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Light paths and edges in families of outer-1-planar graphs ☆

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ABSTRACT

It is proved that (1) every maximal outer-1-planar graph of order at least k contains a path on k-vertices with all vertices of degree at most $2k+1$ (being sharp for $k \le 3$), and a path on k-vertices with degree sum at most $5k-1$, and further, (2) every maximal outer-1-planar graph contains an edge xy with $d(x)+d(y) \le 7$, and every outer-1-planar graph with minimum degree at least 2 contains an edge xy with $d(x)+d(y) \le 9$. Here the bounds 7 and 9 are sharp.

1. Introduction

All graphs in this paper are finite and simple. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. Set $v(G) = |V(G)|$ and $e(G) = |E(G)|$. The *neighborhood* (resp. *degree*) of a vertex v in G, denoted by $N_G(v)$ or $N(v)$ (resp. $d_G(v)$ or $d(v)$), is the set (resp. number) of vertices that are adjacent to v. For a subset S of $V(G)$, $G[S]$ denotes the subgraph induced by S. For a subgraph H of G, by $\Delta_G(H) = \max_{x\in V(H)}\{d_G(x)\}$ (resp. $W_G(H) = \sum_{x\in V(H)} d_G(x)$), we denote the *maximum degree* (resp. *degree sum*) of H in G. We use P_k to denote a k-path, that is a path on k-vertices.

Let H be a connected graph and let $\mathcal{G}$ be a family of graphs. If every graph $G \in \mathcal{G}$ of order at least $|V(H)|$ contains a subgraph $K \cong H$ such that

$$\Delta_G(K) \le t_h < +\infty \quad \text{and } W_G(K) \le t_w < +\infty, \qquad (1)$$

then we say that H is *light* in $\mathcal{G}$, and otherwise *heavy* in $\mathcal{G}$. The smallest integers t_h and t_w satisfying (1) are the *height* and *weight* of H in the family $\mathcal{G}$, denoted by $h(H,\mathcal{G})$ and $w(H,\mathcal{G})$, respectively.

The study of the lightness of certain subgraph in a given family initiates since the proposing of the well-known Four Coloring Conjecture. The first beautiful theorem on the theory of light subgraphs was contributed by Kotzig [13] in 1955, who proved that every 3-connected plane graph contains an edge of weight at most 13 (being sharp).

In general, finding light subgraphs in a given family is sometimes an essential stage when one considers some graph coloring and partition problems, since light subgraphs are possibly reducible (meaning that they cannot occur in a minimal counterexample to the desired conclusion, see [4]). Besides, the theory of light subgraphs also has many potential applications in both geometrical and combinatorial problems, see [12] for a survey.

A graph is *outerplanar* if it can be immersed into the plane so that all vertices lie on the outerface boundary. Harary [9] proved that every outerplanar graph contains a vertex v with $d(v) \le 2$. Hackmann and Kemnitz [8] proved that every outerplanar graph G with minimum degree 2 contains an edge xy with $d(x) \le d(y) \le 4$ and

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Table 1
Main results of this paper.

	$h(P_k, \mathcal{O}1\mathcal{P}^*)$	$w(P_k, \mathcal{O}1\mathcal{P}^*)$	$h(P_k, \mathcal{O}1\mathcal{P}_2)$	$w(P_k, \mathcal{O}1\mathcal{P}_2)$
$k=2$	5 [Corollary 2.8]	7 [Corollary 3.7]	7 [Corollary 3.4]	9 [Corollary 3.4]
$k=3$	7 [Corollary 2.9]	≤ 14 [Theorem 2.10]		
$k\geq 4$	$\leq 2k+1$ [Theorem 2.7]	$\leq 5k-1$ [Theorem 2.10]		

$d(x)+d(y)\leq 6$. Fabrici [6] showed that every 2-connected outerplanar graph G of order at least $k\geq 3$ contains a k-path P_k with $\Delta_G(P_k)\leq k+3$. All upper bounds in those conclusions are the best possible. Hence, if $\mathcal{OP}_2$ is the family of 2-connected outerplanar graphs, then $h(P_1,\mathcal{OP}_2)=2$, $h(P_2,\mathcal{OP}_2)=4$ and $h(P_k,\mathcal{OP}_2)=k+3$ for $k\geq 3$. In other words, P_k with $k\geq 1$ is light in the family of 2-connected outerplanar graphs.

A graph is *1-immersed into a plane*, or *1-planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of the 1-planarity was introduced by Ringel [14] in connection with the problem of the simultaneous coloring of vertices and faces of plane graphs in which adjacent/incident elements receive different colors. Fabrici and Madaras [7] showed that each 3-connected 1-planar graph contains an edge with both endvertices of degree at most 20, and the bound 20 is the best possible. Hudák and Šugerek [11] proved that each 1-planar graph with minimum degree at least 4 contains an edge of type (with degrees of its endvertices) $(4,\leq 13)$ or $(5,\leq 9)$ or $(6,\leq 8)$ or $(7,7)$. The global or local structures of 1-planar graphs and their applications to coloring problems were also studied by many authors including [7,10,15,17,18].

A graph is *outer-1-planar* if it can be 1-immersed into a plane such that all vertices are on the outer face. For example, $K_{2,3}$ and K_4 are outer-1-planar graphs. Outer-1-planar graphs were first introduced by Eggleton [5] who called them *outerplanar graphs with edge crossing number one*, and also investigated under the notion of *pseudo-outerplanar graph* by Zhang, Liu and Wu [16,19]. They proved that every outer-1-planar graph contains a vertex of degree at most 3, and the bound is the best possible.

It is clear that the class of outer-1-planar graphs lies between the classes of 1-planar graphs and outerplanar graphs (see [1]). Given an outer-1-planar graph G, we add a new vertex adjacent to every other vertex of G and obtain a graph G'. Clearly, G' is a 1-planar graph with minimum degree one larger than that of G. Moreover, if G is 2-connected, then G' is 3-connected, and by the result of Fabrici and Madaras [7] mentioned above, G' contains an edge uv with $\max\{d_{G'}(u),d_{G'}(v)\}\leq 20$. If $|G|\geq 21$, then $u,v\in V(G)$ and thus G contains an edge uv with $\max\{d_G(u),d_G(v)\}\leq 19$. This concludes that P_2 is light (with height at most 19) in the family of 2-connected outer-1-planar graphs. In this paper, we generalize this result by showing that P_2 is light (with height exactly 7) in the family of outer-1-planar graphs with minimum degree at least two.

A drawing of an outer-1-planar graph in the plane, so that its outer-1-planarity is preserved and the number of crossings is as few as possible, is an *outer-1-plane* graph, and we call such a drawing *good*. Let G be a 2-connected outer-1-plane graph. Denote by $v_1,v_2,\dots,v_{|G|}$ the vertices of G lying clockwise on the outer boundary. Let $\mathcal{V}[v_i,v_j]=\{v_i,v_{i+1},\dots,v_j\}$ and let $\mathcal{V}(v_i,v_j)=\mathcal{V}[v_i,v_j]\setminus\{v_i,v_j\}$, where the subscripts are taken modulo $|G|$. Set $\mathcal{V}[v_i,v_i]=V(G)$. An edge v_iv_j in G is a *chord* if $j-i\neq 1$ (mod $|G|$). By $\mathcal{C}[v_i,v_j]$, we denote the set of chords xy with $x,y\in\mathcal{V}[v_i,v_j]$.

An outer-1-planar graph G is *maximal* if adding any edge would disturb its outer-1-planarity. The structures and colorings of maximal outer-1-planar graphs were studied in many papers including [2,16]. By $\mathcal{O}1\mathcal{P}^*$ and $\mathcal{O}1\mathcal{P}_2$, we denote the family of maximal outer-1-planar graphs and the family of outer-1-planar graphs with minimum degree at least 2, respectively.

In this paper, we investigate the lightness of P_k in $\mathcal{O}1\mathcal{P}^*$ and $\mathcal{O}1\mathcal{P}_2$, and the main results are listed in Table 1.

2. Light paths in maximal outer-1-planar graphs

Zhang, Liu and Wu [16] proved that the vertex connectivity of any outer-1-planar graph besides K_4 is at most 2. It is known that every 2-connected outerplanar graph is hamiltonian [3], and Zhang, Liu and Wu [16] pointed out that this fact does not hold for 2-connected outer-1-planar graphs (e.g. $K_{2,3}$ is 2-connected and non-hamiltonian). For this reason, we consider maximal outer-1-planar graphs below.

Lemma 2.1. *Every maximal outer-1-planar graph of order at least 3 is 2-connected.*

Proof. Let G be a maximal outer-1-planar G with blocks $G_1,G_2,\dots,G_t$. If $t=1$, then the conclusion is trivial. If $t\geq 2$, then choose an end-block (i.e., a block incident with only one cut-vertex) G_i that is incident with a block G_j. Clearly, G_i and G_j have exactly one common vertex, say v_1.

Let $v_1,v_2\dots,v_{v(G_i)}$ be vertices of G_i in a clockwise ordering on the outer boundary of a good drawing of G_i and let $v_1,u_2\dots,u_{v(G_j)}$ be vertices of G_j in an anticlockwise ordering on the outer boundary of a good drawing of G_j. It is easy to check that the graph obtained from G by adding an edge u_2v_2 is still outer-1-planar, which contradicts the fact that G is maximal. □

Lemma 2.2. *Every maximal outer-1-planar graph with order at least 3 is hamiltonian.*

Proof. Let $v_1,v_2,\dots,v_{|G|}$ be vertices lying in a cyclic ordering on the outer boundary of an outer-1-planar drawing of G. Since G is maximal, $v_1v_2\dots v_{|G|}v_1$ forms a cycle C. Clearly, C is a hamiltonian cycle of G. □

Lemma 2.3. *[16] Every hamiltonian outer-1-planar graph G can be 1-immersed in the plane so that the hamiltonian cycle in G is the outer boundary.*

Lemma 2.4. *[16] Every outer-1-planar graph with minimum degree at least two contains at least two vertices of degree at most 3.*

Lemma 2.5. *Every outer-1-planar graph of order at least two contains at least two vertices of degree at most 3.*

Proof. Let G be an outer-1-planar graph. If G is not connected, then it is enough to consider one component of G, and if $\delta(G) \geq 2$, then the conclusion holds by Lemma 2.4. Therefore, we assume that G is connected and $\delta(G) = 1$. Let xy be an edge with $d_G(x) = 1$. If G has two vertices of degree 1 or $d_{G-x}(y) = 1$, then we are done. Therefore, we assume that $G - x$ is an outer-1-planar graph with minimum degree at least two, so it contains a vertex $z \neq y$ with $d_{G-x}(z) = d_G(z) \leq 3$ by Lemma 2.4. Hence G contains two vertices x and z of degree at most 3. □

Lemma 2.6. *[1] If G is an outer-1-planar graph, then* $e(G) \leq \frac{5}{2}v(G) - 4$.

Theorem 2.7. *Every maximal outer-1-planar graph G of order at least k contains a k-path P_k with $\Delta_G(P_k) \leq 2k + 1$. In other words, $h(P_k, \mathcal{O}1\mathcal{P}^*) \leq 2k + 1$*

Proof. If $k = 1$, then the conclusion holds since every outer-1-planar graph contains a vertex of degree at most 3. Hence we assume $k \geq 2$ in the following arguments.

Suppose, to the contrary, that every k-path in G contains a vertex of degree at least $2k + 2$, saying a *big* vertex. Vertices of degree at most $2k + 1$ in G are *small* vertices. Let $v_1, v_2, \ldots, v_{|G|}$ be vertices of G lying clockwise on the outer boundary of G. Since G is maximal, $v_1 v_2 \ldots v_{|G|} v_1$ forms a hamiltonian cycle C.

Let $v_{i_1}, \ldots, v_{i_t}$ be the big vertices of G in a clockwise ordering on the outer boundary. They split C into at most t paths, and those paths contain only small vertices, so each of them contains at most $k-1$ vertices. First we have $t \geq 2$, because otherwise $|G| = k$ and $d_G(v_{i_1}) \leq k - 1 < 2k + 2$, a contradiction.

Let

$$B_s = \{v_{i_1}, \ldots, v_{i_s}\}$$

with $s \in [t] = \{1, 2, \ldots, t\}$ and let

$$S = \left\{ xy \in E(G) \mid x = v_{i_s}, y \notin \mathcal{V}[v_{i_{s-1}}, v_{i_{s+1}}] \cup B_t, s \in [t] \right\}.$$

Thus, S is a set of edges that are incident with a big vertex v_{i_s} for some $s \in [t]$ and a small vertex not belonging to $\mathcal{V}(v_{i_{s-1}}, v_{i_{s+1}})$.

Case 1. $S \neq \emptyset$.

Choose an edge (indeed, a chord) $xy \in S$ with big vertex $x = v_{i_s}$ and small vertex $y \in \mathcal{V}(v_{i_{r-1}}, v_{i_r})$ so that the *boundary distance* between x and y, which is the distance between x and y on C, is as small as possible.

Without loss of generality, assume that $x = v_{i_1}$, and that the boundary distance between x and y is exactly $|\mathcal{V}[x, y]| - 1$. We then have $r \geq 3$ by the definition of S.

Claim 1. *$r \geq 6$, and exactly $r - 3$ vertices among $v_{i_2}, \ldots, v_{i_{r-1}}$ have degrees at least 4 in the graph induced by B_{r-1}.*

Proof. Since a big vertex v_{i_s} with $2 \leq s \leq r - 1$ is adjacent to at most $2(k - 1)$ small vertices in $\mathcal{V}[x, y]$ by the choice of xy and at most one vertex in $\mathcal{V}(y, x)$, v_{i_s} is adjacent to at least $(2k + 2) - (2k - 2) - 1 = 3$ vertices in the graph induced by B_{r-1}. Moreover, if v_{i_s} has exactly three neighbors in the graph induced by B_{r-1}, then xy is crossed by an edge incident with v_{i_s}. Since xy can be crossed at most once, among $v_{i_2}, \ldots, v_{i_{r-1}}$, there is at most one vertex having degree 3 in $G[B_{r-1}]$, and at least $r - 3$ vertices having degrees at least 4 in $G[B_{r-1}]$. Since $G[B_{r-1}]$ has $r - 1$ vertices and its minimum (resp. maximum) degree is at least 3 (resp. 4), $r \geq 6$. On the other hand, if all vertices among $v_{i_2}, \ldots, v_{i_{r-1}}$ have degrees at least 4 in $G[B_{r-1}]$, then $G[B_{r-1}]$ has at most one vertex of degree at most 3, which is impossible by Lemma 2.5. Therefore, exactly $r - 3$ vertices among $v_{i_2}, \ldots, v_{i_{r-1}}$ have degrees at least 4 in the graph induced by B_{r-1}. □

Claim 2. *If z is a small vertex in $\mathcal{V}(x, y)$, then z has no neighbor in $\mathcal{V}(y, x)$.*

Proof. If this does not hold, then xy is crossed by an edge incident with z, and thus cannot be crossed by an edge incident with some vertex among $v_{i_2}, \ldots, v_{i_{r-1}}$. This implies that all vertices among $v_{i_2}, \ldots, v_{i_{r-1}}$ have degrees at least 4 in $G[B_{r-1}]$ by the same argument as the one in the proof of Claim 1. However, this contradictions Claim 1. □

Claim 3. *If z_1, z_2 are two small vertices in $\mathcal{V}[x, y]$, then $z_1 z_2 \in E(G)$ only if $z_1, z_2 \in \mathcal{V}(v_{i_a}, v_{i_{a+1}})$ for some $1 \leq a \leq r - 1$.*

Proof. Suppose that $z_1 z_2 \in E(G)$, $z_1 \in \mathcal{V}[v_{i_a}, v_{i_{a+1}}]$ and $z_2 \in \mathcal{V}[v_{i_b}, v_{i_{b+1}}]$, where $1 \leq a < b \leq r - 1$. Let $B' = \{v_{i_{a+1}}, \ldots, v_{i_b}\}$. Since $z_1 z_2 \in E(G)$ and $z_1, z_2 \in \mathcal{V}[x, y]$, xy cannot be crossed by an edge incident with B'. Since a big vertex v_{i_s} with $a + 1 \leq s \leq b$ is adjacent to at most $2(k - 1)$ small vertices in $\mathcal{V}[x, y]$ by the choice of xy, and adjacent to no vertex in $\mathcal{V}(y, x)$, v_{i_s} is adjacent to at least $(2k + 2) - (2k - 2) = 4$ vertices in the graph induced by B'. This implies $|B'| \geq 5$ and $\delta(G[B']) \geq 4$. However, $G[B']$ is an outer-1-planar graph with minimum degree at most 3 by Lemma 2.5. This is a contradiction. □

Claim 4. *For any small vertex $z \in \mathcal{V}[x, y] \setminus \{y\}$, if $z \in \mathcal{V}[v_{i_a}, v_{i_{a+1}}]$ for some $1 \leq a \leq r - 1$, then $N_G(z) \subseteq \mathcal{V}[v_{i_a}, v_{i_{a+1}}]$.*

Proof. By Claim 2 and by the choice of xy, if z is adjacent to a big vertex in G, then this big vertex is either v_{i_a} or $v_{i_{a+1}}$. By Claims 2 and 3, if z is adjacent to a small vertex in G, then this small vertex belongs to $\mathcal{V}(v_{i_a}, v_{i_{a+1}})$. □

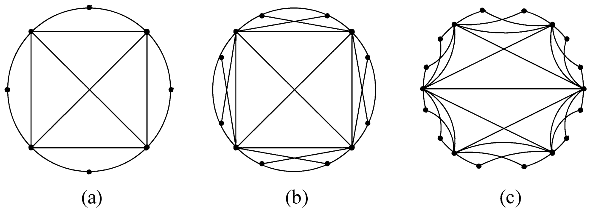

Fig. 1. Extremal outer-1-planar graphs.

Since G is maximal, $v_{i_1}v_{i_2} \in E(G)$ by Claim 4. If $v_{i_1}v_{i_3} \in E(G)$, then v_{i_2} has degree at most $2(k-1)+3 = 2k+1$ in G, a contradiction. Hence $v_{i_1}v_{i_3} \notin E(G)$.

If xy is crossed by an edge incident with v_{i_2}, then by the proof of Claim 1, v_{i_2} has degree 3 in $G[B_{r-1}]$ and all vertices among $v_{i_3}, \dots, v_{i_{r-1}}$ have degrees at least 4 in $G[B_{r-1}]$. Clearly, v_{i_2} is the unique neighbor of v_{i_1} in $G[B_{r-1}]$. Therefore, the outer-1-planar graph derived from $G[B_{r-1}]$ by deleting v_{i_1} has exactly one vertex of degree at most 3 in that graph, contradicting Lemma 2.5.

In what follows, we assume, without loss of generality, that xy is not crossed by an edge incident with v_{i_2}.

Since G is maximal and $v_{i_1}v_{i_3} \notin E(G)$, there is an edge $v_{i_2}v_{i_a}$ for some $4 \le a \le r-1$ by Claim 4 and by the choice of xy (otherwise we can add a new edge $v_{i_1}v_{i_3}$ to G so that the resulting graph is still outer-1-planar).

First, suppose that there is no edge in the form $v_{i_b}v_{i_c}$ with $|c-b| \ne 1$, besides $v_{i_2}v_{i_a}$, in the graph induced by $\{v_{i_2}, \dots, v_{i_a}\}$. Under this assumption, for any vertex v_{i_b} with $3 \le b \le a-1$, it is adjacent to at most $2(k-1)$ small vertices in G by the choice of xy, at most two big vertices in $\{v_{i_2}, \dots, v_{i_a}\}$, and at most one vertex in $\mathcal{V}(v_{i_a}, v_{i_2})$, which implies that the degree of v_{i_b} in G is at most $2k+1$, a contradiction.

At last, we assume that $v_{i_b}v_{i_c}$ with $|c-b| \ne 1$ is an edge in the graph induced by $\{v_{i_2}, \dots, v_{i_a}\}$ so that $|c-b|$ is minimum. Without loss of generality, assume that $b < c$. By similar argument as the one in the previous paragraph, one can conclude that any vertex v_{i_s} with $b < s < c$ has degree at most $2k+1$ in G, a contradiction.

Case 2. $S = \emptyset$.

Since the graph induced by B_t is an outer-1-planar graph, it has a vertex, say v_{i_s}, of degree at most 3 in $G[B_t]$ by Lemma 2.5. Since $S = \emptyset$, v_{i_s} is adjacent to at most $2(k-1)$ small vertices in G. Therefore, the degree of v_{i_s} in G is at most $2k+1$, a contradiction. □

Corollary 2.8. *Every maximal outer-1-planar graph G of order at least 2 contains an edge uv with $\max\{d_G(u), d_G(v)\} \le 5$, and the bound 5 is the best possible. In other words, $h(P_2, \mathcal{O}1\mathcal{P}^*) = 5$.*

Proof. The existence of such an edge uv is guaranteed by Theorem 2.7. The sharpness of the upper bound 5 is implied by picture (a) of Fig. 1, where each edge contains at least one vertex of degree 5. □

Corollary 2.9. *Every maximal outer-1-planar graph G of order at least 3 contains a path uvw with $\max\{d_G(u), d_G(v), d_G(w)\} \le 7$, and the bound 7 is the best possible. In other words, $h(P_3, \mathcal{O}1\mathcal{P}^*) = 7$.*

Proof. The existence of such a path uvw is guaranteed by Theorem 2.7. The sharpness of the upper bound 7 is implied by picture (b) of Fig. 1, where each edge contains at least one vertex of degree 7. □

Theorem 2.10. *Every maximal outer-1-planar graph G of order at least k contains a k-path P_k with $W_G(P_k) \le 5k-1$. In other words, $w(P_k, \mathcal{O}1\mathcal{P}^*) \le 5k+1$.*

Proof. Assume that G is drawn in the plane so that $x_1, x_2, \dots, x_{|V(G)|}$ are vertices lying cyclicly on the outer boundary of G. Since G is maximal, $x_1x_2\dots x_{v(G)}x_1$ is a cycle. For $i = 1, \dots, v(G)$, let Q_i be the k-path $x_ix_{i+1}\dots x_{i+k-1}$ (indices modulo $v(G)$). Since $e(G) \le \frac{5}{2}v(G) - 4$ by Lemma 2.6, we have

$$\frac{1}{v(G)}\sum_{i=1}^{v(G)} W_G(Q_i) = \frac{k}{v(G)}\sum_{x\in V(G)} d_G(x) = \frac{k}{v(G)}2e(G)$$
$$\le 5k - \frac{8}{v(G)} < 5k,$$

which implies that there is a k-path Q_j for some $j \in \{1, \dots, v(G)\}$ with $W_G(Q_j) \le 5k-1$. □

3. Light edges in outer-1-planar graphs

Lemma 3.1. *Let $v_1v_2\dots v_n$ be an n-path with $n \ge 4$. If there is no pair of chords v_iv_j and v_kv_l with $1 \le i < k < j < l \le n$, then among $v_2, \dots, v_{n-1}$, there is a 2-valent vertex adjacent to a vertex of degree at most 4 in the graph induced by $v_1, v_2, \dots, v_n$.*

Proof. For any two vertices v_i and v_j with $i < j$. Denote by G_{ij} the graph induced by $\{v_i, v_{i+1}, \dots, v_j\}$. For simplify, let $G = G_{1n}$.

If $n = 4$, then the required result is trivial since $4 \le d_G(v_2) + d_G(v_3) \le 5$. Therefore, we assume that it holds for any n'-path with $4 \le n' < n$, and then prove it for n-path by induction.

Let v_iv_j be a chord with $j - i \ge 2$. If $j - i \ge 3$, then by the induction hypothesis, there are two adjacent vertices v_a and v_b with $i+1 \le a, b \le j-1$ so that $d_{G_{ij}}(v_a) = 2$ and $d_{G_{ij}}(v_b) \le 4$. Since v_iv_j is non-crossed, $d_G(v_a) = d_{G_{ij}}(v_a) = 2$ and $d_G(v_b) = d_{G_{ij}}(v_b) \le 4$. Now we find a 2-valent vertex v_a adjacent to a vertex v_b of degree at most 4 in G. Therefore, for any chord v_iv_j with $i < j$, we have $j - i = 2$, and moreover, $d_G(v_{i+1}) = 2$.

Suppose that v_i is adjacent to v_j with $j - i > 1$ only if $i = 1$ and $j = n$. Clearly, v_2 and v_3 are two adjacent 2-valent vertices, and we complete the proof. Therefore, we assume that there is a chord v_iv_j with $j - i > 1$ so that either $i \ne 1$ or $j \ne n$. Without loss of generality, assume that $j \ne n$. If $d_G(v_j) \le 4$, then we are done. If $d_G(v_j) \ge 5$, then there exists a chord v_jv_s with $|s - j| \ge 3$, which is impossible by the conclusion of the above paragraph. □

Definition 1. Let G be a 2-connected outer-1-plane graph with vertices $v_1, v_2, \ldots, v_{|G|}$ lying clockwise on its outer boundary. If v_iv_j crosses v_kv_l with $l-j = j-k = k-i = 1 \pmod{|G|}$, $v_iv_k, v_jv_l \in E(G)$ and $v_kv_l \notin E(G)$, then we say that v_iv_j **co-crosses** v_kv_l.

Lemma 3.2. *Let G be a 2-connected (resp. maximal) outer-1-plane graph with vertices $v_1, v_2, \ldots, v_n$ lying clockwise on its outer boundary, where $n = |G| \geq 5$. Among $v_2, \ldots, v_{n-1}$, there is a 2-valent vertex adjacent to a vertex of degree at most 7 (resp. 5), or a 3-valent vertex adjacent to another one 3-valent vertex in G.*

Proof. Let v_i and v_j be two vertices with $i < j$. Denote by G_{ij} the graph induced by $\mathcal{V}[v_i, v_j]$. Note that in the following arguments we do not distinguish whether G is 2-connected or maximal, unless we state specially.

If $n = 5$, then there is a pair of chords v_iv_j and v_kv_l with $1 \leq i < k < j < l \leq 5$, because otherwise $v_1v_2v_3v_4v_5$ is a path, and then by Lemma 3.1 we will find, among v_2, v_3 and v_4, a 2-valent vertex adjacent to a vertex of degree at most 4. Since G is 2-connected or maximal, $\delta(G) \geq 2$. By symmetry, we consider three cases. First, if v_1v_3 crosses v_2v_4, then either $v_2v_3 \in E(G)$ and $d_G(v_2) + d_G(v_3) \leq 6$, or $v_2v_3 \notin E(G)$ and $v_1v_2, v_3v_4 \in E(G)$, which implies that $d_G(v_3) = 2$ and $d_G(v_4) \leq 4$. Second, if v_1v_3 crosses v_2v_5, then either $v_2v_3 \in E(G)$ and $d_G(v_2) + d_G(v_3) \leq 6$, or $v_2v_3 \in E(G)$ and $d_G(v_2) + d_G(v_3) = 7$, which implies that $v_3v_4 \in E(G)$, $d_G(v_3) \leq 4$ and $d_G(v_4) = 2$, or $v_2v_3 \notin E(G)$, which implies that $v_3v_4 \in E(G)$, $d_G(v_3) \leq 3$ and $d_G(v_4) = 2$. Third, if v_1v_4 crosses v_2v_5, then $d_G(v_3) = 2$, and v_3 is adjacent to either v_2 or v_4, any of which has degree at most 4 in G.

In what follows, we prove the conclusion by induction on n, assuming that it holds for any 2-connected (resp. maximal) outer-1-plane graph with order less than n.

Claim 1. *If v_iv_j is a chord crossed by v_kv_l with $i < k < j < l$, then we can find two adjacent vertices among $\mathcal{V}(v_i, v_l)$ that satisfy the conclusion of Lemma 3.2, unless v_iv_j co-crosses v_kv_l.*

Proof. If $k - i \geq 4$, then by the induction hypothesis, there are two adjacent vertices $v_a, v_b \in \mathcal{V}[v_{i+1}, v_{k-1}]$ so that $d_{G_{ik}}(v_a) = d_{G_{ik}}(v_b) = 3$, or $d_{G_{ik}}(v_a) = 2$ and $d_{G_{ik}}(v_b) \leq 7$ ($d_{G_{ik}}(v_b) \leq 5$ if G is maximal). Since there is no edge between $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$ by the outer-1-planarity of G, $d_G(u) = d_{G_{ik}}(u)$ for any vertex $u \in \mathcal{V}(v_i, v_k)$. Hence v_a and v_b are the two required vertices in G. Therefore, $k - i \leq 3$, and similarly, we have $j - k \leq 3$ and $l - j \leq 3$.

If $j - k = 3$, then v_kv_{j-1} crosses $v_{k+1}v_j$ (otherwise $d_G(v_{k+1}) + d_G(v_{j-1}) \leq 5$ and $v_{k+1}v_{j-1} \in E(G)$, so we finish the proof). If $v_{k+1}v_{j-1} \in E(G)$, then $d_G(v_{k+1}) = d_G(v_{j-1}) = 3$ and we complete the proof. Hence we assume that $v_{k+1}v_{j-1} \notin E(G)$ and thus G is not maximal. Now we have that $d_G(v_{j-1}) = 2$. Since $l - j \leq 3$, we have $d_G(v_j) \leq 7$, and thus v_{j-1} and v_j are the required vertices. Hence $j - k \leq 2$.

If $k - i = 2$, then $d_G(v_{k-1}) = 2$. Since $j - k \leq 2$, $d_G(v_k) \leq 5$ and thus v_{k-1}, v_k are the required vertices. Hence $k - i \neq 2$, and by symmetry, $l - j \neq 2$.

If $j - k = 2$, then $d_G(v_{j-1}) = 2$. If $l - j = 1$, then $d_G(v_j) \leq 4$, and v_{j-1}, v_j are the required vertices. If $l - j = 3$, then v_jv_{l-1} crosses $v_{j+1}v_l$ (otherwise $d_G(v_{j+1}) + d_G(v_{l-1}) \leq 5$ and $v_{j+1}v_{l-1} \in E(G)$, we complete the proof). If G is maximal, then $d_G(v_{j+1}) = d_G(v_{l-1}) = 3$ and $v_{j+1}v_{l-1} \in E(G)$, so v_{j+1} and v_{l-1} are the required vertices. If G is not maximal but is 2-connected, then $d_G(v_j) \leq 6$, so v_{j-1} and v_j are the required vertices. Hence $j - k \neq 2$ and thus

$$j - k = 1.$$

Suppose that $k - i = 1$. If $l - j = 1$, then either v_iv_j co-crosses v_kv_l, or v_k and v_j are two adjacent 3-valent vertices, and we are done. If $l - j \neq 1$, then $l - j = 3$, and moreover, v_jv_{l-1} crosses $v_{j+1}v_l$ (otherwise $d_G(v_{j+1}) + d_G(v_{l-1}) \leq 5$ and $v_{j+1}v_{l-1} \in E(G)$, we complete the proof). If $v_{j+1}v_{l-1} \in E(G)$, then $d_G(v_{j+1}) = d_G(v_{l-1}) = 3$, and v_{j+1}, v_{l-1} are the required vertices. If $v_{j+1}v_{l-1} \notin E(G)$, then $v_jv_{j+1} \in E(G)$, $d_G(v_j) \leq 5$ and $d_G(v_{j+1}) = 2$, so v_j and v_{j+1} are the required vertices. Hence $k - i \neq 1$, and by symmetry, $l - j \neq 1$. This implies that

$$l - j = k - i = 3.$$

Since $k - i = 3$, v_iv_{k-1} crosses $v_{i+1}v_k$ (otherwise $d_G(v_{i+1}) + d_G(v_{k-1}) \leq 5$ and $v_{i+1}v_{k-1} \in E(G)$, we are done). If $v_{i+1}v_{k-1} \in E(G)$, then $d_G(v_{i+1}) = d_G(v_{k-1}) = 3$, and v_{i+1}, v_{k-1} are the required vertices. If $v_{i+1}v_{k-1} \notin E(G)$, then $v_{k-1}v_k \in E(G)$, $d_G(v_{k-1}) = 2$ and $d_G(v_k) \leq 5$, so v_{k-1} and v_k are the required vertices. □

Claim 2. *If v_iv_j is a non-crossed chord with $j - i \geq 4$, then there are two adjacent vertices in $\mathcal{V}(v_i, v_j)$ that satisfy the conclusion of Lemma 3.2.*

Proof. Since $j - i \geq 4$, $|\mathcal{V}[v_i, v_j]| \geq 5$. By the induction hypothesis, there are two adjacent vertices $v_a, v_b \in \mathcal{V}(v_i, v_j)$ so that $d_{G_{ij}}(v_a) = d_{G_{ij}}(v_b) = 3$, or $d_{G_{ij}}(v_a) = 2$ and $d_{G_{ij}}(v_b) \leq 7$ ($d_{G_{ik}}(v_b) \leq 5$ if G is maximal). Since v_iv_j is non-crossed, $d_G(v_a) = d_{G_{ij}}(v_a)$ and $d_G(v_b) = d_{G_{ij}}(v_b)$. Therefore, v_a and v_b are the two adjacent vertices in $\mathcal{V}(v_i, v_j)$ that satisfy the conclusion. □

If no crossing exists in G, then by Lemma 3.1, we can find a 2-valent vertex adjacent to a vertex of degree at most 4 among $v_2, \ldots, v_{n-1}$. Hence we assume that G contains at least one crossing.

Suppose that v_iv_j co-crosses v_kv_l with $i < k < j < l = i + 4$ by Claim 1. Clearly, since $v_kv_j \notin E(G)$ (otherwise v_k and v_l are two adjacent 3-valent vertices in G, and the proof is completed), G is not maximal. Since $n \geq 5$, either $i \neq 1$ or $l \neq n$. Without loss of generality, assume that $l \neq n$. Since $d_G(v_k) = d_G(v_j) = 2$, $d_G(v_i) \geq 8$ and $d_G(v_l) \geq 8$, because otherwise we can find the desired two vertices. This implies either a chord v_lv_s with $l < s$, or a chord v_tv_i with $t < i$. By symmetry, we assume that the chord v_lv_s with $l < s$ exists.

If v_lv_s is crossed by a chord v_av_b with $l < a < s$, then by Claim 1, v_lv_s co-crosses v_av_b. This implies that

$b-s=1$ and $d_G(v_a)=d_G(v_s)=2$. Since $d_G(v_l)\geq 8$, there is a chord v_lv_r with $b<r$, or a chord v_lv_t with $t<i$. By symmetry, we assume that the chord v_lv_r with $b<r$ exists. By Claim 1, v_lv_r is non-crossed. By Claim 2, there are two adjacent vertices $v_\alpha, v_\beta \in \mathcal{V}(v_l, v_r)$ that satisfy the conclusion.

On the other hand, suppose that v_lv_s is non-crossed. If $s-l\geq 4$, then by Claim 2, one can find two adjacent vertices $v_\alpha, v_\beta \in \mathcal{V}(v_l, v_s)$ that satisfy the conclusion. If $s-l=3$, then v_lv_{s-1} co-crosses $v_{l+1}v_s$ (otherwise $v_{l+1}v_{s-1}\in E(G)$ and $d_G(v_{l+1})+d_G(v_{s-1})\leq 6$, we complete the proof), and we meet the same condition as the one appearing in the previous paragraph. Hence we lastly consider the case when $s-l=2$, which implies that $d_G(v_{l+1})=2$.

Since $d_G(v_l)\geq 8$, there is a chord v_lv_t with $t<i$ or $t>s$. If $t<i$, then v_lv_t is non-crossed by Claim 1, so one can find two adjacent vertices $v_\alpha, v_\beta \in \mathcal{V}(v_t, v_l)$ that satisfy the conclusion by Claim 2. Therefore, $t>s$. By Claim 1, v_lv_t is non-crossed. If $t-l\geq 4$, then one can find two adjacent vertices $v_\alpha, v_\beta \in \mathcal{V}(v_l, v_t)$ that satisfy the conclusion by Claim 2. If $t-l=3$, then $d_G(v_s)=3$, $d_G(v_{s-1})=2$ and $v_{s-1}v_s\in E(G)$. Hence v_{s-1} and v_s are the required vertices. □

Theorem 3.3. *Every outer-1-planar graph with minimum degree at least two contains an edge xy with $d(x)=d(y)=3$, or $d(x)=2$ and $d(y)\leq 7$.*

Proof. Let B be an end-block of G with vertices $v_1, v_2, \ldots, v_{|B|}$ lying consecutively on the outer boundary of a good drawing of B. Without loss of generality, let v_1 be the unique cut-vertex on B. Since $\delta(G)\geq 2$, $|B|\geq 3$.

If $|B|\geq 5$, then by Lemma 3.2, there are two adjacent vertices v_i and v_j among $v_2, v_3, \ldots, v_{|B|-1}$ so that $d_B(v_i)=d_B(v_j)=3$, or $d_B(v_i)=2$ and $d_B(v_j)\leq 7$. Since $d_G(v_i)=d_B(v_i)$ and $d_G(v_j)=d_B(v_j)$, we complete the proof by letting $x:=v_i$ and $y:=v_j$.

If $|B|=4$, then either $v_2v_3\in E(G)$ and $d_G(v_2)+d_G(v_3)\leq 6$, or $v_3v_4\in E(G)$ and $d_G(v_3)+d_G(v_4)\leq 6$, so we let $\{x,y\}:=\{v_2,v_3\}$, or let $\{x,y\}:=\{v_3,v_4\}$, respectively.

If $|B|=3$, then $v_2v_3\in E(G)$ and $d_G(v_2)=d_G(v_3)=2$. Hence we complete the proof by letting $x:=v_2$ and $y:=v_3$. □

Corollary 3.4. *Every outer-1-planar graph with minimum degree at least 2 contains an edge xy with $d(x)+d(y)\leq 9$ and $\max\{d(x),d(y)\}\leq 7$, and the upper bounds 9 and 7 are the best possible. In other words, $w(P_2, \mathcal{O}1\mathcal{P}_2)=9$ and $h(P_2, \mathcal{O}1\mathcal{P}_2)=7$.*

Proof. Theorem 3.3 directly implies this result, and the sharpness of the two upper bounds can be confirmed by picture (c) of Fig. 1, where the weight of each edge is at least 9 and each edge contains a vertex of degree at least 7. □

Corollary 3.5. *If G is an outer-1-planar graph of order at least 2 so that the distance of any two 1-valent vertices is at least 3, then G contains an edge xy with $d(x)+d(y)\leq 9$, moreover, the upper bound 9 is the best possible.*

Proof. If $\delta(G)\geq 2$, then Theorem 3.3 implies this result. Hence we assume that G is a minimal (in terms of the order) counterexample to this with $\delta(G)=1$. Let uv be an edge with $d_G(u)=1$. If $d_G(v)\leq 2$, then we complete the proof, so we assume that $d_G(v)\geq 3$.

By the minimality of G, $H:=G-u$ contains an edge xy with $d_H(x)+d_H(y)\leq 9$. If none of x and y is the vertex v, then $d_G(x)+d_G(y)=d_H(x)+d_H(y)\leq 9$, a contradiction. If x or y, say x, is v, then consider two cases. First, if $d_H(x)\leq 7$, then $d_G(x)=d_H(x)+1\leq 8$ and $d_G(x)+d_G(u)\leq 9$, a contradiction. Second, if $d_H(x)\geq 8$, then $d_H(y)=1$ and thus $d_G(y)=1$, which is impossible since the distance of the two 1-valent vertices u and y is exactly 2.

The sharpness of the upper bound 9 can be confirmed by picture (c) of Fig. 1. □

Note that the assumptions on the minimum degree and on the distance of two 1-valent vertices in Corollaries 3.4 and 3.5 cannot be removed, since the star $K_{1,n}$ is an outer-1-planar graph with the degree sum of any edge being $n+1$.

Theorem 3.6. *Every maximal outer-1-planar graph contains an edge xy with $d(x)=d(y)=3$, or $d(x)=2$ and $d(y)\leq 5$.*

Proof. By Lemma 2.1, G is 2-connected. By Lemmas 2.2 and 2.3, G has an outer-1-planar drawing so that its outer boundary forms a hamiltonian cycle. If G contains no crossing, then the conclusion follows from Lemma 3.1. If G contains crossings, then it follows from Lemma 3.2. □

Corollary 3.7. *Every maximal outer-1-planar graph of order at least 2 contains an edge xy with $d(x)+d(y)\leq 7$, and the upper bound 7 is the best possible. In other words, $w(P_2, \mathcal{O}1\mathcal{P}^*)=7$.*

Proof. Theorem 3.6 directly implies this result, and the sharpness of the upper bound can be confirmed by picture (a) of Fig. 1, where the weight of each edge is at least 7. □

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