List vertex arboricity of graphs without forbidden minors*

Xin Zhang†

School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

Abstract

The notions of L-tree-coloring and list vertex arboricity of graphs are introduced in the paper, while a sufficient condition for a plane graph admitting an L-tree-coloring are given. Further, it is proved that every graph without K_5-minors or $K_{3,3}$-minors has list vertex arboricity at most 3, and this upper bound is sharp.

Keywords: vertex arboricity, K_5-minor-free, $K_{3,3}$-minor-free, minor closed.

1 Introduction

All graphs in this paper are undirected, finite and simple. A graph is *planar* if it can be drawn on the plane in such a way that no edges cross each other. Such a drawing of a planar graph is called a *plane* graph. A cycle C in a plane graph is *separating* if both the interior and exterior of C contains vertices of G. A plane graph G is a *near-triangulation* if the boundary of every face, except possibly the outer face, is a cycle on three vertices, and is *triangulation* if the boundary of every face is a cycle on three vertices.

A graph H is called a *minor* of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges. The theory of graph minors began with the well-known Wagner's theorem [4] that a graph is planar if and only if its minors do not include the complete graph K_5 nor the complete bipartite graph $K_{3,3}$.

A *k-tree-coloring* of G is function φ from the vertex set $V(G)$ to $\{1, 2, \ldots, k\}$ so that the graph induced by $\varphi^{-1}(i)$ is an union of trees for every $1 \le i \le k$. The minimum integer k so that G admits a k-tree-coloring is the *vertex arboricity* of

*Supported by National Natural Science Foundation of China (No. 11301410), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130203120021), and the Fundamental Research Funds for the Central Universities (No. JB150714).

†Email address: xzhang@xidian.edu.cn.

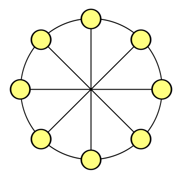

Figure 1: Wagner graph

G, denoted by $va(G)$. Chartrand, Kronk and Wall [3] showed that $va(G) \leq 3$ for any planar graph G.

Naturally, we can consider the list version of vertex arboricity. Let $L(v)$ be a *list* of colors assigned to each vertex $v \in V(G)$. An *L-tree-coloring* of G is a function $\varphi : V(G) \rightharpoonup \bigcup_v L(v)$ so that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and the graph induced by $\varphi^{-1}(i)$ is an union of trees for every $i \in \bigcup_v L(v)$. A graph G is *list k-tree-colorable* if G has an L-tree-coloring as long as one assign to each vertex $v \in V(G)$ an arbitrary list $L(v)$ with size k. The minimum integer k so that G is list k-tree-colorable is the *list vertex arboricity* of G, denoted by $va_l(G)$. Clearly, $va(G) \leq va_l(G)$, but whether there is a gap between these two parameters is unknown.

In this paper, we first give a sufficient condition for a plane graph admitting an L-tree-coloring (see Theorem 6), and further, prove that $va_l(G) \leq 3$ if G is K_5-minor-free, or $K_{3,3}$-minor-free (see Theorem 11).

2 Main results and their proofs

By $G_1 \cap G_2$ (resp. $G_1 \cup G_2$), we denote the graph with vertex set $V(G_1) \cap V(G_2)$ (resp. $V(G_1) \cup V(G_2)$) and edge set $E(G_1) \cap E(G_2)$ (resp. $E(G_1) \cup E(G_2)$). If G_1 and G_2 are subgraphs of G so that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a complete graph on k-vertices, then we say that G is the *clique k-sum* of G_1 and G_2.

An H-minor-free graph G is *edge-maximal* if the graph derived form G by joining any two nonadjacent vertices has at least one H-minor. A planar graph G is *edge-maximal* if joining any two nonadjacent vertices of G will disturb the planarity. *Wagner graph* is a 3-regular graph with 8 vertices and 12 edges named after Klaus Wagner, see Figure 1.

Lemma 1. *(Wagner [4]) A graph is K_5-minor-free if and only if it can be obtained by clique 0-, 1-, 2-, 3-summing stating from planar graphs and the Wagner graph.*

Lemma 2. *(Wagner [4]) A graph is $K_{3,3}$-minor-free if and only if it can be obtained by clique 0-, 1-, 2-summing stating from planar graphs and K_5.*

The following corollaries directly follow from Lemmas 1 and 2, respectively.

Corollary 3. *If G is an edge-maximal K_5-minor-free graph, then it can be obtained by clique 2-, 3-summing starting from edge-maximal planar graphs and the Wagner graph.* □

Corollary 4. *If G is an edge-maximal $K_{3,3}$-minor-free graph, then it can be obtained by clique 2-summing starting from edge-maximal planar graphs and K_5.* □

Lemma 5. *Let G be a near-triangulation with outer face $C = v_1v_2 \dots v_nv_1$. Assume that $L(u)$ is a list of at least two colors for $u \in V(C)$, and at least three colors for $u \in V(G) \setminus V(C)$. If φ is an L-tree-coloring of $\{v_1, v_2\}$, then φ can be extended to an L-tree-coloring of G.*

Proof. We prove it by induction on n. First, the conclusion is trivial when $n = 3$, so we assume that it holds for any near-triangulation with order less than n and consider near-triangulation G with order $n \geq 4$.

If C contains a chord v_iv_j with $1 \leq i < j \leq n$, then let G_1 be the subgraph induced by $\{v_i, v_{i+1}, \dots, v_j\}$ and let G_2 be the subgraph induced by $\{v_j, v_{j+1} \dots, v_n, v_1, \dots, v_i\}$.

Since G is a near-triangulation, G_1 is a near-triangulation with outer face $C_1 = v_iv_{i+1} \dots v_jv_i$ and G_2 is a near-triangulation with outer face $C_2 = v_jv_{j+1} \dots v_nv_1 \dots v_i v_j$. Without loss of generality, assume that $v_1, v_2 \in V(C_1)$.

By the induction hypothesis, φ can be extended to an L-tree-coloring λ_1 of G_1, and then the coloring on $\{v_i, v_j\}$ can be extended to an L-tree-coloring λ_2 of G_2. Combining the L-tree-colorings λ_1 with λ_2, we obtain a coloring λ of G. If λ is not an L-tree-coloring of G, then there is a monochromatic cycle in G that is incident with the chord v_iv_j. This implies a monochromatic cycle in either G_1 or G_2, which are all impossible since λ_1 and λ_2 are L-tree-colorings. Therefore, λ is an L-tree-coloring of G to which φ is extended.

Hence we assume that C contains no chord.

Let $v_1, u_1, u_2, \dots, u_k$ and v_{n-1} be the neighbors of v_n in that clockwise order around v_n. Since G be a near-triangulation and C is chordless, $C' = v_1v_2 \dots v_{n-1}u_k u_{k-1} \dots u_1v_1$ is a cycle.

Let G' be the subgraph induced by the vertices of C'. Clearly, G' is a near-triangulation with outer face C'. Let $a \in L(v_n) \setminus \{\varphi(v_1)\}$ and let $L'(u_i) = L(u_i) \setminus \{a\}$ for every $1 \leq i \leq k$. Further, let $L'(w) = L(w)$ for every $w \in V(G') \setminus \{u_1, \dots, u_k\}$. This follows that $|L'(w)| \geq 2$ for every $w \in V(C')$ and $|L'(w)| \geq 3$ for every $w \in V(G') \setminus V(C')$. Hence by the induction hypothesis, φ can be extended to an L'-tree-coloring λ' of G'. At last, we color v_n with a and get a coloring λ of G. Since λ' is an L'-tree-coloring and at most one neighbor of v_n is colored with a under λ', λ is an L-tree-coloring of G, as required. □

The following theorem is an immediate corollary of the above lemma.

Theorem 6. *Let G be a plane graph with outerface C. If L is a list of colors so that $|L(v)| = 2$ for every $v \in V(C)$ and $|L(v)| = 3$ for every $v \in V(G) \setminus V(C)$, then G has an L-tree-coloring.* □

Lemma 7. *Let G be a near-triangulation and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_3 and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.*

Proof. We prove it by induction on the order n of a near-triangulation. First, the conclusion is trivial when $n = 3$, so we assume that it holds for any near-triangulation with order less than n and consider near-triangulation G with order $n \geq 4$.

If $H = uvwu$ (i.e., a cycle on three vertices) is not separating, then we may redraw the graph G and add some necessary edges so that the resulting drawing, also denoted by G, is a triangulation with outer face H. Let $u, w_1, w_2, \ldots, w_k$ and v be the neighbors of w in that clockwise order around w. Since G is a triangulation, $uw_1 \ldots w_k vu$ is a cycle C. Let G' be the subgraph induced by the vertices of C'. Clearly, G' is a near-triangulation with outer face C'. Let $L'(w_i) = L(w_i) \setminus \{\varphi(w)\}$ for every $1 \leq i \leq k$ and $L'(x) = L(x)$ for every $x \in V(G') \setminus \{w_1, \ldots, w_k\}$. Since $|L'(x)| \geq 2$ for every $x \in V(C')$ and $|L'(x)| \geq 3$ for every $x \in V(G') \setminus V(C')$, the coloring φ of $\{u, v\}$ can be extended to an L-tree-coloring of G' by Lemma 5. Clearly, this L-tree-coloring of G' along with the coloring φ of the vertex w form an L-tree-coloring of G.

If $H = uvwu$ is separating, then the subgraph G_1 induced by the vertices inside or on H is a triangulation with outer face H. By the induction hypothesis, the coloring φ on H can be extended to an L-tree-coloring λ_1 of G_1. On the other hand, the subgraph G_2 induced by the vertices outside or on H is a near-triangulation with a non-separating cycle H on three vertices, so by the same argument as the one in the previous paragraph, the coloring φ on H can be extended to an L-tree-coloring λ_2 of G_2. Combining the L-tree-colorings λ_1 with λ_2, we obtain an L-tree-coloring of G. □

Lemma 8. *Let G be a near-triangulation and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_2 and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.*

Proof. If $H = xy$ is an edge on the outer face of G, then the conclusion follows from Lemma 5. If $H = xy$ is not an edge on the outer face of G, then there is a vertex z so that $H' = xyzx$ forms a K_3. Clearly, φ can be extended to an L-tree-coloring φ' of H' via coloring z with a color different from $\varphi(x)$ and $\varphi(y)$. By Lemma 7, φ' can be extended to an L-tree-coloring of G. □

Similar conclusion also holds for Wagner graph and the complete graph K_5. The proof of the following lemma is quite basic, so we omit it here.

Lemma 9. *Let G be the Wagner graph (resp. the complete graph K_5) and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_2 or K_3, and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.* □

Lemma 10. *Let G be an edge-maximal K_5-minor-free graph (resp. $K_{3,3}$-minor-free graph) and let $L(v)$ be a list of at least three colors for every $v \in V(G)$. If G has a subgraph H isomorphic to K_2 or K_3 (resp. isomorphic to K_2), and φ is an L-tree-coloring of H, then φ can be extended to an L-tree-coloring of G.*

Proof. If G is an edge-maximal planar graph, then G can be drawn as a triangulation, so the conclusion holds by Lemmas 7 and 8 (resp. by Lemma 8). If G is the Wagner graph (resp. the complete graph K_5), then it holds by Lemma 9. In the following, we assume that G is neither edge-maximal planar graph nor Wagner graph (resp. the complete graph K_5), and prove it by induction on the order of G.

By Corollary 3, $G = G_1 \cup G_2$, where G_1 is an edge-maximal K_5-minor-free graph (resp. edge-maximal $K_{3,3}$-minor-free graph) and G_2 is an edge-maximal planar graph or a Wagner graph (resp. K_5) so that $G_1 \cap G_2 = H'$ that is isomorphic to K_2 or K_3 (resp. K_2).

If $H \subseteq G_1$, then by the induction hypotheses, φ can be extended to an L-tree-coloring λ_1 of G_1. Whereafter, the L-tree-coloring of H' can be extended to an L-tree-coloring λ_2 of G_2 by Lemmas 7, 8 and 9 (resp. by Lemmas 8 and 9). Combining the coloring λ_1 with λ_2, we obtain an L-tree-coloring of G.

If $H \subseteq G_2$, then by Lemmas 7, 8 and 9 (resp. by Lemmas 8 and 9), φ can be extended to an L-tree-coloring λ_2 of G_2. Whereafter, the L-tree-coloring of H' can be extended to an L-tree-coloring λ_1 of G_1 by the induction hypotheses. Combining the coloring λ_1 with λ_2, we obtain an L-tree-coloring of G. □

Theorem 11. *If G is a K_5-minor-free graph, or a $K_{3,3}$-minor-free graph, then $va_l(G) \leq 3$. Moreover, this upper bound 3 is best possible.*

Proof. Since every K_5-minor-free graph (resp. $K_{3,3}$-minor-free graph) is a subgraph of an edge-maximal K_5-minor-free graph (resp. edge-maximal $K_{3,3}$-minor-free graph), this conclusion directly follows from Lemma 10.

Since there exists planar graph with vertex arboricity exactly 3 (see [2]) and every planar graph is K_5-minor-free and $K_{3,3}$-minor-free, the upper bound 3 in this theorem is best possible. □

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, GTM 244, 2008.

[2] G. Chartrand, H. V. Kronk. The point-arboricity of planar graphs. *J. Lond. Math. Soc*, 44 (1969), 612–616.

[3] G. Chartrand, H. V. Kronk, C. E. Wall, The point-arboricity of a graph. *Israel J. Math.*, 6 (1968), 169–175.

[4] K. Wagner, über eine Eigenschaft der ebenen Komplexe, *Math. Ann.*, 114 (1937), 570–590,