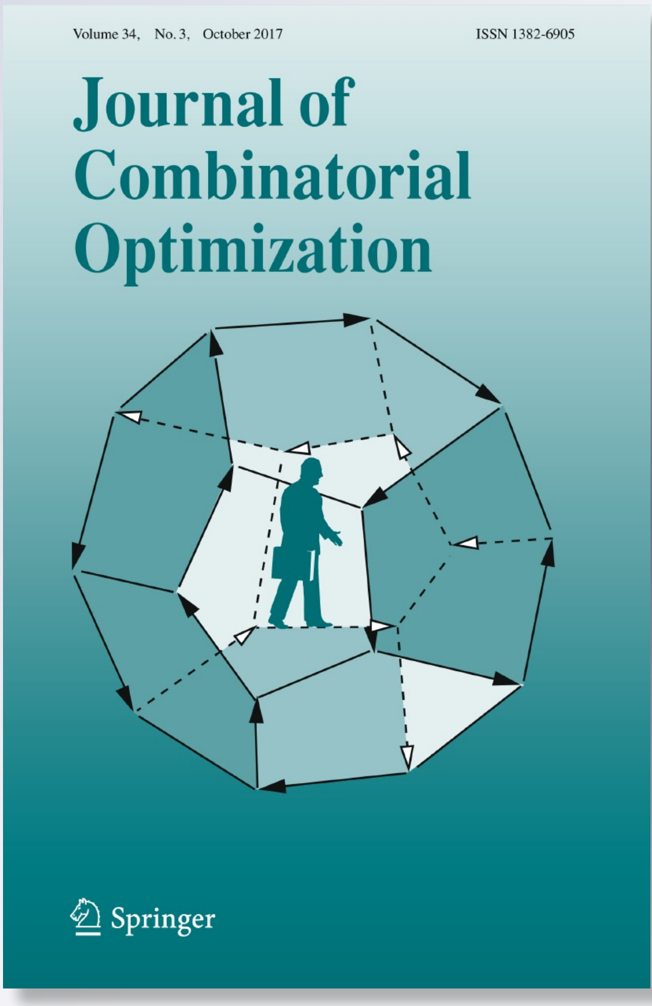
Total coloring of outer-1-planar graphs with near-independent crossings

Xin Zhang

Journal of Combinatorial Optimization

ISSN 1382-6905
Volume 34
Number 3

J Comb Optim (2017) 34:661-675
DOI 10.1007/s10878-016-0093-x



Springer

J Comb Optim (2017) 34:661–675
DOI 10.1007/s10878-016-0093-x

Total coloring of outer-1-planar graphs with near-independent crossings

Xin Zhang[1]

Published online: 26 November 2016

Abstract A graph G is *outer-1-planar with near-independent crossings* if it can be drawn in the plane so that all vertices are on the outer face and $|M_G(c_1) \cap M_G(c_2)| \le 1$ for any two distinct crossings c_1 and c_2 in G, where $M_G(c)$ consists of the end-vertices of the two crossed edges that generate c. In Zhang and Liu (Total coloring of pseudo-outerplanar graphs, arXiv:1108.5009), it is showed that the total chromatic number of every outer-1-planar graph with near-independent crossings and with maximum degree at least 5 is $\Delta + 1$. In this paper we extend the result to maximum degree 4 by proving that the total chromatic number of every outer-1-planar graph with near-independent crossings and with maximum degree 4 is exactly 5.

Keywords Outerplanar graph · Outer-1-planar graph · Local structure · Total coloring

1 Introduction

Graph coloring is an important optimization problem with many applications in computer science, such as frequency assignment in optical communication networks, computation of Hessian matrix, and pattern matching. There are various kinds of coloring, such as vertex coloring, edge coloring, total coloring, and so on.

A *total coloring* of a graph G is an assignment of colors to the vertices and edges of G such that every pair of adjacent or incident elements receive different colors. A *total k-coloring* of a graph G is a total coloring of G from a set of k colors. The minimum positive integer k for which G has a total k-coloring, denoted by $\chi''(G)$, is the *total chromatic number* of G. It is easy to see that $\chi''(G) \ge \Delta(G) + 1$ for any graph G

✉ Xin Zhang
xzhang@xidian.edu.cn

[1] School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

by looking at the color of a vertex with maximum degree and its incident edges. On the other side, it is natural to look for a Brooks'-typed or Vizing-typed upper bound for the total chromatic number in terms of maximum degree. However, it turns out that the total coloring version of maximum degree upper bound is a difficult problem and has eluded mathematicians for nearly 50 years. The most well-known speculation is the *total coloring conjecture*, independently raised by Behzad (1965) and Vizing (1968), which asserts that every graph of maximum degree Δ admits a total $(\Delta + 2)$-coloring. This conjecture remains open although many beautiful results concerning it have been obtained (cf. Yap 1996). In particular, the total chromatic number of outerplanar graphs has been determined completely by Zhang et al. (1988) and that of series-parallel graphs has been determined completely by Wu and Hu (2004).

A graph is *outer-1-planar (o1p)* if it can be drawn in the plane such that all vertices are in the outer face and each edge is crossed at most once, see Auer et al. (2016). For example, $K_{2,3}$ and K_4 are outer-1-planar graphs.

Outer-1-planar graphs were first introduced by Eggleton (1986) who called them *outerplanar graphs with edge crossing number one*, and were also investigated under the notion of *pseudo-outerplanar graphs* by Zhang et al. (2012), who proved that the edge chromatic number of every outer-1-planar graph with maximum degree $\Delta \geq 4$ is Δ and the linear arboricity of every outer-1-planar graph with maximum degree $\Delta \geq 5$ is $\lceil \Delta/2 \rceil$. Zhang and Liu [10] showed that the total chromatic number of every outer-1-planar graph with maximum degree $\Delta \geq 5$ is $\Delta + 1$, and this result was recently generated to its list version by Zhang (2013).

A drawing of an outer-1-planar graph in the plane so that its outer-1-planarity is satisfied and the number of crossings is as few as possible is an *outer-1-plane* graph, and we call this drawing a *good* drawing. Note that every crossing in an outer-1-plane graph G is generated by two mutually crossed chords, thus for every crossing c there exists a vertex set $M_G(c)$ of size four, where $M_G(c)$ consists of the end-vertices of the two chords that generate c. For two distinct crossings c_1 and c_2 in an outer-1-plane graph G, it is easy to check that $|M_G(c_1) \cap M_G(c_2)| \leq 2$ by the definition of the outer-1-planarity.

A graph G is *outer-1-planar with near-independent crossings (Nicop)* if G is outer-1-planar and $|M_G(c_1) \cap M_G(c_2)| \leq 1$ for any two distinct crossings c_1 and c_2 in G. We define *outer-1-plane graph with near-independent crossings* as a "good" drawing of a *Nicop* graph. Note that every *Nicop* graph is *o1p*, and on the other hand, *Nicop* can also be seen as the combination of outerplanar and planar with near-independent crossings that was introduced by Zhang (2014).

In this paper, we first investigate the local structures of *Nicop* graphs, and then prove that every *Nicop* graph with maximum degree $\Delta \geq 4$ has total chromatic number $\Delta+1$.

2 Preliminaries

From now on, when saying that a graph is *Nicop* we always mean that it is an outer-1-plane graph with near-independent crossings.

Let G be a 2-connected *Nicop* graph. Denote by $v_1, \ldots, v_{|G|}$ the vertices of G with clockwise ordering on its boundary. Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \ldots, v_j\}$ and

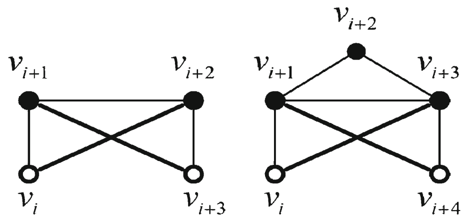


Fig. 1 Graphs defining co-crossed chords

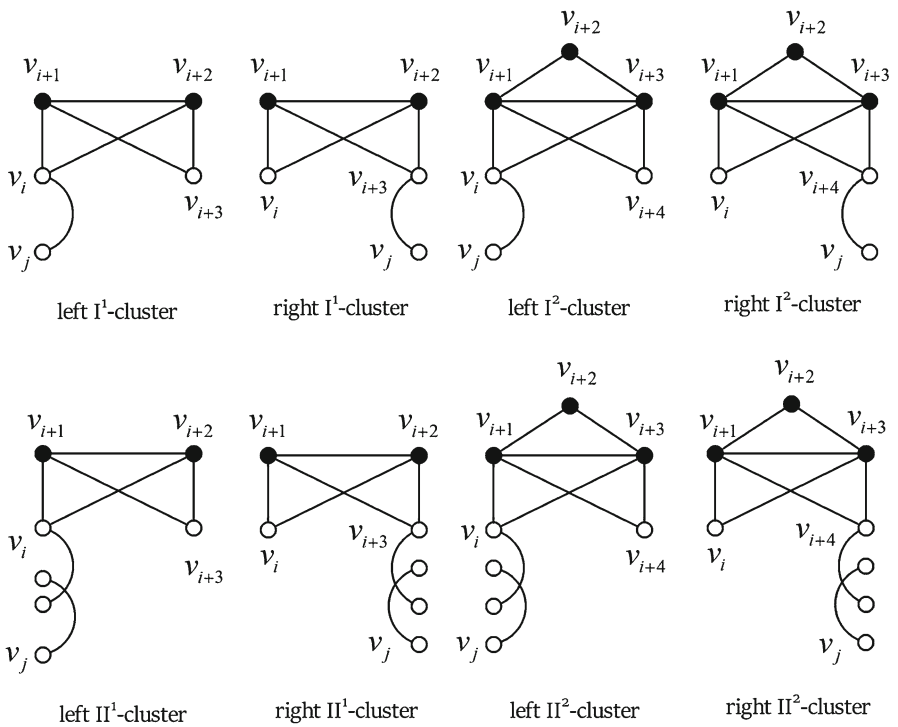


Fig. 2 The definitions of I-clusters and II-clusters

$\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j]\backslash\{v_i, v_j\}$, where the subscripts are taken modulo $|G|$. Set $\mathcal{V}[v_i, v_i] = V(G)$ and $\mathcal{V}(v_i, v_i) = V(G)\backslash\{v_i\}$.

A vertex set $\mathcal{V}[v_i, v_j]$ with $i \neq j$ is *non-edge* if $j = i + 1$ and $v_i v_j \notin E(G)$, is *path* if $v_k v_{k+1} \in E(G)$ for all $i \leq k < j$, and is *subpath* if $j > i + 1$ and some edges in the form $v_k v_{k+1}$ for $i \leq k < j$ are missing. An edge $v_i v_j$ in G is a *chord* if $j - i \neq 1$ or $1 - |G|$. By $\mathcal{C}[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$. For a vertex set V, $G[V]$ denotes the subgraph of G induced by V. In any figure of this paper, the degree of a solid (or hollow) vertex is exactly (or at least) the number of edges that are incident with it, respectively. Moreover, solid vertices are distinct but two hollow vertices may be identified unless stated otherwise, and the edges drawn as crossed have to cross and the curving edges are chords.

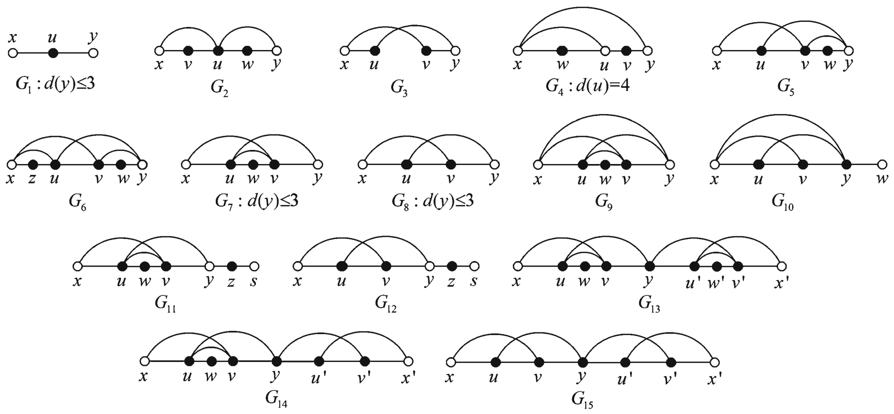


Fig. 3 Local structures in *Nicop* graphs with maximum degree at most 4

Let $v_i v_j$ and $v_k v_l$ be two chords in a *Nicop* graph G so that $v_i v_j$ crosses $v_k v_l$ and v_i, v_k, v_j and v_l lie in a clockwise ordering. If $v_i v_k, v_k v_j, v_j v_l \in E(G), l-j = j-k = k-i = 1$ and $d(v_k) = d(v_j) = 3$, or $v_i v_k, v_k v_j, v_j v_l, v_k v_{k+1}, v_{k+1} v_j \in E(G)$, $l-j = k-i = 1$, $j-k = 2$, $d(v_k) = d(v_j) = 4$ and $d(v_{k+1}) = 2$, then we call that $v_i v_j$ *co-crosses* $v_k v_l$, and $v_i v_j$, $v_k v_l$ are *co-crossed chords* in G, see Fig. 1.

We call H a *I-cluster* in G if H is either a left I^1-cluster, or a right I^1-cluster, or a left I^2-cluster, or a right I^2-cluster, see Fig. 2. The *II-cluster* is defined similarly.

We use $[v_L, v_R]_1$ and $[v_L, v_R]_2$ to denote a I-cluster and II-cluster, respectively, where L and R are the subscripts of the far left vertex and the far right vertex on the boundary of G (see in a clockwise direction from left to right). The *width* of a cluster in Fig. 2 is defined to be the value of $|\mathcal{V}[v_L, v_R]|$. For example, the width of the left I^1-cluster $[v_j, v_{i+3}]_1$ in the figure is $(i+3) - j + 1 = i - j + 4$, and the width of the right I^1-cluster $[v_i, v_j]_1$ in the figure is $j - i + 1$. Of course, the final values should be taken modulo by $|G|$.

We now introduce some useful lemmas that are frequently used in the next sections. From now on, when mentioning the configuration G_i with $1 \le i \le 15$ we always refer to the corresponding picture in Fig. 3. Saying that G contains G_i, we mean that G contains a subgraph isomorphic to G_i such that the degree in G of any solid vertex in that picture is exactly the number of edges that are incident with it there. In particular, saying that a good drawing of a *Nicop* graph G contains G_{10}, we also mean that G contains a subgraph isomorphic to G_{10} with yw not being a chord (so yw is not crossed).

Lemma 2.1 (Zhang et al. 2012) *Let v_i and v_j be vertices of a 2-connected Nicop graph G. If there are no crossed chords in $\mathcal{C}[v_i, v_j]$ and no edges between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then $\mathcal{V}[v_i, v_j]$ is either non-edge or path.*

Lemma 2.2 *Let $\mathcal{V}[v_i, v_j]$ with $j - i \ge 3$ be path in a 2-connected Nicop graph G with $\Delta(G) \le 4$. If there are no crossed chords in $\mathcal{C}[v_i, v_j]$ and no edges between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then G contains G_1 or G_2 as a subgraph.*

Proof If $\mathcal{C}[v_i, v_j]\backslash\{v_i v_j\} = \emptyset$, then $d(v_{i+1}) = d(v_{i+2}) = 2$ and G_1 appears. If there is at least one chord in $\mathcal{C}[v_i, v_j]\backslash\{v_i v_j\}$, then choose one, say $v_a v_b$ with $a < b$, so that there are no other chords in $\mathcal{C}[v_a, v_b]$. By the absences of the configurations G_1 and G_2, we have $b - a = 2$, $d(v_{a+1}) = 2$ and $d(v_a), d(v_b) \geq 4$. Without loss of generality, assume that $b \neq j$. Let v_c be a vertex so that $v_b v_c$ is a chord. If $b < c$, then $c - b = 2$ and $d(v_{b+1}) = 2$, otherwise G_1 occurs. This implies the appearance of the configuration G_2 in G. If $b > c$, then $a \neq i$, which implies that there is a vertex v_d so that $v_a v_d$ is a chord with $d < a$. Afterwards, a copy of G_1 occurs if $a - d \geq 3$, and G_2 occurs if $a - d = 2$, in which case we have $d(v_{a-1}) = 2$. □

Lemma 2.3 *Let $v_i v_j$ and $v_k v_l$ with $i < k < j < l$ be two crossed chords in a 2-connected Nicop graph G with $\Delta(G) \leq 4$ so that $v_i v_j$ crosses $v_k v_l$ and there is no other pair of crossed chords contained in the graph induced by $\mathcal{V}[v_i, v_l]$. We have*

(1) *at most one of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is non-edge;*
(2) *if one of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is non-edge, then G has a subgraph isomorphic to either G_1 or G_3;*
(3) *if all of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are paths, then either $v_i v_j$ co-crosses $v_k v_l$ in G, or G has a subgraph isomorphic to one of the configurations among G_1, G_2, G_4, G_5 and G_6.*

Proof The results (1) and (2) are proved in Zhang et al. (2012). We now prove (3).

Suppose that G is a counterexample. By Lemma 2.2, $\max\{k - i, j - k, l - j\} \leq 2$. Set $X = \mathcal{C}[v_i, v_l]\backslash\{v_i v_j, v_k v_l\}$ and let $x = |X|$. It is clear that $x \leq 2$ since $\Delta(G) \leq 4$.

Assume that $v_i v_j$ does not co-cross $v_k v_l$. If $x = 0$, then G_1 appears. If $x = 1$, then G_1 or G_5 appears. If $x = 2$, then one of the configurations among G_1, G_4 and G_6 appears. All are contradictions. □

3 Local structures

Let G be a 2-connected *Nicop* graph with $\Delta(G) \leq 4$. If there are no crossings in G, then G is outerplanar, and the following result is immediate.

Lemma 3.1 (Wang and Zhang 1999) *If there are no crossings in G, then G contains either G_1 or G_2.*

Suppose that G contains a crossing. Choose one pair of crossed chords $v_i v_j$ and $v_k v_l$ such that

(a) $v_i v_j$ crosses $v_k v_l$ in G and v_i, v_k, v_j and v_l lie in a clockwise ordering,
(b) there are no crossed chords contained in $\mathcal{C}[v_i, v_l]$ besides $v_i v_j$ and $v_k v_l$.

Lemma 3.2 *If chords $v_i v_j$ and $v_k v_l$ satisfy the conditions (a) and (b), then $v_i v_j$ co-crosses $v_k v_l$ unless G contains one of the configurations among $G_1 – G_6$.*

Proof This follows directly from Lemmas 2.1 and 2.3. □

In the following arguments, we assume that G does not contain any of the configurations among $G_1 – G_{15}$.

By Lemma 3.2, we assume, without loss of generality, that $v_i v_j$ co-crosses $v_k v_l$ in G and $i = 1$. Since $G_7 - G_{10}$ are absent, $d(v_l) = 4$ and thus there is a chord $v_l v_s$ with $s > l$.

Looking at the graph induced by $\mathcal{V}[v_i, v_l]$ and $v_l v_s$, we can find that it is a I-cluster in G. If there is another I-cluster contained in the graph induced by $\mathcal{V}[v_i, v_s]$ with shorter width, then we consider that instead of the previous one. Hence the following assumption is natural.

Assumption 1 $[v_i, v_s]_1$ is the shortest I-cluster contained in the graph induced by $\mathcal{V}[v_i, v_s]$.

Claim 3.3 *$v_l v_s$ is a crossed chord.*

Proof If $v_l v_s$ is not crossed, then there are no edges between $\mathcal{V}(v_l, v_s)$ and $\mathcal{V}(v_s, v_l)$. If $s - l = 2$, then $d(v_{l+1}) = 2$, which implies the appearance of G_{11} or G_{12}. If $s - l \geq 3$, then by Lemma 2.2, there is a pair of co-crossed chords $v_{i'} v_{j'}$ and $v_{k'} v_{l'}$ with $l \leq i' < k' < j' < l' \leq s$, otherwise G_1 or G_2 appears. Since $\Delta(G) \leq 4$ and $G_7 - G_{10}$ are forbidden in G, $i' \neq l$ and by Assumption 1, there is a chord $v_i' v_t'$ with $l' < t' \leq s$. Since G is *Nicop*, there is no chord in the form $v_{l'} v_{s'}$ with $s' > t'$. By the absences of $G_7 - G_{10}$, there is a chord $v_{l'} v_{s'}$ with $l' < s' \leq t'$, which contradicts Assumption 1. □

Suppose that $v_l v_s$ is crossed by a chord $v_r v_t$ with $l < r < s$. Since G is a *Nicop* graph, $t \neq i$ and thus $t > s$. Note that the graph induced by $\mathcal{V}[v_i, v_l]$ and the chords $v_l v_s$, $v_r v_t$ is a II-cluster denoted by $[v_i, v_t]_2$. Again, the following assumption is natural.

Assumption 2 $[v_i, v_t]_2$ is the shortest II-cluster contained in the graph induced by $\mathcal{V}[v_i, v_t]$.

Claim 3.4 *There are no crossed chords in $\mathcal{C}[v_l, v_r]$.*

Proof Suppose that $v_a v_b$ crosses $v_c v_d$ with $l \leq a < c < b < d \leq r$. By Lemma 2.2, we can properly choose $v_a v_b$ and $v_c v_d$ so that one co-crosses the other. By Assumption 1, the fact that $\Delta(G) \leq 4$ and the absences of $G_7 - G_{10}$, there are chords $v_a v_\beta$ and $v_d v_\alpha$ with $l \leq \alpha < a$ and $d < \beta \leq r$. This contradicts the definition of the *Nicop*. □

Claim 3.5 $r - l = 1$.

Proof By Lemma 2.1 and Claim 3.4, $\mathcal{V}[v_l, v_r]$ is non-edge or path. If $\mathcal{V}[v_l, v_r]$ is non-edge, then $r - l = 1$. If $\mathcal{V}[v_l, v_r]$ is path, then by Lemma 2.2 and the absences of G_{11} and G_{12}, we also have $r - l = 1$. □

Claim 3.6 *There are no crossed chords in $\mathcal{C}[v_r, v_s]$.*

Proof Suppose that $v_a v_b$ crosses $v_c v_d$ with $r \leq a < c < b < d \leq s$. By Lemma 2.2, we can assume that $v_a v_b$ co-crosses $v_c v_d$. If $a = r$, then $d \neq s$ by the definition of the *Nicop*. Since $G_7 - G_{10}$ are forbidden, there is a chord $v_d v_\alpha$ with $d < \alpha \leq s$, which contradicts Assumption 1. Hence $a \neq r$, and by similar reason, $d \neq s$. By Assumption 1 and the absences of $G_7 - G_{10}$, there are chords $v_a v_\beta$ and $v_d v_\alpha$ with $r \leq \alpha < a$ and $d < \beta \leq s$. This again contradicts the definition of the *Nicop*. □

Claim 3.7 *$v_r v_s \in E(G)$ and $s - r \leq 2$ with equality only if $v_l v_r, v_r v_{r+1}, v_{r+1} v_s \in E(G)$.*

Proof By Lemma 2.1 and Claim 3.6, $\mathcal{V}[v_r, v_s]$ is non-edge or path.

If $\mathcal{V}[v_r, v_s]$ is non-edge, then $v_l v_r \in E(G)$, otherwise v_r has degree one, contradicting the 2-connectivity of G. This implies that v_r has degree two, and thus G_{11} or G_{12} appears.

If $\mathcal{V}[v_r, v_s]$ is path, then Lemma 2.2, $s - r \leq 2$. If $s - r = 1$, then $v_r v_s \in E(G)$. If $s - r = 2$, then the graph induced by v_r, v_{r+1} and v_s is a triangle, and moreover, $v_l v_r \in E(G)$, otherwise $d(v_{r+1}) = 2$ and $d(v_r) \leq 3$, which implies a copy of G_1. □

Claim 3.8 *There are no crossed chords in $\mathcal{C}[v_s, v_t]$.*

Proof If this claim is false, then there is a pair of co-crossed chords $v_{i'} v_{j'}$ and $v_{k'} v_{l'}$ in $\mathcal{C}[v_s, v_t]$ with $s \leq i' < k' < j' < l' \leq t$. By the absences of $G_7 – G_{10}$, there are chords $v_{l'} v_{s'}$ and $v_{i'} v_{t'}$ with $s' \neq k', i'$ and $t' \neq j', l'$.

First, assume that $l' < s' \leq t$ and, without loss of generality, that there are no chords in the form $v_{l'} v_{s''}$ with $l' < s'' < s'$. By Claim 3.3, $v_{l'} v_{s'}$ is crossed. By Assumption 2, we have to assume that $v_{l'} v_{s'}$ is crossed by a chord $v_m v_{m'}$ with $l' < m < s'$ and $s \leq m' \leq i'$. By the definition of the *Nicop*, $m' \neq i'$. This implies that $m' \leq t' < i'$, and then by Claim 3.3 and Assumption 2, $v_{i'} v_{t'}$ is crossed by a chord incident with $v_{l'}$, contradicting the definition of the *Nicop*.

Hence, if $l' \neq t$, then $s \leq s' \leq t' < i'$. By Claim 3.3 and Assumption 2, $v_{l'} v_{s'}$ is crossed by a chord incident with $v_{l'}$, which contradicts the definition of the *Nicop*. Therefore, $l' = t$, and $i' = s$ by symmetry. However, the definition of the *Nicop* declines this case. □

Claim 3.9 *$\mathcal{V}[v_s, v_t]$ is path and $t - s \leq 2$.*

Proof By Lemma 2.1 and Claim 3.8, $\mathcal{V}[v_s, v_t]$ is either non-edge or path. Suppose that $\mathcal{V}[v_s, v_t]$ is non-edge.

If $s - r = 2$, then by Claim 3.7, $d(v_{s-1}) = 2$ and $d(v_s) = 3$, which implies a copy of G_1.

If $s - r = 1$, then $v_r v_s \in E(G)$ by Claim 3.7. This implies that $d(v_s) = 2$ and $d(v_r) \leq 3$. Hence G_1 appears, a contradiction.

Therefore, $\mathcal{V}[v_s, v_t]$ is path, and $t - s \leq 2$ by Lemma 2.2. □

Lemma 3.10 *The graph induced by $\mathcal{V}[v_i, v_t]$ contains one of the configurations among G_1 – G_{15}, and furthermore, v_i or v_t cannot be the solid vertex of any such configuration.*

Proof If $t - s = 1$, then $v_s v_t \in E(G)$ by Claim 3.9. If $s - r = 1$ and $v_r v_s \in E(G)$, then $v_l v_r \in E(G)$ by Claim 3.5, otherwise G_1 appears. This contradiction implies that the graph induced by $\mathcal{V}[v_i, v_t]$ contains a copy of G_{14} or G_{15}. Hence $s - r = 2$ and $v_r v_{r+1}, v_{r+1} v_s, v_r v_s \in E(G)$ by Claim 3.7. Under this condition we have $v_l v_r \in E(G)$ by Claim 3.5, otherwise G_1 appeas, a contradiction. This implies that the graph induced by $\mathcal{V}[v_i, v_t]$ contains a copy of G_{13}. Therefore, $t - s = 2$ by Claim 3.9. This implies that $d(v_{s+1}) = 2$.

If $v_s v_t \in E(G)$, then $s - r = 1$ and $v_r v_s \in E(G)$ by Claim 3.7, otherwise v_s has degree five, contradicting the fact that $\Delta(G) \leq 4$. If $v_l v_r \notin E(G)$, then v_r has degree two and G_3 appears. If $v_l v_r \in E(G)$, then the graph induced by $\mathcal{V}[v_l, v_t]$ is a copy of G_5. In each case we obtain contradictions. Hence $v_s v_t \notin E(G)$.

If $s - r = 1$, then by Claim 3.7, $v_r v_s \in E(G)$. This implies that v_s has degree three and thus G_1 appears. Hence $s - r = 2$.

By Claim 3.7, v_r, v_{r+1}, v_s induce a triangle with $d(v_{r+1}) = 2$. Now the graph induced by $\mathcal{V}[v_r, v_t]$ is a copy of G_4, a contradiction.

At last, one can check that the proofs of any claims or lemmas in the previous arguments grantee that v_i or v_t is not a solid vertex in such configuration. □

Theorem 3.11 *Every Nicop graph with minimum degree at least 2 and with maximum degree at most 4 contains one of the configurations among G_1 – G_{15}.*

Proof Let G be a *Nicop* graph with maximum degree at most 4. If G is 2-connected, then this result holds by Lemma 3.10. Hence we assume that G is not 2-connected and is a counterexample to the result.

Choose an end-block H of G with an unique cut vertex, say v_1, and let $v_1, \ldots, v_{|H|}$ be the vertices of H with clockwise ordering on its boundary.

Suppose that there is a pair of crossed chords $v_i v_j$ and $v_k v_l$ so that v_i, v_k, v_j and v_l lie in a clockwise ordering. Without loss of generality, assume that $1 \leq i < k < j < l$.

If there is another pair of crossed chords $v_{i'} v_{j'}$ and $v_{k'} v_{l'}$ with $i' < k' < j' < l'$ contained in $\mathcal{C}[v_i, v_l]$, then set $i := i'$, $j := j'$, $k := k'$ and $l := l'$. In other words, we always assume that there are no crossed chords contained in $\mathcal{C}[v_i, v_l]$ besides $v_i v_j$ and $v_k v_l$. Hence by Lemma 3.2, $v_i v_j$ co-crosses $v_k v_l$.

By the absences of G_7 – G_{10}, there are chords $v_l v_s$ and $v_i v_m$ with $s \neq k, i$ and $t \neq j, l$. By the definition of *Nicop*, $v_l v_s$ does not cross $v_i v_t$, thus either $v_1 \notin \mathcal{V}(v_l, v_s)$ or $v_1 \notin \mathcal{V}(v_m, v_i)$. By symmetry, we assume, without loss of generality, that $v_1 \notin \mathcal{V}(v_l, v_s)$.

If $s \neq 1$, then by Claim 3.3, $v_l v_s$ is crossed by a chord $v_r v_t$ with $l < r < s$, otherwise the graph induced by $\mathcal{V}[v_i, v_s]$ (and thus G) contains one of the required configurations. By Lemma 3.10, $t \geq 1$, because otherwise the graph induced by $\mathcal{V}[v_i, v_t]$ (and thus G) contains one of the required configurations. This implies that $1 \leq m < i$. By Claim 3.3, $v_i v_m$ is crossed by a chord $v_p v_q$ with $q < m < p < i$. Since $t \geq 1$, $q \geq 1$, which implies that $v_1 \notin \mathcal{V}(v_q, v_l)$. Hence by Lemma 3.10, the graph induced by $\mathcal{V}[v_q, v_l]$ (and thus G) contains one of the required configurations.

Therefore, $s = 1$, which implies that $v_1 \notin \mathcal{V}(v_m, v_i)$. Hence by similar arguments as previous paragraph, we also have $m = 1$. This implies that $v_l v_s$ is not crossed, and thus by the proof of Claim 3.3, the graph induced by $\mathcal{V}[v_i, v_s]$ (and thus G) contains one of the required configurations.

Hence, H is outerplanar.

If H contains no chords, then G contains two adjacent 2-valent vertices, a contradiction.

Let $v_i v_j$ with $1 \leq i < j$ be a chord so that it is the unique chord in $\mathcal{C}[v_i, v_j]$. It is easy to see that $j - i = 2$ and $d(v_{i+1}) = 2$, because otherwise G contains two adjacent 2-valent vertices. By the absence of G_1, there are chords $v_i v_s$ and $v_j v_t$. Since H is outerplanar, either $v_1 \notin \mathcal{V}(v_j, v_t)$ or $v_1 \notin \mathcal{V}(v_s, v_i)$.

Without loss of generality, assume that $v_1 \notin \mathcal{V}(v_j, v_t)$. By same reason as above, we have $t - j = 2$ and $d(v_{j+1}) = 2$. This implies a copy of G_2 in the graph induced by $\mathcal{V}[v_i, v_t]$, a contradiction, and this completes the proof of the theorem. □

In what follows, we list some important facts that are frequently used in the arguments of the next section.

Fact 1 *Deleting edges or vertices from a Nicop graph results in a Nicop graph.*

Proof This is obvious by the definition of the *Nicop*. □

Fact 2 *If G is a Nicop graph with $uv \notin E(G)$ and $G + uv$ is an outer-1-plane graph so that uv is non-crossed, then $G + uv$ is a Nicop graph.*

Proof Since this operation does not generate new crossings and $M_G(c) = M_{G+uv}(c)$ for any crossing c in G, $G + uv$ is a *Nicop* graph. □

Fact 3 *If G is good drawing of a Nicop graph with maximum degree 4 and G contains the configuration G_{10}, then the graph obtained from G by deleting u, v and identifying x and y is still a Nicop graph.*

Proof If $d(x) = 3$, then the conclusion is obvious. Hence we assume that $d(x) = 4$ and let z be the fourth neighbor of x.

Let G' be the graph obtained from G by deleting u, v and identifying x and y into a common vertex x_y. It is clear that this operation do not generate new crossings and thus G' is outer-1-planar. One can also check that x_y has degree exactly two in G', and $x_y w$ is a non-crossed edge in G' by the definition of the configuration G_{10} (recall its definition mentioned before Lemma 2.1).

If $x_y z$ is not crossed in G', then for any crossing c in G', we have $x_y \notin M_{G'}(c)$, which implies that $M_{G'}(c) = M_G(c)$. Hence $|M_{G'}(c_1) \cap M_{G'}(c_2)| = |M_G(c_1) \cap M_G(c_2)| \le 1$ for any two distinct crossings c_1 and c_2 in G', and thus G' is a *Nicop* graph.

If $x_y z$ is crossed in G', then $x_y \in M_{G'}(c_0)$, where c_0 is the crossing on $x_y z$. Let c_1 and c_2 be two distinct crossings in G'. If $c_1 \ne c_0$ and $c_2 \ne c_0$, then $x_y \notin M_{G'}(c_1)$ and $x_y \notin M_{G'}(c_2)$, which implies that $|M_{G'}(c_1) \cap M_{G'}(c_2)| = |M_G(c_1) \cap M_G(c_2)| \le 1$. If $c_1 = c_0$, then we also have $|M_{G'}(c_1) \cap M_{G'}(c_2)| \le 1$. Otherwise, we assume that $M_{G'}(c_0) \cap M_{G'}(c_2) = \{s_1, s_2\}$. Since $x_y \in M_{G'}(c_0)$ and $x_y \notin M_{G'}(c_2)$, we have $\{s_1, s_2\} \subseteq V(G) \backslash \{u, v, x, y\}$. This implies that $|M_G(c_1) \cap M_G(c_2)| = |\{s_1, s_2\}| = 2$, a contradiction. □

4 Total coloring

Let G be a *Nicop* graph with maximum degree at most 4 satisfying:

(1) G does not admit any total 5-coloring, and

(2) any *Nicop* graph H with maximum degree at most 4 and with smaller order or size than G has a total 5-coloring.

It is easy to see that G is 2-connected.

Lemma 4.1 *G does not contain a 2-valent vertex adjacent to a 3-valent vertex.*

Proof Suppose, to the contrary, that $uv \in E(G)$, $d(u) = 2$ and $d(v) \leq 3$. By Fact 1 and (2), $G - uv$ has a total 5-coloring.

Delete the color on u and denote the resulting partial coloring by c. Since uv is incident with at most four colored elements under c, uv can be colored properly. Since u is incident with four colored elements after coloring uv, u can also be colored properly. Therefore, we obtain a total 5-coloring of G, a contradiction. □

Lemma 4.2 *G does not contain a cycle of length 4 with two nonadjacent vertices of degree 2.*

Proof Suppose, to the contrary, that there is a cycle $uxvy$ of length four with $d(u) = d(v) = 2$. By Fact 1 and (2), $G - \{u, v\}$ has a total 5-coloring c.

Since every edge in the cycle is incident with at most three colors, there are at least two available colors for each of them, which is sufficient for coloring the edges of the cycle $uxvy$. At last, we color u and v properly. This can be easily done since $d(u) = d(v) = 2$ and $uv \notin E(G)$. Therefore, we obtain a total 5-coloring of G, a contradiction. □

Lemma 4.3 *G does not contain a triangle uvw with $d(v) = 2$ and u adjacent to a vertex x of degree 2.*

Proof Suppose, to the contrary, that there is a triangle uvw with $d(v) = 2$ and u adjacent to a vertex x of degree 2. By Lemma 4.1, $d(u) = d(w) = 4$. Let y and z be vertices with $uy, xz \in E(G)$. By Fact 1 and (2), $G - ux$ has a total 5-coloring. Remove the colors on x and v, and denote the resulting partial coloring by c.

If $\{c(u), c(uv), c(uw), c(uy), c(xz)\} \subset \{1, 2, 3, 4, 5\}$, then ux can be colored properly and afterwards, x and v can be colored since $d(x) = d(v) = 2$. Therefore, we assume, without loss of generality, that $c(u) = 1$, $c(uv) = 2$, $c(uw) = 3$, $c(uy) = 4$ and $c(xz) = 5$. If $c(vw) \neq 5$, then recolor uv with 5 and color ux with 2. If $c(vw) = 5$, then exchange the colors on vw and uw, and color ux with 3. In each case we obtain a total 5-coloring of G after coloring x and v properly, which can be easily done since $d(x) = d(v) = 2$. □

We now consider the configurations $\mathcal{A}$ and $\mathcal{B}$ represented by G_7 and G_8 without the degree constraints.

Lemma 4.4 *If G contains $\mathcal{A}$, then $xy \notin E(G)$.*

Proof Suppose that G contains $\mathcal{A}$ and $xy \in E(G)$. By the 2-connectivity of G, $d(x) = d(y) = 4$, otherwise G is the graph induced by $\mathcal{A}$. At this stage, we obtain a total 5-coloring of G by coloring x, vy, uw with 1, u, vw, xy with 2, w, y, uv with 3, v, ux with 4, and xv, uy with 5. Let x' and y' be the fourth neighbor of x and y, respectively. If $x' = y'$, then G is the graph induced by u, v, w, x, y and x' by the 2-connectivity of G. We color x, vy, uw with 1, u, x', vw, xy with 2, w, y, uv, xx' with 3, v, ux, yx' with 4, and xv, uy with 5, and get a total 5-coloring of G. Hence in the following we assume that $x' \neq y'$.

Let G' be the graph obtained from G by deleting w. By Fact 1 and (2), G' admits a total 5-coloring c. Without loss of generality, assume that $c(x) = 1$, $c(y) = 2$ and

$c(xy) = 3$. If $c(xx') = c(yy')$, then one can easily check that G' cannot be properly totally colored with only five colors, a contradiction. Hence $c(xx') \neq c(yy')$. We erase the colors on u, v and on its incident edges from c, and extend this partial total 5-coloring to G according to the following three cases.

If $c(xx_1) = c(y) = 2$ and $c(yy_1) = c(x) = 1$, then recolor y with a color in $\{4, 5\}\backslash\{c(y')\}$, say 4. Next, we construct a total 5-coloring of G by coloring w, uv with 1, u, vy with 2, v, uw with 3, vw, ux with 4, and uy, vx with 5.

If $c(xx_1) = c(y) = 2$ and $c(yy_1) \neq c(x)$ (here assume that $c(yy_1) = 4$), then we construct a total 5-coloring of G by coloring uw, vy with 1, w, uv with 2, u, vw with 3, v, ux with 4, and uy, vx with 5.

If $c(xx_1) \neq c(y)$ and $c(yy_1) \neq c(x)$ (here assume that $c(xx_1) = 4$ and $c(yy_1) = 5$), then we construct a total 5-coloring of G by coloring vw, uy with 1, uw, vx with 2, w, uv with 3, u, vy with 4, and v, ux with 5.

Hence $xy \notin E(G)$ if G contains $\mathcal{A}$. □

Corollary 4.5 *G does not contain any of the configurations among G_1 – G_4 and G_9.*

Proof This is a direct corollary from Lemmas 4.1, 4.2, 4.3 and 4.4. □

Lemma 4.6 *If G contains $\mathcal{A}$ or $\mathcal{B}$, then $d(x) = d(y) = 4$*

Proof Suppose that G contains $\mathcal{A}$ and $d(x) \leq 3$. By the 2-connectivity of G, we shall assume that $d(x) = 3$. Otherwise G is isomorphic to the graph induced by u, v, w, x and y, and this graph is totally 5-colorable, since it is just the graph obtained from K_5 by removing two adjacent edges. Without loss of generality, assume that $d(y) = 4$. Let x_1, y_1, y_2 be vertices with $xx_1, yy_1, yy_2 \in E(G)$. Delete u, v, w from G and add a new edge xy. By Fact 1, Fact 2 and (2), the resulting graph G' has a total 5-coloring c. Without loss of generality, assume that $c(x) = 1$, $c(y) = 2$ and $c(xy) = 5$.

If $c(xx_1) = 2$ and $1 \in \{c(yy_1, c(yy_2))\}$ (here assume that $c(yy_1) = 1$ and $c(yy_2) = 3$), then recolor x with a color in $\{4, 5\}\backslash\{c(x_1)\}$, say 4, and then extend this updated partial coloring to a total 5-coloring of G by coloring u, xv with 1, w, uv with 2, v, uw with 3, vw, uy with 4, and ux, vy with 5.

If $c(xx_1) = 2$ and $1 \notin \{c(yy_1, c(yy_2))\}$ (here assume that $c(yy_1) = 3$ and $c(yy_2) = 4$), then extend c to a total 5-coloring of G by coloring vw, uy with 1, w, uv with 2, u, xv with 3, v, uw with 4, and ux, vy with 5.

If $c(xx_1) \neq 2$ and $1 \in \{c(yy_1, c(yy_2))\}$ (here assume that $c(xx_1) = 3$ and $c(yy_1) = 1$), then extend c to a total 5-coloring of G by coloring w, uv with 1, vw, ux with 2, v, uw with $c(yy_2)$, u, vy with $\{3, 4\}\backslash\{c(yy_2)\}$, and vx, uy with 5. Note that $c(yy_2) \in \{3, 4\}$.

If $c(xx_1) \neq 2$ and $1 \notin \{c(yy_1, c(yy_2))\}$ (here assume that $c(xx_1) = c(yy_1) = 3$ and $c(yy_2) = 4$), then extend c to a total 5-coloring of G by coloring vw, uy with 1, w, uv with 2, v, uw with 3, u, xv with 4, and ux, vy with 5.

Hence $d(x) = 4$, and $d(y) = 4$ by symmetry. By similar arguments, the same result hold if G contains $\mathcal{B}$. □

Lemma 4.7 *If G contains $\mathcal{A}$ or $\mathcal{B}$, then the neighbors of x and y have degrees at least 3.*

Proof By Lemma 4.6, $d(x) = d(y) = 4$. Let x_1, x_2, y_1, y_2 be vertices with $xx_1, xx_2, yy_1, yy_2 \in E(G)$. Suppose, to the contrary, that G contains $\mathcal{A}$ with $d(x_1) = 2$. Let x_3 be a vertex with $x_1x_3 \in E(G)$.

Delete u, v from G and add three new edges xy, wx and wy. It is easy to check by Facts 1 and 2 that the resulting graph G' is a *Nicop* graph with maximum degree at most 4, and thus by (2), G' admits a total 5-coloring c. Without loss of generality, assume that $c(yy_1) = 1$, $c(yy_2) = 2$, $c(y) = 3$ and $c(xy) = 5$.

First, suppose that the color on x is either 1 or 2, say 1.

If $3 \in \{c(xx_1), c(xx_2)\}$, then $2 \notin \{c(xx_1), c(xx_2)\}$, otherwise wx and wy are colored with 4, a contradiction. Therefore, we assume that $c(xx_1) = 3$ and $c(xx_2) = 4$. Remove the color on x_1 (which can be easily completed at the last stage) and recolor xx_1 with a color in $\{2, 5\}\backslash\{c(x_1x_3)\}$. Denote the resulting coloring still by c. We extend c to a total 5-coloring of G by coloring w, uv with 1, vw with 2, uw, xv with 3, v, uy with 4, vy with 5, u with $c(xx_1)$, and ux with $\{2, 5\}\backslash\{c(xx_1)\}$.

If $3 \notin \{c(xx_1), c(xx_2)\}$, then $\{c(xx_1), c(xx_2)\} = \{2, 4\}$, and c can be extended to a total 5-coloring of G by coloring w, uv with 1, u, vw with 2, uw, xv with 3, v, uy with 4, and ux, vy with 5.

Second, suppose that the color on x is neither 1 nor 2. This implies that $c(x) = 4$ and $\{c(xx_1), c(xx_2)\} \cap \{1, 2\} \neq \emptyset$. By symmetry, assume that $c(xx_1) = 1$.

If $c(xx_2) = 3$, then extend c to a total 5-coloring of G by coloring v, uw with 1, u, xv with 2, w, uv with 3, vw, uy with 4, and ux, vy with 5.

If $c(xx_2) \neq 3$, then $c(xx_2) = 2$. Remove the color on x_1 (which can be easily completed at the last stage) and recolor xx_1 with a color in $\{3, 5\}\backslash\{c(x_1x_3)\}$. Denote the resulting partial coloring still by c. If $c(xx_1) = 3$, then extend c to a total 5-coloring of G by coloring v, ux with 1, u, vw with 2, w, uv with 3, uw, vy with 4, and xv, uy with 5. If $c(xx_1) = 5$, then extend c to a total 5-coloring of G by coloring v, ux with 1, w, uv with 2, uw, xv with 3, vw, uy with 4, and u, vy with 5.

Therefore, $d(x_1) \geq 3$, and by symmetry, $d(x_2), d(y_1), d(y_2) \geq 3$. This implies that the neighbors of x and y have degree at least 3. By similar arguments as above, one can prove the same result if G contains $\mathcal{B}$. □

Corollary 4.8 *G does not contain any of the configurations among G_7, G_8, G_{11} and G_{12}.*

Proof This is a direct corollary from Lemmas 4.6 and 4.7. □

Lemma 4.9 *G does not contain the configuration G_6.*

Proof Suppose this claim is false. If $xy \in E(G)$ or $\min\{d(x), d(y)\} = 3$, then by the 2-connectivity of G, G is isomorphic to the graph induced by x, z, u, v, w and y. At this stage, we get a total 5-coloring of G by coloring z, w, ux, vy with 1, y, uv, xz with 2, v, uz with 3, u, vx, wy with 4, x, vw, uy with 5, and xy with 3 if it exists, a contradiction. Hence $xy \notin E(G)$ and $d(x) = d(y) = 4$.

Let x_1, y_1 be the vertices with $xx_1, yy_1 \in E(G)$. If $x_1 = y_1$, then by the 2-connectivity of G, G is isomorphic to he graph induced by x, z, u, v, w, y and x_1. However, we can obtain a total 5-coloring of this special graph by coloring x, uv, yx_1 with 1, y, uz, vw, xx_1 with 2, v, xz, uy with 3, z, w, ux, vy with 4, u, x_1, vx, wy with 5, a contradiction. Hence $x_1 \neq y_1$.

Delete z, u, v, w from G and add a new edge xy. It is easy to check by Facts 1 and 2 that the resulting graph is a *Nicop* graph with maximum degree at most 4. Hence by (2), G' admits a total 5-coloring c. Without loss of generality, assume that $c(x) = 1$, $c(y) = 2$ and $c(xy) = 3$.

If $c(xx_1) = c(y) = 2$ and $c(yy_1) = c(x) = 1$, then we construct a total 5-coloring of G by coloring uv with 1, uz, vw with 2, v, xz, uy with 3, z, w, ux, vy with 4, u, vx, wy with 5.

If $c(xx_1) = c(y) = 2$ and $c(yy_1) \neq c(x)$ (here assume that $c(yy_1) = 4$), then we construct a total 5-coloring of G by coloring u, w, vy with 1, uv with 2, xz, vw, uy with 3, v, z, ux with 4, and uz, wy, vx with 5.

If $c(xx_1) \neq c(y)$ and $c(yy_1) \neq c(x)$, then we consider two subcases. First, if $c(xx_1) \neq c(yy_1)$ (here assume that $c(xx_1) = 4$ and $c(yy_1) = 5$), then we construct a total 5-coloring of G by coloring w, uz, vy with 1, xz, uv with 2, u, vx, wy with 3, vw, uy with 4, and v, z, ux with 5. Second, if $c(xx_1) = c(yy_1) = 4$, then we construct a total 5-coloring of G by coloring vw, uy with 1, uv, xz with 2, z, w, ux, vy with 3, v, uz with 4, and u, vx, wy with 5. □

Lemma 4.10 *G does not contain the configuration* G_5.

Proof Suppose this claim is false. If $xy \in E(G)$, or $d(y) = 3$, or $d(x) = 2$, then by the 2-connectivity of G, G is isomorphic to the graph induced by u, v, w, x and y. At this stage, we get a total 5-coloring of G by coloring w, ux, vy with 1, y, uv with 2, v with 3, u, vx, wy with 4, x, vw, uy with 5, and xy with 3 if it exists, a contradiction. Hence $xy \notin E(G)$, $d(y) = 4$ and $2 \leq d(x) \leq 3$.

If $d(x) = 3$, then by highly similar arguments as in the proof of Lemma 4.9, one can obtain a total 5-coloring of G, a contradiction. Hence we have $d(x) = 4$. Let x_1, x_2, y_1 be the vertices with $xx_1, xx_2, yy_1 \in E(G)$.

Delete u, v, w from G and add a new edge xy. By Fact 1, Fact 2 and (2), the resulting graph G' has a total 5-coloring c. Without loss of generality, assume that $c(xx_1) = 1$, $c(xx_2) = 2$, $c(x) = 3$ and $c(xy) = 5$.

First, suppose that the color on y is either 1 or 2, say 1. We first extend c to a partial coloring c' of G by coloring u, vw with 1, v with 2, uv with 3, xv with 4, and xu, vy, w with 5. Afterwards, extend c' to a total 5-coloring of G by coloring wy, uy with 3, 4 if $c(yy_1) = 2$, 2, 4 if $c(yy_1) = 3$, and 3, 2 if $c(yy_1) = 4$, respectively.

Second, suppose that the color on y is neither 1 nor 2. This implies that $c(y) = 4$. We first extend c to a partial coloring c' of G by coloring uy, vw with 2, w with 3, u, xv with 4, and v, xu, wy with 5. Afterwards, extend c' to a total 5-coloring of G by coloring uv, vy with 1, 3 if $c(yy_1) = 1$, and 3, 1 if $c(yy_1) = 3$, respectively. □

Lemma 4.11 *G does not contain the configuration* G_{10}.

Proof Suppose, to the contrary, that G contains G_{10}. If $d(x) = 3$, then w is a cut-vertex or a vertex of degree one, which contradicts the fact that G is 2-connected. Hence $d(x) = 4$. By z, we denote the fourth neighbor of x in G. If $z = w$, then by the 2-connectivity of G, G is isomorphic to the the graph derived from K_5 by removing two adjacent edges, which admits a total 5-coloring. Hence $z \neq w$. Delete u, v from G and identify x with y. By G' and x_y we denote the resulting graph and the common vertex in G' indicating x or y. By Fact 3 and (2), G' admits a total 5-coloring c.

Without loss of generality, assume that $c(x_y) = 1$, $c(x_y z) = 2$ and $c(x_y w) = 3$. Remove the color on x_y. We now extend this partial coloring to a total 5-coloring of G as follows. First, color x with 1, and y with a color in $\{4, 5\} \backslash \{c(w)\}$, say 4. Afterwards, we color uy with 1, u, vy with 2, v, ux with 3, vx with 4, and uv, xy with 5. This results in a total 5-coloring of G, a contradiction. □

Lemma 4.12 *G does not contain any of the configurations among G_{13}, G_{14} and G_{15}.*

Proof We just prove that G does not contain G_{13} by contradiction, since another two results can be obtained by highly similar arguments. By the definition of the *Nicop*, $x \neq x'$. By Lemma 4.6, $d(x) = d(x') = 4$.

If $xx' \in E(G)$, then let x_1 and x'_1 be the vertices with $xx_1, x'x'_1 \in E(G)$. Construct a graph G' from G by deleting u, v, w, u', v', w' and y. One can easily check by Fact 1 that the resulting graph G is a *Nicop* graph with maximum degree at most 4. Hence by (2), G' has a total 5-coloring c.

Without loss of generality, assume that $c(x) = 1$, $c(x') = 2$ and $c(xx') = 3$. We extend this partial total 5-coloring to G according to the following cases.

If $c(xx_1) = c(x') = 2$ and $c(x'x'_1) = c(x) = 1$, then we construct a total 5-coloring of G by coloring $w, w', uv, u'v'$ with 1, $u, vw, u'y, v'w'$ with 2, $v, u', uw, v'y$ with 3, $v', ux, vy, u'x'$ with 4, and $uy, vx, v'x', u'w'$ with 5.

If $c(xx_1) = c(x') = 2$ and $c(x'x'_1) \neq c(x)$ (here assume that $c(x'x'_1) = 4$), then we construct a total 5-coloring of G by coloring $w, uv, u'w', v'x'$ with 1, $u, vw, v'w', u'y$ with 2, $v, u', uw, v'y$ with 3, $w', ux, vy, u'v'$ with 4, and $v', vx, uy, u'x'$ with 5.

If $c(xx_1) \neq c(x')$ and $c(x'x'_1) \neq c(x)$, then we consider two subcases. First, if $c(xx_1) \neq c(x'x'_1)$ (here assume that $c(xx_1) = 4$ and $c(x'x'_1) = 5$), then we construct a total 5-coloring of G by coloring $w, uv, u'w', v'x'$ with 1, $w', ux, vy, u'v'$ with 2, $v, v', uw, u'y$ with 3, $u, vw, v'y, u'x'$ with 4, and $u', vx, uy, v'w'$ with 5. Second, if $c(xx_1) = c(x'x'_1) = 4$, then we construct a total 5-coloring of G by coloring $w, uv, u'w', v'x'$ with 1, $w', ux, vy, u'v'$ with 2, $v, v', uw, u'y$ with 3, $u, u', vw, v'y$, with 4, and $vx, uy, v'w', u'x'$ with 5.

Hence we assume that $xx' \notin E(G)$.

Let x_1, x_2, x'_1, x'_2 be vertices with $xx_1, xx_2, x'x'_1, x'x'_2 \in E(G)$. Delete u, v, w, u', v', w' from G and add three new edges yx, yx' and xx'. It is easy to check by Facts 1 and 2 that the resulting graph G' is a *Nicop* graph with maximum degree at most 4, hence by (2), G' admits a total 5-coloring c. Without loss of generality, assume that $c(x'x'_1) = 1$, $c(x'x'_2) = 2$, $c(x') = 3$ and $c(xx') = 5$.

First, suppose that the color on x is either 1 or 2, say 1.

If $3 \in \{c(xx_1), c(xx_2)\}$, then $2 \notin \{c(xx_1), c(xx_2)\}$, otherwise xy and $x'y$ are colored with 4, a contradiction. Therefore, we assume that $c(xx_1) = 3$ and $c(xx_2) = 4$. Extend c to a total 5-coloring of G by coloring $w, w', y, uv, u'v'$ with 1, $u', ux, vy, v'w'$ with 2, $u, vw, u'w', v'y$ with 3, $v, uw, u'y, v'x'$ with 4, and $v', uy, vx, u'x'$ with 5.

If $3 \notin \{c(xx_1), c(xx_2)\}$, then $\{c(xx_1), c(xx_2)\} = \{2, 4\}$, and extend c to a total 5-coloring of G by coloring $w, w', y, uv, u'v'$ with 1, $u, u', vy, v'w'$ with 2, $ux, vw, u'w', v'y$ with 3, $v, uw, u'y', v'x'$ with 4, and $v', uy, vx, u'x'$ with 5.

Second, suppose that the color on x is neither 1 nor 2. This implies that $c(x) = 4$ and $\{c(xx_1), c(xx_2)\} \cap \{1, 2\} \neq \emptyset$. By symmetry, assume that $c(xx_1) = 1$, and thus $c(xx_2) \in \{2, 3\}$.

If $c(xx_2) = 2$, then extend c to a total 5-coloring of G by coloring $u, v', vw, u'y$ with 1, $v, u', uw, v'y$ with 2, $w', ux, vy, u'v'$ with 3, $w, uv, v'w', u'x'$ with 4, and $vx, uy, u'w', v'x'$ with 5.

If $c(xx_2) = 3$, then extend c to a total 5-coloring of G by coloring $u, v', vw, u'y$ with 1, $v, u', ux, v'y$ with 2, $w', uw, vy, u'v'$ with 3, $w, uv, v'w', u'x'$ with 4, and $vx, uy, u'w', v'x'$ with 5. □

Theorem 4.13 *Every Nicop graph with maximum degree at most 4 is total 5-colorable.*

Proof Let G be a minimum counterexample to this claim in terms of $|V(G)|+|E(G)|$. By Corollaries 4.5 and 4.8, and Lemmas 4.9, 4.10, 4.11 and 4.12, G does not contain any configurations among $G_1 - G_{15}$. This contradicts Theorem 3.11. □

Corollary 4.14 *If G is a Nicop graph with maximum degree $\Delta \geq 4$, then $\chi''(G) = \Delta + 1$.*

Proof Zhang's result in [10] implies this claim for the case when $\Delta \geq 5$, and Theorem 4.13 implies this for the case when $\Delta = 4$. □

Since K_4 is a *Nicop* graph with maximum degree 3 and total chromatic number 5, the lower bound for Δ in Corollary 4.14 is sharp.

Acknowledgements The author appreciates the anonymous referees sincerely for their helpful comments. This paper is supported by National Natural Science Foundation of China (No. 11301410), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130203120021), and the Fundamental Research Funds for the Central Universities (No. JB150714).

References

Auer C, Bachmaier C, Brandenburg FJ et al (2016) Outer 1-planar graphs. Algorithmica 74(4):1293–1320

Behzad M (1965) Graphs and their chromatic numbers. Doctoral thesis, Michigan State University

Eggleton RB (1986) Rectilinear drawings of graphs. Utilitas Math 29:149–172

Vizing V (1968) Some unsolved problems in graph theory. Uspekhi Mat Nauk 23:117–134

Wang W, Zhang K (1999) Δ-Matchings and edge-face chromatic numbers. Acta Math Appl Sin 22:236–242

Wu JL, Hu D (2004) Total coloring of series-parallel graphs. ARS Comb 73:209–211

Yap HP (1996) Total colourings of graphs. Lecture notes in mathematics 1623. Springer, Berlin

Zhang X (2013) List total coloring of pseodo-outerplanar graphs. Discret Math 313:2297–2306

Zhang X (2014) Drawing complete multipartite graphs on the plane with restrictions on crossings. Acta Math Sin (Engl Ser) 30(12):2045–2053

Zhang X, Liu G Total coloring of pseudo-outerplanar graphs. arXiv:1108.5009

Zhang X, Liu G, Wu JL (2012) Edge covering pseudo-outerplanar graphs with forests. Discret Math 312:2788–2799

Zhang Z, Zhang J, Wang J (1988) The total chromatic number of some graphs. Sci Sin Ser A 31:1434–1441