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(2,1)-total labelling of planar graphs with large maximum degree

Yong Yu
School of Mathematics
Shandong University
Jinan
Shandong 250100
P. R. China

Xin Zhang *
School of Mathematics and Statistics
Xidian University
Xi'an
Shaanxi 710071
P. R. China

Guanghui Wang
Guizhen Liu
School of Mathematics
Shandong University
Jinan
Shandong 250100
P. R. China

Jinbo Li
College of Sciences
China University of Mining and Technology
Xuzhou
Jiangsu 221116
P. R. China

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**E-mail:* `xzhang@xidian.edu.cn (Corresponding author)`

Abstract

In this paper, we prove for planar graph G with maximum degree $\Delta \geq 12$ that the (2,1)-total labelling number $\lambda_2(G)$ is at most $\Delta + 2$.

***Keywords**: (2,1)-total labelling, Planar graphs, Discharging*

1. Introduction

In this paper, all graphs considered are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ (or simply V, E, δ, and Δ) to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. For a plane graph G, $F(G)$ denotes the face set of G and $d(f)$ denotes the degree of a face $f \in F(G)$, which is the number of edges incident with it, where cut edge is counted twice. A k-, k^+- and k^--vertex (or face) in a graph G is a vertex (or face) of degree k, at least k and at most k, respectively. If a vertex v is adjacent to a k-vertex u, then we say that u is a *k-neighbor* of v. For $f \in F(G)$, we call f a $[d(v_1), d(v_2), \cdots, d(v_k)]$*-face* if $v_1, v_2, \cdots, v_k$ are the boundary vertices of f in clockwise order. A 3-face is usually called a *triangle face*. Readers are referred to [2] for other undefined terms and notations.

A *k-$(d, 1)$-total labelling* of a graph G is a function c from $V(G) \cup E(G)$ to the color set $\{0, 1, \cdots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$, $c(e) \neq c(e')$ if e and e' are two adjacent edges, and $|c(u) - c(e)| \geq d$ if vertex u is incident to the edge e. The minimum k such that G has a k-$(d, 1)$-total labelling, denoted by $\lambda_d^T(G)$, is the *$(d, 1)$-total labelling number*. The notion of $(d, 1)$-total labelling of graphs, which is a generation of the total coloring of graphs, was introduced by Havet and Yu [5]. Readers can referr to [1, 4, 6, 7, 9] for further research. In particular, Havet and Yu gave the following $(d, 1)$-Total Labelling Conjecture, which can be seen as the generation of the well-known Total Coloring Conjecture.

Conjecture 1: *If G be a simple graph with maximum degree D, then* $\lambda_d^T(G) \leq \min\{\Delta + 2d - 1, 2\Delta + d - 1\}$.

Assuming $d = 2$, we obtain the following weaker conjecture.

Conjecture 1′: *If G be a simple graph with maximum degree D, then* $\lambda_2^T(G) \leq \Delta + 3$.

Additionally, the following risky conjecture was proposed in [1].

Conjecture 2 ([1]): *If G is a planar triangle-free graph with maximum degree $D \geq 3$, then* $\lambda_d^T(G) \leq \Delta + d$.

Let χ and χ′ denote the chromatic number and the edge chromatic number, respectively. The following results was first mentioned in [5].

Proposition 3 ([5]): *If G is a graph with maximum degree Δ, then*

(1) $\lambda_d^T(G) \leq \chi + \chi' + d - 2$;

(2) $\lambda_d^T(G) \geq \Delta + d - 1$;

(3) $\lambda_d^T(G) \geq \Delta + d$ if $d \geq \Delta$ or G is Δ-regular.

For planar graph with large maximum degree Δ, Bazzaro, Montassier and Raspaud [1] proved that if $\Delta \geq 8d + 2$ then $\lambda_d^T(G) \leq \Delta + 2d - 2$. Recently, this lower bound for Δ in the above result was improved to $6d + 2$ by Zhang, Liu and Yu [11]. Indeed, for planar graph with maximum degree at least 7, $(d, 1)$-Total Labelling Conjecture is meaningful only for d with $\Delta + 2d - 1 \leq \Delta + d + 2$, i.e. $1 \leq d \leq 3$ by (1) of Proposition 3, since $\chi \leq 4$ and $\chi' = \Delta$ [8].

In this paper, we consider the (2,1)-total labellings of planar graph with large maximum degree. Our main result, shown as in Theorem 4, is an improvement of the above mentioned results of Bazzaro, Montassier and Raspaud or of Zhang, Liu and Yu when $d = 2$. On the other hand, it is also can be seen as a support for Conjecture 2 and $(d, 1)$-Total Labelling Conjecture when $d = 2$. Furthermore, the upper bound $\Delta + 2$ for $\lambda_2^T(G)$ in Theorem 4 is sharp because planar graph with arbitrary maximum degree and $\lambda_2^T(G) = \Delta + 2$ was given in [1].

Theorem 4: *If G is a planar graph with maximum degree $\Delta \geq 12$, then* $\Delta + 1 \leq \lambda_2^T(G) \leq \Delta + 2$.

The lower bound of our result is trivial by (2) of Proposition 3. For the upper bound, we prove a conclusion that is slightly stronger as follows.

Theorem 5: *If G is a planar graph with maximum degree $\Delta \leq M$, where $M \geq 12$ is a fixed integer, then* $\lambda_2^T(G) \leq M + 2$. *In particular,* $\lambda_2^T(G) \leq \Delta + 2$ *if* $M = \Delta$.

The interesting case of Theorem 5 is when $M = \Delta$. Indeed, Theorem 5 is only a technical strengthening of Theorem 4, without which we would get complications when considering a subgraph $H \subset G$ with $\Delta(H) < \Delta(G)$.

Let G be a minimal counterexample in terms of $|V| + |E|$ to Theorem 5. By the minimality of G, any proper subgraph of G is (2,1)-total labelable.

It is not difficult to see that G is connected. In Section 2, we obtain some structural properties of the minimal counterexample G. In Section 3, we complete the proof with discharging method.

2. Structural Properties

From now on, we use without distinction the terms *color* and *label*. For a set X, we usually denote the cardinality of X by $|X|$. A partial (2,1)-total labelling of G is a function Φ from $X \subseteq V(G) \cup E(G)$ to the color interval $C = \{0, 1, \cdots, k\}$ with $|C| = k+1 = M+3$ such that the color of the element $x \in X$, denoted by $\Phi(x)$, satisfies all the conditions in the definition of (2,1)-total labelling of graphs. Next, we need some notations to make our description concise.

$E_\Phi(v) = \{\Phi(e) \mid e \in E \textit{ is incident with vertex } v\}$ for $v \in V$;

$I_\Phi(x) = \{\Phi(x)-1, \Phi(x), \Phi(x)+1\} \cap C$ for $x \in V \cup E$;

$F_\Phi(v) = E_\Phi(v) \cup I_\Phi(v)$ for $v \in V$;

$A_\Phi(uv) = C \setminus \left(F_\Phi(u) \cup F_\Phi(v)\right)$ for $uv \in E$;

$A_\Phi(u) = C \setminus \left(\left(\cup_{x \in N(u)} \Phi(x)\right) \cup \left(\cup_{e \ni u} I_\Phi(e)\right)\right)$ for $u \in V$.

In all the notations above, only elements got colors under the partial (2,1)-total labelling Φ are counted in our notations. For example, if v is not colored under Φ, then $F_\Phi(v) = E_\Phi(v)$ by our definition. It is not difficult to see that $A_\Phi(uv)$ (resp. $A_\Phi(u)$) is just the set of colors which are still available for labelling uv (resp. u) under the partial (2,1)-total labelling Φ. Thus, if $|A_\Phi(uv)| \geq 1$ (resp. $|A_\Phi(u)| \geq 1$), then we can (2,1)-total labelling edge uv (resp. vertex u) properly under Φ.

Lemma 6: *For each $uv \in E$, we have* $d(u) + d(v) \geq M - 1$.

Proof: Assume that there is an edge $uv \in E$ such that $d(u) + d(v) \leq M - 2$. By the minimality of G, $G - e$ has a (2,1)-total labelling Φ with color interval C. Since $|A_\Phi(uv)| = |C| - |F_\Phi(u) \cup F_\Phi(v)| \geq |C| - (d(u) + d(v) - 2 + 3 \times 2) \geq |C| - (M+2) \geq 1$, we can extend Φ from subgraph $G - e$ to G, a contradiction.

Lemma 7: *For any edge $e = uv \in E$ with* $\min\{d(u), d(v)\} \leq \left\lfloor \frac{M+2}{4} \right\rfloor$, *we have* $d(u) + d(v) \geq M + 2$.

Proof: Suppose there is an edge $uv \in E$ such that $d(u) \le \left\lfloor \frac{M+2}{4} \right\rfloor$ and $d(u)+d(v) \le M+1$. By the minimality of G, $G-e$ is (2,1)-total labelable with color interval C. Erase the color of vertex u, and denote this partial (2,1)-total labelling by Φ. Then $|A_\Phi(uv)| \ge |C| - |F_\Phi(u)| - |F_\Phi(v)| = |C| - |E_\Phi(u)| - |F_\Phi(v)| \ge |C| - (d(u)+d(v)-2+3) \ge |C| - (M+2) \ge 1$ which implies that uv can be properly colored. We still denote the labelling by Φ after uv is colored. Next, for vertex u, $|A_\Phi(u)| \ge |C| - |\cup_{x\in N(u)} \Phi(x)| - |\cup_{e \ni u} I_\Phi(e)| \ge M+3-4d(u) \ge 1$. Thus, we can extend the partial (2,1)-total labelling Φ to G, a contradiction.

A *k-alternator* $\left(3 \le k \le \left\lfloor \frac{M+2}{4} \right\rfloor\right)$ is a bipartite subgraph $B(X, Y)$ of graph G such that $d_B(x) = d_G(x) \le k$ for each $x \in X$ and $d_B(y) \ge d_G(y) + k - M$ for each $y \in Y$. This concept was first introduced by Borodin, Kostochka and Woodall [3] and generalized by Wu and Wang [10].

Lemma 8 ([3]): *A bipartite graph G is edge f-choosable where f(uv) = max{d(u), d(v)} for any uv ∈ E(G).*

Lemma 9: *There is no k-alternator B(X, Y) in G for any integer k with* $3 \le k \le \left\lfloor \frac{M+2}{4} \right\rfloor$.

Proof: Suppose that there exits a k-alternator $B(X, Y)$ in G. Obviously, X is an independent set of vertices in graph G by Lemma 7. By the minimality of G, the subgraph $G[V(G)\setminus X]$ has a (2,1)-total labelling Φ with color interval C. Then for each $xy \in B(X, Y)$, $|A_\Phi(xy)| \ge |C| - |F_\Phi(y)| - |F_\Phi(x)| \ge |C| - \left(d_G(y) - d_B(y) + 3\right) - 0 \ge M+3-\left(M - d_B(y) + 3\right) \ge d_B(y)$ and $|A_\Phi(xy)| \ge |C| - \left(d_G(y) - d_B(y) + 3\right) \ge M+3-(M+3-k) \ge k$ because $B(X, Y)$ is a k-alternator. Therefore, $|A(xy)| \ge \max\{d_B(y), d_B(x)\}$. By Lemma 8, it follows that $E(B(X,Y))$ can be colored properly. Denote this new partial (2,1)-total labelling by Φ'. Then for each vertex $x \in X$, $|A_{\Phi'}(x)| \ge |C| - |\cup_{z\in N(x)} \Phi'(z)| - |\cup_{e \ni x} I_{\Phi'}(e)| \ge |C| - 4d(x) \ge M+3-(M+2) \ge 1$ because $d_G(x) \le k \le \left\lfloor \frac{M+2}{4} \right\rfloor$. Thus, we can extend the partial (2,1)-total labelling Φ to G, a contradiction.

Lemma 10: *Let* $X_k = \{x \in V(G) \mid d_G(x) \le k\}$ *and* $Y_k = \cup_{x\in X_k} N(x)$ *for any integer k with* $3 \le k \le \left\lfloor \frac{M+2}{4} \right\rfloor$. *If* $X_k \ne \emptyset$, *then there exists a bipartite subgraph*

M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x)=1$ for each $x \in X_k$ and $d_{M_k}(y) \le k-1$ for each $y \in Y_k$.

Proof: The proof is omitted here since it is almost the same with the proof of Lemma 2.4 in Wu and Wang [10].

We call y the *k-master* of x if $xy \in M_k$ and $x \in X_k$, $y \in Y_k$. By Lemma 7, if $uv \in E(G)$ satisfies $d(v) \le \left\lfloor \frac{M+2}{4} \right\rfloor$ and $d(u) = M - i$, then $d(v) \ge M + 2 - d(u) \ge i + 2$. Together with Lemma 10, it follows that each $(M - i)$-vertex can be a j-master of at most $j - 1$ vertices, where $2 \le i+2 \le j \le \left\lfloor \frac{M+2}{4} \right\rfloor$. Each i-vertex has a j-master where $2 \le i \le j \le \left\lfloor \frac{M+2}{4} \right\rfloor$.

Lemma 11: *The minimal counterexample G to Theorem 5 has the following structural properties.*

(a) A 4-vertex is adjacent to 8^+-vertices;

(b) There is no $[d(v_1), d(v_2), d(v_3)]$-face with $d(v_1) = 5$, $\max\{d(v_2), d(v_3)\} \le 6$;

(c) If $f = [d(v_1), d(v_2), d(v_3)]$ is a triangle face with $d(v_1) = 5$, $d(v_2) = 6$ and $d(v_3) = 7$, then v_1 has no other 6-neighbors besides v_2.

(d) If a vertex v is adjacent to two vertices v_1, v_2 such that $2 \le d(v_1) = d(v_2) = M + 2 - d(v) \le 3$, then every face incident with vv_1 or vv_2 is a 4^+-face.

(e) Each Δ-vertex is adjacent to at most one 2-vertex.

Proof:

(*a*) Otherwise, suppose that there is $uv \in E$ such that $d(u) = 4$ and $d(v) \le 7$. By the minimality of G, $H = G - uv$ is (2,1)-total labelable with color interval C. Erase the color of vertex u, and denote this partial (2,1)-total labelling by Φ. Then $|A_\Phi(u)| \ge |C| - |\cup_{x \in N(u)} \Phi(x)| - |$

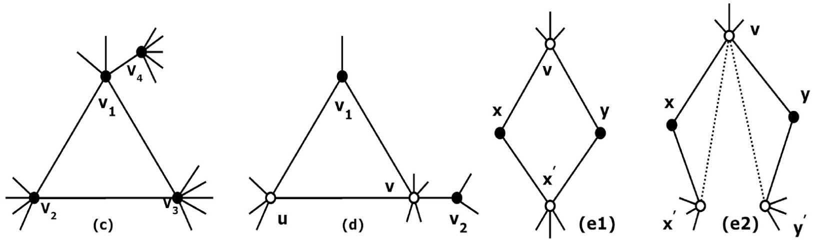

Figure 1

Reducible configurations of Lemma 1.1.

$\cup_{e \ni u} I_\Phi(e)| \geq M+3-4-3\times 3 \geq 15-13 \geq 2$ and $|A_\Phi(uv)| \geq |C|-|E_\Phi(u)| -|F_\Phi(v)| \geq 15-3-(6+3) \geq 3$. Choose $\alpha \in A_\Phi(u)$ to color u. If $A_\Phi(uv) \neq \{\alpha-1, \alpha, \alpha+1\}$, then we can choose $\gamma \in A_\Phi(uv) \setminus \{\alpha-1, \alpha, \alpha+1\}$ to color edge uv. Otherwise, $A_\Phi(uv) = \{\alpha-1, \alpha, \alpha+1\}$. Then we choose $\beta \in A_\Phi(u) \setminus \{\alpha\}$ to color u. Since $A_\Phi(uv) \neq \{\beta-1, \beta, \beta+1\}$, we can choose $\gamma' \in A_\Phi(uv) \setminus \{\beta-1, \beta, \beta+1\}$ to color edge uv. Thus, we extend Φ from subgraph H to G, a contradiction.

(*b*) By Lemma 6, it is enough to prove that there is no [5, 6, 6]-face. Otherwise, let $d(v_1) = 5$, $d(v_2) = d(v_3) = 6$ and let $H = G - \{v_1v_2, v_1v_3\}$. Then H has a (2,1)-total labelling Φ with interval C.

Case 1: $\Phi(v_1) \in F_\Phi(v_2) \cup F_\Phi(v_3)$. Without loss of generality, suppose that $\Phi(v_1) \in F_\Phi(v_2)$, i.e. $|F_\Phi(v_1) \cup F_\Phi(v_2)| \leq |F_\Phi(v_1)| + |F_\Phi(v_2)| - 1$. Then $|A_\Phi(v_1v_2)| = M+3-|F_\Phi(v_1) \cup F_\Phi(v_2)| \geq 15-(3+3+5+3-1) \geq 2$ and $|A_\Phi(v_1v_3)| = M+3-|F_\Phi(v_1) \cup F_\Phi(v_3)| \geq 15-(3+3+5+3) \geq 1$ which implies that we can extend Φ to G, a contradiction.

Case 2. $\Phi(v_1) \notin F_\Phi(v_2) \cup F_\Phi(v_3)$. That is, $\Phi(v_1) \in A_\Phi(v_2v_3)$. Recolor v_2v_3 with color $\Phi(v_1)$ and denote this new partial (2,1)-total labelling by Φ'. Then $|F_{\Phi'}(v_1) \cup F_{\Phi'}(v_2)| \leq |F_{\Phi'}(v_1)| + |F_{\Phi'}(v_2)| - 1$. Analogous to Case 1, we can extend Φ' to G, a contradiction.

(*c*) Suppose on the contrary that G contains such a configuration (see Fig. 1 (c)). By the minimality of G $H = G - \{v_1v_2, v_1v_3\}$ has a (2,1)-total labelling Φ with color interval C.

Claim 1: $\Phi(v_1) \neq \Phi(v_2v_3)$. Otherwise, $|F_\Phi(v_1) \cap F_\Phi(v_2)| \geq 1$. Then $|A_\Phi(v_1v_2)| = M+3-|F_\Phi(v_1) \cup F_\Phi(v_2)| \geq 15-(3+3+5+3-1) \geq 2$ and $|A_\Phi(v_1v_3)| = M+3-|F_\Phi(v_1) \cup F_\Phi(v_3)| \geq 15-(3+3+6+3-1) \geq 1$ which implies that we can extend the partial (2,1)-total labelling Φ to G, a contradiction.

Claim 2. $E_\Phi(v_1) \subseteq F_\Phi(v_2) \cup F_\Phi(v_3)$. Otherwise, we can choose a color $\alpha \in E_\Phi(v_1) \setminus \left(F_\Phi(v_2) \cup F_\Phi(v_3)\right) \neq \emptyset$ to recolor edge v_2v_3. Denote this new coloring of H by Φ'. Then $\alpha \in F_{\Phi'}(v_1) \cap F_{\Phi'}(v_2) \cap F_{\Phi'}(v_3)$. Therefore, $|A_{\Phi'}(v_1v_2)| \geq 2$ and $|A_{\Phi'}(v_1v_3)| \geq 1$ which implies that we can extend Φ' to G, a contradiction.

Claim 3. $E_\Phi(v_1) \subseteq F_\Phi(v_2)$. Otherwise, we have $E_\Phi(v_1) \cap F_\Phi(v_3) \neq \emptyset$ by Claim 2. Assume that $E_\Phi(v_1) \subseteq F_\Phi(v_3)$. We then have $|F_\Phi(v_1) \cap F_\Phi(v_3)| \geq 3$, which implies that $|A_\Phi(v_1v_3)| = M+3-|F_\Phi(v_1) \cup F_\Phi(v_3)| \geq 15-(3+3+6+3-3) \geq 3$ and $|A_\Phi(v_1v_2)| = M+3-|F_\Phi(v_1)$

$\cup F_{\Phi}(v_2)|\geq 15-(3+3+5+3)\geq 1$. Therefore, we can extend Φ from subgraph H to G, a contradiction.

By Claim 2 and Claim 3, we have $|F_{\Phi}(v_1)\cap F_{\Phi}(v_2)|\geq|E_{\Phi}(v_1)\cap F_{\Phi}(v_2)|=3$ and $E_{\Phi}(v_1)\cap F_{\Phi}(v_3)=\varnothing$. Since $\Phi(v_1v_4)\in E_{\Phi}(v_1)$, we have $\Phi(v_1v_4)\notin F_{\Phi}(v_3)$. For edge v_1v_4, $|A_{\Phi}(v_1v_4)|=M+3-|F_{\Phi}(v_1)\cup F_{\Phi}(v_4)|\geq 15-(3+3+5+3)\geq 1$. Therefore, we choose $\alpha\in A_{\Phi}(v_1v_4)$ to recolor v_1v_4 and denote this new partial (2,1)-total labelling by Φ'. Obviously, $F_{\Phi'}(v_1)=F_{\Phi}(v_1)\cup\{\alpha\}\setminus\{\Phi(v_1v_4)\}$, $F_{\Phi'}(v_2)=F_{\Phi}(v_2)$ and $F_{\Phi'}(v_3)=F_{\Phi}(v_3)$. Thus, $\Phi(v_1v_4)\notin F_{\Phi'}(v_1)\cup F_{\Phi'}(v_3)$ which implies that we can color v_1v_3 with $\Phi(v_1v_4)$. For edge v_1v_2, we have $|A_{\Phi'}(v_1v_2)|=M+3-|F_{\Phi'}(v_1)\cup F_{\Phi'}(v_2)|\geq 15-(3+3+5+3-2)\geq 3$ because $|F_{\Phi'}(v_1)\cap F_{\Phi'}(v_2)|\geq 2$. Therefore, we choose $\Phi(v_1v_4)$ and $\beta\in A_{\Phi'}(v_1v_2)\setminus\{\Phi(v_1v_4)\}$ to color v_1v_3 and v_1v_2, respectively. Then we obtain a (2,1)-total labelling of G, a contradiction.

(*d*) Assume that there is a triangle face $f=uvv_1$ such that $vv_2\in E(G)$ and $2\leq d(v_1)=d(v_2)=M+2-d(v)\leq 3$ (see Fig. 1 (d)). By the minimality of G, $H=G-\{vv_1, vv_2\}$ is (2,1)-total labelable with color interval C. Erase the colors of v_1 and v_2, and denote this partial (2,1)-total labelling by Φ. Then $|A_{\Phi}(vv_1)|\geq M+3-|E_{\Phi}(v_1)|-|F_{\Phi}(v)|\geq M+3-(d(v_1)-1)-(d(v)-2+3)\geq M+3-(d(v_1)+d(v))\geq 1$. Similarly, $|A_{\Phi}(vv_2)|\geq 1$. If $\max\{|A_{\Phi}(vv_1)|,|A_{\Phi}(vv_2)|\}\geq 2$ or $A_{\Phi}(vv_1)\neq A_{\Phi}(vv_2)$, then we can label vv_1 and vv_2 properly by choosing colors from $A_{\Phi}(vv_1)$ and $A_{\Phi}(vv_2)$, respectively. If $A_{\Phi}(vv_1)=A_{\Phi}(vv_2)=\{\alpha\}$, then $\left(E_{\Phi}(v_1)\cup E_{\Phi}(v_2)\right)\cap F_{\Phi}(v)=\varnothing$ and $E_{\Phi}(v_1)=E_{\Phi}(v_2)$. Therefore we can exchange the colors of uv_1 and uv. Denote this new partial (2,1)-total labelling by Φ'. It is not difficult to see that $A_{\Phi'}(vv_1)=A_{\Phi}(vv_1)=\{\alpha\}$ and $|A_{\Phi'}(vv_2)|\geq M+3-(|E_{\Phi'}(v_2)|+|F_{\Phi'}(v)|-1)\geq M+3-(d(v_2)+d(v)-1)\geq 2$, then we can label vv_1 and vv_2 properly by choosing colors from $A_{\Phi'}(vv_1)$ and $A_{\Phi'}(vv_2)$, respectively.

At last, we extend the above partial labelling to a (2,1)-total labelling of G by labelling the 2-vertices v_1 and v_2 properly. This can be easily done since $|A_{\Phi}(v_1)|\geq M+3-4d(v_1)\geq M-9\geq 3$ and $|A_{\Phi}(v_2)|\geq M+3-4d(v_2)\geq 3$.

(*e*) Suppose that v is a Δ-vertex adjacent to two 2-vertices x and y. Let x' (resp. y') be the neighbor of x (resp. y) different from v.

Case 1: $x'=y'$, i.e. $vxx'y$ forms a 4-cycle (see Fig. 1 (e1)). By the minimality of G, $H=G-\{x, y\}$ has a (2,1)-total labelling Φ with color interval C. Then $|A_{\Phi}(vx)|\geq M+3-|F_{\Phi}(v)|\geq M+3-(\Delta-2+3)\geq 2$.

Similarly, $|A_{\Phi}(vy)| \geq 2, |A_{\Phi}(x'x)| \geq 2, |A_{\Phi}(x'y)| \geq 2$. Since $\chi_l'(C_4) = 2$, we can choose colors to label all the edges of 4-cycle $vxx'y$ properly. Denote this new partial (2,1)-total labelling by Φ'. Now consider the 2-vertices x and y. Since $|A_{\Phi'}(x)| \geq M+3-4d(x) \geq M-5 \geq 7$ and $|A_{\Phi'}(y)| \geq M+3-4d(y) \geq M-5 \geq 7$, we can extend Φ' from H to G, a contradiction.

Case 2: $\left\{vx', vy'\right\} \cap E(G) \neq \varnothing$. This case is impossible by Lemma 11 (d).

Case 3. $\left\{vx', vy'\right\} \cap E(G) = \varnothing$ (see Fig. 1 (e2)). By the minimality of G, $H = G - \left\{x, y\right\} \cup \left\{vx', vy'\right\}$ has a (2,1)-total labelling Φ, which implies that $\Phi(vx') \notin F_{\Phi}(x') \cup F_{\Phi}(v)$ and $\Phi(vy') \notin F_{\Phi}(y') \cup F_{\Phi}(v)$. Color xx',vy with $\Phi(vx')$ and color yy', vx with $\Phi(vy')$. Then we obtain a partial (2,1)-total labelling Φ' of G. Since $|A_{\Phi'}(x)| \geq M+3-4d(x) \geq M-5 \geq 7$ and $|A_{\Phi'}(y)| \geq M+3-4d(y) \geq M-5 \geq 7$, we can choose colors to label x and y properly. Thus we extend Φ' to graph G, a contradiction.

In the next section, we call a [5, 6, 7]-face a *special 3-face* and the other 3-face a *normal 3-face*. Lemma 11 implies that each 5-vertex is incident with at most two special 3-faces.

3. Discharging Part

Proof of Theorem 5: Let G be a minimal counterexample in terms of $|V| + |E|$ with $M \geq 12$. By the Lemmas of Section 2, we conclude that

- (C1) G is connected;
- (C2) For each $uv \in E$, $d(u) + d(v) \geq M - 1$;
- (C3) If $uv \in E$ and $\min\{d(u), d(v)\} \leq \left\lfloor \frac{M+2}{4} \right\rfloor$, then $d(u) + d(v) \geq M - 2$.
- (C4) Each i-vertex (if exists) has one j-master, where $2 \leq i \leq j \leq 3$;
- (C5) Each $(M - i)$-vertex (if exists) can be a j-master of at most $j - 1$ vertices, where $2 \leq i + 2 \leq j \leq 3$.
- (C6) G satisfies (a) – (e) of Lemma 11.

We define the initial charge function $w(x) := d(x) - 4$ for all element $x \in V \cup F$. By Euler's formula $|V| - |E| + |F| = 2$, we have $\sum_{x \in V \cup F} w(x) =$

$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(d(f)-4)=-8<0.$ The discharging rules are defined as follows.

(R1) Each 2-vertex receives charge $\frac{1}{2}$ from each of its incident Δ-vertex and receives charge 1 from its 3-master.

(R2) Each 3-vertex receives charge 1 from its 3-master.

(R3) Each 5-vertex transfer charge $\frac{1}{4}$ to each of its incident special 3-face and transfer $\frac{1}{6}$ to each of its incident normal 3-face.

(R4) Each k-vertex with $6 \le k \le 7$ transfer charge $\frac{k-4}{k}$ to each 3-face that incident with it.

(R5) Each 8^+-vertex transfer charge $\frac{1}{2}$ to each 3-face that incident with it.

Let v be a k-vertex of G. If $k = 2$, then $w'(v)=w(v)+1+\frac{1}{2}\times 2=-2+1+1=0$ by ($R1$) and ($C3$); If $k = 3$, then $w'(v)=w(v)+1=0$ since it receives 1 from its 3-master by ($R2$) and ($C4$); If $k = 4$, then $w'(v)=w(v)=0$ since we never change the charge by our rules; If $k = 5$, then $w'(v)\ge w(v)-\frac{1}{4}\times 2-\frac{1}{6}\times 3=0$ by ($R3$) and Lemma 11 (c); If $6 \le k \le 7$, then $w'(v)\ge w(v)-k\frac{k-4}{k}=0$ by ($R4$); If $8\le k\le M-2$, then $w'(v)\ge w(v)-k\frac{1}{2}\ge 0$ by ($R5$) and ($C3$);

By Lemma 7, it is not difficult to prove that $\delta(G)\ge 2$ when $\Delta = M$ and $\delta(G)\ge 3$ otherwise. If $M\ge\Delta+2$, then $M-2\ge\Delta$. Thus, $w(v) \ge 0$ for all $v \in V(G)$. Otherwise, $\Delta \le M \le \Delta + 1$. Consider the k-vertex v with $M - 1 \le d(v) = k \le \Delta$.

If $M = \Delta + 1$, then $\delta \ge 3$ and $k = \Delta = M - 1$. Lemma 11 (d) implies that ($M - 1$)-vertex is incident with at most $M - 4$ triangle faces if it has at least two 3-neighbors. Thus, together with rules ($R2$) and ($R5$), we have $w'(v)\ge \min\left\{w(v)-\frac{1}{2}\Delta-1, w(v)-\frac{1}{2}(\Delta-3)-2\right\}=\frac{M-1}{2}-5 \ge \frac{1}{2}$.

If $M = \Delta$, then $\delta \ge 2$. If $k = \Delta - 1 = M - 1$, then do the similar arguments as above . If $k = \Delta = M$, then $w'(v) \ge \min\left\{w(v)-\frac{1}{2}\Delta-1-\frac{1}{2}, w(v)-\frac{1}{2}(\Delta-3)-2-\frac{1}{2}\right\} = \frac{M-11}{2} > 0$ by Lemma 11 (d), (e) and rules ($R1$), ($R2$), ($R5$).

Let f be a k-face of G. If $k \ge 4$, then $w'(f)=w(f)\ge 0.$ since we never change the charge of them by our rules. If $k = 3$, then assume that $f = [d(v_1), d(v_2), d(v_3)]$ with $d(v_1) \le d(v_2) \le d(v_3)$. It is easy to see $w(f) = -1$. If

$d(v_1) \leq 3$, then $\min\{d(v_2), d(v_3)\} \geq M + 2 - d(v_1) \geq M - 1 \geq 11$ by ($C3$). Thus, $w'(f) = w(f) + \frac{1}{2} \times 2 = 0$ by ($R5$). If $d(v_1) = 4$, then $d(v_3) \geq d(v_2) \geq 8$ by Lemma 11 (a). Therefore, $w'(f) = w(f) + \frac{1}{2} \times 2 = 0$ by ($R5$). If $d(v_1) = 5$, then $d(v_2) = 6$, $d(v_3) \geq 7$ or $d(v_3) \geq d(v_2) \geq 7$ by Lemma 11 (b). If f is a special 3-face, then $w'(f) \geq w(f) + \frac{1}{4} + \frac{1}{3} + \frac{3}{7} = \frac{1}{84} > 0$ by ($R2$) and ($R3$). If f is a normal 3-face, then $d(v_2) = 6, d(v_3) \geq 8$ or $d(v_3) \geq d(v_2) \geq 7$. Therefore, $w'(f) \geq w(f) + \frac{1}{6} + \min\left\{\frac{1}{3} + \frac{1}{2}, \frac{3}{7} \times 2\right\} \geq 0$ by ($R3$) – ($R5$). If $d(v_1) = m \geq 6$, then $d(v_1) \geq d(v_1) \geq 6$. Therefore, $w'(f) \geq w(f) + 3 \times \min\left\{\frac{m-4}{m}, \frac{1}{2}\right\} = 0$ by ($R4$) and ($R5$).

Thus, we have $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) > 0$ since $w(v) > 0$ when $d(v) = \Delta$. This contradiction completes the proof.

Actually, the above proof implies the following immediate corollary.

Corollary 12: *If G is a graph embedded in a surface of nonnegative Euler characteristic with maximum degree $\Delta \geq 12$, then $\lambda_2^T(G) \leq \Delta + 2$.*

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