting 1-planar graph is either 20 or 3, and moreover, any two 3-vertices are not adjacent. Therefore, the degree sum of each it edge is at least 23. Note that this 1-planar graph was also constructed by Fabrici and Madaras[6]. To our knowledge, Liu et al.[8] showed that every 1-planar graph with $\delta(G) \geq 3$ contains an edge uv with $d_G(u) + d_G(v) \leq 29$.

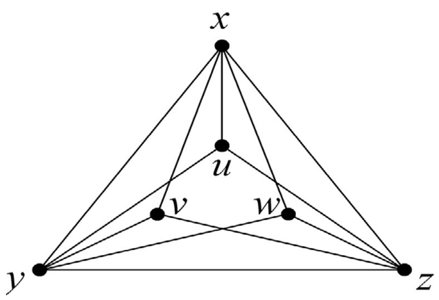

Figure 2. A 1-planar Graph

Theorem 2.3. *If G is a NIC-planar graph with minimal degree $\delta(G) \geq 3$, then G contains*
(a) an edge uv with $d_G(u) + d_G(v) \leq 17$, or
(b) there is a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_1) = d_G(v_3) = 3$.

Proof. Suppose, to the contrary, that G is a counterexample. If $\Delta(G) \le 8$, then each edge uv of G satisfies $d_G(u)+d_G(v) \le 2\Delta(G) \le 16$, which implies (a), a contradiction. Hence we assume that $\Delta(G) \ge 9$. By the absence of (a), the neighbors of a 3-vertex in G are all 15^+-vertices.

In what follows, we call a true vertex of $G^\times$ *big* if $d_{G^\times}(v) \ge 15$, *middle* if $4 \le d_{G^\times}(v) \le 14$, and *small* if $d_{G^\times}(v) = 3$. A middle vertex is an M^{8-}*-vertex* if $4 \le d_{G^\times}(v) \le 8$, and an M^{10+}*-vertex* if $10 \le d_{G^\times}(v) \le 14$. We use F, B, M^{8-}, M^{10+}, M and S to represent false vertex, big vertex, M^{8-}-vertex, M^{10+}-vertex, M-vertex and small vertex, respectively, and then use these notations to represent the structure of a face of $G^\times$.

We now apply the discharging method to the associated plane graph $G^\times$ of G. Formally, for each vertex $v \in V(G^\times)$, let $c(v) := d_{G^\times}(v) - 4$ be its initial charge, and for each face $f \in F(G^\times)$, let $c(f) := d_{G^\times}(f) - 4$ be its initial charge. Clearly, $\sum\limits_{x\in V(G^\times)\cup F(G^\times)} c(x) = -8 < 0$ by the well-known Euler's formula. The discharging rules are defined as follows.

R1 every middle vertex v sends $\frac{d_{G^\times}(v)-4}{d_{G^\times}(v)}$ to each of its incident faces.
R2 every big vertex sends $\frac{11}{15}$ to each of its incident faces.
R3 let f be a face in $G^\times$ incident with an edge $uv \in E(G^\times)$.
- **R3.1** if u is a M^{8-}-vertex, and uv is incident with a 3-face f' such that w is a false vertex (in this case v must be an M^{10+}- or B-vertex), then f sends $\frac{3}{14}$ to f' through uv (see the left picture of Figure 3).
- **R3.2** if v is a false vertex, and uv is incident with a 3-face f' such that w is a M^{8-}-vertex (in this case u must be an M^{10+}- or B-vertex), then f sends $\frac{1}{14}$ to f' through uv (see the middle picture of Figure 3).
- **R3.3** if u is a big vertex, v is a false vertex, and uv is incident with a 3-face f' such that w is a small vertex, then f sends $\frac{4}{15}$ to f', and $\frac{1}{30}$ to w, both through uv (see the right picture of Figure 3).

R4 every burdened true 3-face of $G^\times$ sends $\frac{7}{15}$ to each of its incident small vertices(if exists).
R5 every burdened 4-face f of $G^\times$ sends to each of its incident small vertices $\frac{11}{30}$ if f is an (B, S, F, S)-face, $\frac{1}{2}$ if f is an (B, S, F, M^{8-})-face, and at least $\frac{19}{30}$ otherwise.
R6 every burdened 5^+-face of $G^\times$ sends $\frac{19}{30}$ to each of its incident small vertices.

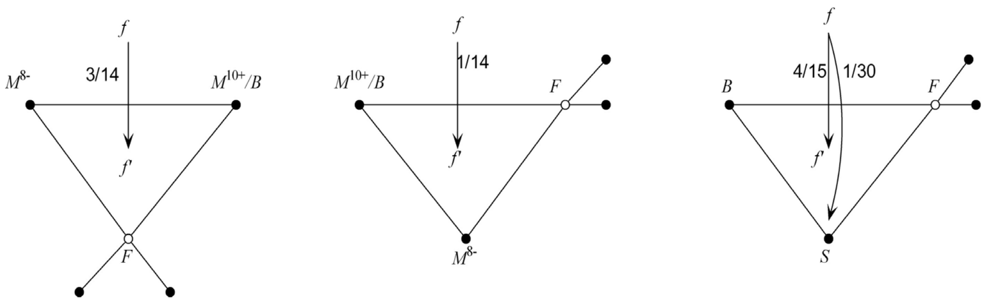

Figure 3. R3.1, R3.2 and R3.3

Note that if f is a 3-face in R3.1, then f is true and not burdened, if f is a 3-face in R3.2, then f is of type $(*_1, F, *_2)$, where $*_1$ or $*_2$ stands for M^{10+} or B, and if f is a 3-face in R3.2, then f is of type (B, F, B).

Therefore, the face f' as described in R3.1, R3.2 or R3.3 will not lose charge through uv.

Let $c'(x)$ be the charge of $x \in V(G^\times) \cup F(G^\times)$ after applying the above rules. Since our rules only move charge around, and do not affect the sum, we have

$$\sum_{x\in V(G^\times)\cup F(G^\times)} c'(x) = \sum_{x\in V(G^\times)\cup F(G^\times)} c(x) < 0.$$

Next, we prove that $c'(x) \geq 0$ for each $x \in V(G^\times) \cup F(G^\times)$. This leads to

$$\sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) \geq 0,$$

a contradiction.

We first calculate the final charge of each face $f \in F(G^\times)$.

Case 1. $f = uvwu$ is a true 3-face such that $d_{G^\times}(u) \leq d_{G^\times}(v) \leq d_{G^\times}(w)$.

If $d_{G^\times}(u) = 3$, then v and w are of degree at least 15, and they are big vertices. Note that the sum of the degrees of the two end-vertices of an edge in G is at least 18 by the absence of (a). By R2 and R4, $c'(f) \geq 3 - 4 + 2 \times \frac{11}{15} - \frac{7}{15} = 0$.

If $4 \leq d_{G^\times}(u) \leq 8$, then $d_{G^\times}(v), d_{G^\times}(w) \geq 10$, and thus $c'(f) \geq 3 - 4 + \frac{d_{G^\times}(u)-4}{d_{G^\times}(u)} + 2 \times \min\{\frac{d_{G^\times}(v)-4}{d_{G^\times}(v)}, \frac{11}{15}\} - 2 \times \frac{3}{14} \geq 3 - 4 + \frac{4-4}{4} + 2 \times \frac{14-4}{14} - 2 \times \frac{3}{14} = 0$ by R1, R2 and R3. Note that $d_{G^\times}(u) + d_{G^\times}(v) = d_G(u) + d_G(v) \geq 18$.

If $d_{G^\times}(u) \geq 9$, then $c'(f) \geq 3 - 4 + 3 \times \min\{\frac{9-4}{9}, \frac{11}{15}\} > 0$ by R1 and R2.

Case 2. $f = uvwu$ is a false 3-face such that u is a false vertex, and $d_{G^\times}(v) \leq d_{G^\times}(w)$.

If $d_{G^\times}(v) = 3$, then w is a big vertex, from which f receives $\frac{11}{15}$ by R2. In addition, f would receive another $\frac{4}{15}$ from the other face, besides f, incident with uw by R3.3. Therefore, $c'(f) \geq 3 - 4 + \frac{11}{15} + \frac{4}{15} = 0$.

If $4 \leq d_{G^\times}(v) \leq 8$, then v is an M^{8-}-vertex and w is a 10^+-vertex, thus by R3.1 and R3.2, f receives $\frac{3}{14}$ from the other face, besides f, incident with vw, and $\frac{1}{14}$ from the other face, besides f, incident with uw. Since $d_{G^\times}(v) + d_{G^\times}(w) = d_G(v) + d_G(w) \geq 18$, $c'(f) \geq 3 - 4 + \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} + \min\{\frac{d_{G^\times}(w)-4}{d_{G^\times}(w)}, \frac{11}{15}\} + \frac{3}{14} + \frac{1}{14} \geq 3 - 4 + \frac{4-4}{4} + \frac{14-4}{14} + \frac{3}{14} + \frac{1}{14} = 0$ by R1 and R2.

If $d_{G^\times}(v) = 9$, then $d_{G^\times}(w) \geq 9$ and f would not lose charge through its incident edges, thus $c'(f) \geq 3 - 4 + \frac{5}{9} + \min\{\frac{d_{G^\times}(w)-4}{d_{G^\times}(w)}, \frac{11}{15}\} \geq 3 - 4 + 2 \times \frac{5}{9} > 0$ by R1 and R2.

If $10 \leq d_{G^\times}(v) \leq 14$, then w is a 10^+-vertex, thus $c'(f) \geq 3-4+\frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} + \min\{\frac{d_{G^\times}(w)-4}{d_{G^\times}(w)}, \frac{11}{15}\} - 2 \times \frac{1}{14} \geq 3 - 4 + 2 \times \frac{10-4}{10} - 2 \times \frac{1}{14} > 0$ by R1, R2 and R3.2.

If $d_{G^\times}(v) \geq 15$, then f is incident with at most one (B, F, S)-face, because otherwise (b) occurs. Therefore, R3.3 will be applied to f at most once (note that R3.2 may still be applied to f twice), and thus $c'(f) \geq 3 - 4 + 2 \times \frac{11}{15} - \max\{2 \times \frac{1}{14}, \frac{4}{15} + \frac{1}{30}\} = 0$ by R2, R3.2 and R3.3.

Case 3. f is a true 4-face.

If f is not incident with any small vertex, then f is incident with at least two 9^+-vertices, thus $c'(f) \geq 4 - 4 + 2 \times \min\{\frac{9-4}{9}, \frac{11}{15}\} - 4 \times \frac{3}{14} > 0$ by R1, R2 and R3.1.

If f is incident with exactly one small vertex, then f is adjacent to at least two big vertices, thus $c'(f) \geq 4 - 4 + 2 \times \frac{11}{15} - 2 \times \frac{3}{14} - \frac{19}{30} > 0$ by R2, R3.1 and R5.

If f is incident with exactly two small vertices, then f is adjacent to exactly two big vertices, and R3.1 will not be applied. Therefore, $c'(f) \geq 4 - 4 + 2 \times \frac{11}{15} - 2 \times \frac{19}{30} > 0$ by R2 and R5.

Case 4. $f = uvwyu$ is a false 4-face such that u is a false vertex.

Note that f is incident with exactly one false vertex by the definition of the NIC-planarity.

If f is not incident with small vertex, then we consider two cases.

Firstly, suppose that f is incident with an M^{8-}-vertex.

If v is an M^{8-}-vertex, then w is a 10^+-vertex. Therefore, $c'(f) \geq 4 - 4 + \min\{\frac{10-4}{10}, \frac{11}{15}\} - 2 \times \frac{3}{14} > 0$ by R1, R2 and R3.1 if y is a M^{8-}-vertex, and $c'(f) \geq 4 - 4 + \min\{\frac{10-4}{10}, \frac{11}{15}\} + \min\{\frac{9-4}{9}, \frac{11}{15}\} - \frac{3}{14} - \frac{4}{15} - \frac{1}{30} > 0$ by R1, R2 and R3 if y is a 9^+-vertex.

If w is an M^{8-}-vertex, then v and y are 10^+-vertices. One can see that from v, w and y, f totally receives at least $2 \times \frac{14-4}{14} = \frac{10}{7}$ (this minimum is taken when w is a 4-vertex and v, y are 14-vertices). Therefore, $c'(f) \geq 4 - 4 + \frac{10}{7} - 2 \times \frac{3}{14} - 2 \times \left(\frac{4}{15} + \frac{1}{30}\right) > 0$ by R1, R2 and R3.

Secondly, suppose that f is incident with three 9^+-vertices, i.e, v, w and y are 9^+-vertices. In this case, it is easy to conclude that $c'(f) \geq 4 - 4 + 3 \times \min\{\frac{9-4}{9}, \frac{11}{15}\} - 2 \times \left(\frac{4}{15} + \frac{1}{30}\right) > 0$ by R1, R2, R3.2 and R3.3.

Hence we now consider the case that f is incident with a small vertex.

If v is a small vertex, then w is big. If y is a small vertex, then $c'(f) \geq 4 - 4 + \frac{11}{15} - 2 \times \frac{11}{30} = 0$ by R2 and R5. If y is an M^{8-}-vertex, then $c'(f) \geq 4 - 4 + \frac{11}{15} - \frac{3}{14} - \frac{1}{2} > 0$ by R2, R3.1 and R5. If y is a 9^+-vertex, then $c'(f) \geq 4 - 4 + \min\{\frac{9-4}{9}, \frac{11}{15}\} + \frac{11}{15} - \frac{4}{15} - \frac{1}{30} - \frac{19}{30} > 0$ by R1, R2, R3.3 and R5.

If w is a small vertex, then v and y are big vertices. Note that uv cannot be incident with a 3-face $uvzu$ such that u is a false vertex and z is a small vertex, because otherwise $zvwyz$ is a 4-cycle in G such that $d_G(z) = d_G(w) = 3$. Hence f will not send out charge by R3.3 via uv. Similarly, f will not loss charge by R3.3 via uy. Therefore, $c'(f) \geq 4 - 4 + 2 \times \frac{11}{15} - 2 \times \frac{1}{14} - \frac{19}{30} > 0$ by R2, R3.1 and R5.

Case 5. f is a 5^+-face.

Suppose that f is incident with t big vertices and s small vertices. Since small vertices are not adjacent in G, $s \leq \lfloor \frac{d_{G^\times}(f)}{2} \rfloor$ and there are $2s$ edges on f that are incident with a small vertex.

Let l_1 be the number of edges uv on f such that u is a big vertex and v is an M^{8-}-vertex or a false vertex. Through each of those edges, f may sends out at most at most $\max\{\frac{3}{14}, \frac{1}{14}, \frac{4}{15} + \frac{1}{30}\} = \frac{3}{10}$ by R3.

Let l_2 be the number of edges on f that is incident with neither a small vertex nor a big vertex. Through each of those edges, f may sends out at most $\max\{\frac{3}{14}, \frac{1}{14}\} = \frac{3}{14}$ by R3.1 and R3.2

Since $l_1 + l_2 \leq d_{G^\times}(f) - 2s$ and $l_1 \leq 2t$, by R2 and R6,

$$\begin{aligned}
c'(f) \geq & d_{G^\times}(f) - 4 + \frac{11}{15}t - \frac{19}{30}s - \frac{3}{10}l_1 - \frac{3}{14}l_2 \\
= & d_{G^\times}(f) - 4 + \frac{11}{15}t - \frac{19}{30}s - \frac{3}{14}(l_1 + l_2) - \frac{3}{35}l_1 \\
\geq & d_{G^\times}(f) - 4 + \frac{11}{15}t - \frac{19}{30}s - \frac{3}{14}(d_{G^\times}(f) - 2s) - \frac{3}{35} \cdot 2t \\
= & \frac{11}{14}d_{G^\times}(f) + \frac{59}{105}t - \frac{43}{210}s - 4 \\
\geq & \frac{11}{14}d_{G^\times}(f) + \frac{59}{105}t - \frac{43}{210} \cdot \lfloor \frac{1}{2}d_{G^\times}(f) \rfloor - 4.
\end{aligned}$$

Clearly, $c'(f) \geq 0$ provided that $d_{G^\times}(f) \geq 6$, or $d_{G^\times}(f) = 5$ and $t \geq 1$.

Now suppose that f is a 5-face with $t = 0$. In this case, f is incident with at most one small vertex, i.e., $s \leq 1$.

If $s = 1$, then f is an (S, F, M, M, F)-face, and thus $c'(f) \geq 5 - 4 - \frac{3}{14} - 2 \times \frac{1}{14} - \frac{19}{30} > 0$ by R3.1, R3.2 and R6.

If $s = 0$, then R3.1 will not be applied to f five times, because otherwise two M^{8-} vertices are adjacent in G, and thus (a) occurs. Therefore, $c'(f) \geq 5 - 4 - 4 \times \frac{3}{14} - \frac{1}{14} > 0$ by R3.1 and R3.2.

Now we calculate the final charge of each vertex $v \in V(G^\times)$.

Case 6. v is a 3-vertex.

If v is incident with three 3-faces, then all of those 3-faces are true by Lemma 2.1(2), thus $c'(v) \geq 3 - 4 + 3 \times \frac{7}{15} > 0$ by R4.

If v is incident with two 3-faces, then at least one of them is true by Lemma 2.1(3). If they are both true, then $c'(v) \geq 3-4+2\times\frac{7}{15}+\frac{11}{30} > 0$ by R4, R5 and R6. If v is incident with a false 3-face $f = uvwu$ such that w is a false vertex, then the other face, besides f, incident with uw would sends $\frac{1}{30}$ to v through uw by R3.3. Meanwhile, the 4^+-face incident with v cannot be of the type (B, S, F, S) (otherwise, denote this 4-face by $vwxyv$ such that x is small and y is big, and then $vuxyv$ is a 4-cycle in G such that v and x are 3-vertices, a contradiction). Therefore, $c'(v) \geq 3 - 4 + \frac{1}{30} + \frac{7}{15} + \frac{1}{2} = 0$ by R4, R5 and R6.

Now we assume that v is incident with at most one 3-face.

If v is incident with a 5^+-face, then besides this face, v is incident with another 4^+-face, thus $c'(v) \geq 3 - 4 + \frac{11}{30} + \frac{19}{30} = 0$ by R5 and R6.

If v is not incident with any 5^+-face, then v is incident with at least two 4-faces. Since two small vertices are not adjacent in G, among the two 4-faces incident with v, at least one is not of type (B, S, F, S). If none of them is a (B, S, F, S)-face, then $c'(v) \geq 3 - 4 + 2 \times \frac{1}{2} = 0$ by R5. If exactly one of them is a (B, S, F, S)-face, then v is not incident with (B, S, F, M^{8-})-face (otherwise two 8^--vertices are adjacent in G), thus $c'(v) \geq 3 - 4 + \frac{11}{30} + \frac{19}{30} = 0$ by R5.

Case 7. v is a 4^+-vertex.

If v is a middle vertex, then by R1, $c'(v) \geq d_{G^\times}(v) - 4 - \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} \cdot d_{G^\times}(v) = 0$.

If v is a 15^+-vertex, then by R2, $c'(v) \geq d_{G^\times}(v) - 4 - \frac{11}{15}d_{G^\times}(v) = \frac{1}{15}(4d_{G^\times}(v) - 60) \geq 0$. □

3 Proofs of the Main Theorems

Proof of Theorem 1.2. Actually we prove a slightly stronger result than Theorem 1.2.

Theorem 1.2'. *If G is an NIC-planar graph with $\Delta(G) \leq M$ and $M \geq 14$, then* $\mathrm{la}(G) \leq \lceil \frac{M+1}{2} \rceil$.

Note that the maximum degree of a subgraph of G in Theorem 1.2 may be less than 14 but any subgraph H of G in Theorem 1.2' satisfies $\Delta(H) \leq M$ and $M \geq 14$. This is why we do this slight modification.

Let G be a minimum counterexample to Theorem 1.2', that is, an NIC-planar graph with maximum degree at most M and linear arboricity larger than $\lceil \frac{M+1}{2} \rceil$ such that any proper subgraph of G has linear arboricity at most $\lceil \frac{M+1}{2} \rceil$. Wu[9] proved (see the proof of [9, Theorem 2.1]) that:

(a) $\delta(G) \geq 3$;

(b) for any edge uv, $d_G(u) + d_G(v) \geq 2\lceil \frac{M+1}{2} \rceil + 2 \geq 18$;

(c) G does not contain a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_1) = d_G(v_3) = 3$.

Note that Wu's above result do not need G to be NIC-planar. It is actually a general conclusion for graphs with hereditary property such as planarity, NIC-planarity, etc.

However, we know by Theorem 2.3 that every NIC-planar graph with $\delta(G) \geq 3$ contains either an edge uv with $d_G(u)+d_G(v) \leq 17$, or a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_1) = d_G(v_3) = 3$. This contradicts Wu's result. Therefore such a counterexample to Theorem 1.2' does not exist and thus Theorem 1.2' is proved. □

Proof of Theorem 1.3. Again, we prove a slightly stronger result than Theorem 1.3.

Theorem 1.3'. *If G is an NIC-planar graph with $\Delta(G) \leq M$ and $M \geq 21$, then* $\mathrm{la}(G) \leq \lceil \frac{M}{2} \rceil$.

Wu[9] proved (see the proof of [9, Theorem 2.2]) that any minimum counterexample G to Theorem 1.3' satisfies

(a) $\delta(G) \geq 2$;

(b) for any edge uv, $d_G(u) + d_G(v) \geq 2\lceil \frac{M}{2} \rceil + 2 \geq 24$;

(c) G does not a 2-alternating cycle $v_0v_1 \cdots v_{2n-1}v_0$ such that $d_G(v_0) = d_G(v_2) = \cdots = d_G(v_{2n-2}) = 2$ and $\max\limits_{1\leq i\leq n} \mid N_2(v_{2i-1}) \mid \geq 3$.

Note, again, that Wu's above result do not need G to be NIC-planar. It always holds if G has maximum degree at most M and linear arboricity larger than $\lceil \frac{M}{2} \rceil$, and any proper subgraph of G has linear arboricity at most $\lceil \frac{M}{2} \rceil$. Therefore, it can be seen as a general conclusion for graphs with hereditary property such as planarity, NIC-planarity, etc.

However, Theorem 2.2 tells us that every NIC-planar graph with $\delta(G) \geq 2$ contains either an edge uv with $d_G(u) + d_G(v) \leq 23$, or the above configuration (c). This contradiction implies that such a counterexample to Theorem 1.3' does not exist and thus Theorem 1.3' is proved. □

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