Discrete Mathematics 339 (2016) 1724–1726

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Equitable vertex arboricity of subcubic graphs ☆

Xin Zhang

School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

ARTICLE INFO

Article history:
Received 1 August 2015
Received in revised form 31 January 2016
Accepted 1 February 2016

Keywords:
Equitable coloring
Equitable partition
Vertex arboricity
Subcubic graph

ABSTRACT

An equitable partition of a graph G is a partition of the vertex set of G such that the sizes of any two parts differ by at most one. In this note, we prove that every subcubic graph can be equitably partitioned into k induced forests for any integer $k \geq 2$.

All graphs considered in this note are finite, simple and undirected unless otherwise stated. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. The *degree* of a vertex v in G, denoted by $d(v, G)$, is the number of edges that are incident with v in G. By $e(G)$, we denote the *size* of G, i.e., the number of edges in G. For two disjoint vertex set U and W, we use $e(U, W)$ to denote the set of edges whose end-vertices lie in both U and W. The subgraph of G induced by a vertex set U is denoted by $G[U]$.

The *vertex arboricity* $va(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a forest. This notion was introduced by Chartrand and Kronk [2] in 1969, who proved that $va(G) \leq 3$ for every planar graph.

In 2013, Wu, Zhang and Li [5] introduced an equitable version of the vertex arboricity. An *equitable partition* of a graph G is a partition of the vertex set of G such that the sizes of any two parts differ by at most one. The *equitable vertex arboricity* $va_{eq}(G)$ of a graph G is the minimum number of induced forests into which G can be equitably partitioned, and the *strong equitable vertex arboricity* $va^*_{eq}(G)$ of G is the minimum integer t so that G can be equitably partitioned into t' induced forests for any $t' \geq t$. Note that $va_{eq}(G)$ and $va^*_{eq}(G)$ can vary a lot. For example, $va_{eq}(K_{n,n}) = 2$ and $va^*_{eq}(K_{n,n}) = 2\lfloor(\sqrt{8n+9}-1)/4\rfloor$ if $2n = t(t+3)$ and t is odd, see [5]. Concerning $va^*_{eq}(G)$, Wu, Zhang, and Li made the following two conjectures:

Conjecture 1. *$va^*_{eq}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for every graph G.*

Conjecture 2. *There is a constant c so that $va^*_{eq}(G) \leq c$ for every planar graph G.*

Recently, Esperet, Lemoine and Maffray [3] proved that $va^*_{eq}(G) \leq \chi_a(G) - 1$ for any graph G, where $\chi_a(G)$ denotes the acyclic chromatic number of G. It was proved by Borodin [1] that any planar graph has an acyclic coloring with at most 5 colors. Therefore, the answer to Conjecture 2 is positive. In other words, $va^*_{eq}(G) \leq 4$ for every planar graph G. On the other hand, Conjecture 1 was confirmed for complete bipartite graphs [5] and graphs G with $\Delta(G) \geq |G|/2$ [6].

☆ Supported by SRFDP (No. 20130203120021), NSFC (Nos. 11301410, 11271230), and the Fundamental Research Funds for the Central Universities (No. JB150714).

E-mail address: xzhang@xidian.edu.cn.

http://dx.doi.org/10.1016/j.disc.2016.02.003
0012-365X/

A graph G is *cubic* if G is 3-regular, and is *subcubic* if it is a subgraph of a cubic graph. Grünbaum [4] showed that the acyclic chromatic number of any subcubic graph is at most 4, and this assertion $\chi_a(G) \leq 4$ for any subcubic graph G is best possible. Therefore, by the assertion $va^*_{eq}(G) \leq \chi_a(G) - 1$ of Esperet, Lemoine and Maffray, we immediately deduce that $va^*_{eq}(G) \leq 3$ for every subcubic graph G.

In the remaining of this note, we confirm Conjecture 1 for subcubic graphs by proving

Theorem 3. *Every subcubic graph can be equitably partitioned into k induced forests for any integer $k \geq 2$, i.e., $va^*_{eq}(G) \leq 2$ for every subcubic graph G.*

Actually, since we already have $va^*_{eq}(G) \leq 3$, we just need to prove the following

Theorem 4. *Any subcubic graph G can be equitably partitioned into two induced forests.*

Let G be a counterexample to Theorem 4 with $|E(G)| + |V(G)|$ being as small as possible. It is obvious that $\delta(G) \geq 2$.

Lemma 5. *G is a cubic graph.*

Proof. Suppose, to the contrary, that there is an edge uv with $d(u, G) = 2$ and $2 \leq d(v, G) \leq 3$. By the minimality of G, the vertex set of the graph $G - \{u, v\}$ can be equitably partitioned into two subsets V_1 and V_2 so that each of them induces a forest. Without loss of generality, assume that $d(v, G[V_1]) \leq 1$. Moving v to V_1 and u to V_2, we get an equitable partition $V_1 \cup \{v\}$ and $V_2 \cup \{u\}$ of $V(G)$, each of which induces a forest, a contradiction. □

Let x be a vertex with neighbors x_1, x_2 and x_3 in G. By our assumption, the vertex set of $G - xx_1$ can be equitably partitioned into two subsets V_1 and V_2 so that each of them induces a forest. Since G is cubic by Lemma 5, the order of G is even, and thus $|V_1| = |V_2|$. Let $F_1 = G[V_1]$ and $F_2 = G[V_2]$. If x and x_1 belong to different subsets, or x, x_1 belong to a subset and x_2, x_3 belong to the other subset, then G can be equitably partitioned into two induced forests F_1 and F_2, a contradiction. Therefore, we assume, without loss of generality, that $x, x_1, x_2 \in V_1$.

Lemma 6. *Every vertex in V_2 has at least two neighbors in V_1.*

Proof. If $u \in V_2$ has at most one neighbor in V_1, then move u to V_1 and move x to V_2. It is easy to see that $V_1 \cup \{u\} \setminus \{x\}$ and $V_2 \cup \{x\} \setminus \{u\}$ is an equitable partition of $V(G)$ so that each of them induces a forest, a contradiction. □

By Lemma 6, we split V_2 into two subsets

$$\mathcal{A} = \{u \in V_2 \mid d(u, V_1) = 2\}$$

and

$$\mathcal{B} = \{u \in V_2 \mid d(u, V_1) = 3\}.$$

Let $s = |V_1| = |V_2|$ and $a = |\mathcal{A}|$. We have

$$e(\mathcal{A} \cup \mathcal{B}, V_1) = 2a + 3(s - a) = 3s - a$$

and

$$e(G) = 3s.$$

It follows that

$$e(F_1) + e(G[\mathcal{A}]) = e(G) - e(\mathcal{A} \cup \mathcal{B}, V_1) = a.$$

Note that $\mathcal{B}$ is an independent set and $e(\mathcal{A}, \mathcal{B}) = 0$.

Since every vertex in $\mathcal{A}$ has exactly two neighbors in V_1 and has none neighbor in $\mathcal{B}$, it has exactly one neighbor in $\mathcal{A}$. This implies that

$$e(G[\mathcal{A}]) = \frac{a}{2}$$

and a is even. Therefore,

$$e(F_1 - xx_1) = e(F_1) - 1 = \frac{a}{2} - 1.$$

Lemma 7. *If a vertex $v \in V_1$ has two neighbors $u, w \in \mathcal{A}$, then $uw \in E(G)$.*

Proof. If $uw \notin E(G)$, then move u and w to V_1, and v and x to V_2. Note that $v \neq x$, since x has at most one neighbor in $\mathcal{A}$. Let $F'_1 := F_1 \cup \{u, w\} \setminus \{v, x\}$ and $F'_2 := F_2 \cup \{v, x\} \setminus \{u, w\}$. Since u and w have degrees at most 1 in F'_1, and $uw \notin E(F'_1)$, F'_1 is an induced forest. If $v \neq x_1, x_2, x_3$, then x and v have degrees at most 1 in F'_2, and $xv \notin E(F'_2)$. If $v = x_j$ for some $j = 1, 2, 3$, then x has degree at most 2 in F'_2 and x is the unique neighbor of v in F'_2. In either case, F'_2 is an induced forest. Hence G can be equitably partitioned into two induced forests F'_1 and F'_2, a contradiction. □

Lemma 8. *Every vertex in V_1 has at most two neighbors in $\mathcal{A}$.*

Proof. If a vertex in V_1 has three neighbors u, v, w in $\mathcal{A}$, then by Lemma 7, u, v and w induce a triangle in $G[\mathcal{A}]$. However, this is impossible, since $G[\mathcal{A}]$ is a forest. $\square$

Let $S = \{u \in V_1 \mid \exists v \in \mathcal{A}, \text{s.t.}\, uv \in E(G)\}$ denotes the set of vertices in V_1 that are adjacent to some vertex in $\mathcal{A}$. By Lemma 8, $e(S, \mathcal{A}) \leq 2|S|$. By the definition of $\mathcal{A}$, we have $e(S, \mathcal{A}) = e(V_1, \mathcal{A}) = 2a$. This implies that

$$|S| \geq a.$$

Lemma 9. *If a vertex $u \in \mathcal{A}$ has two neighbors v and w in V_1, then each of v and w is incident with at least one edge in $F_1 - xx_1$. In other words, $d(v, F_1 - xx_1) \geq 1$ for each $v \in S$.*

Proof. Suppose, to the contrary, that $d(v, F_1 - xx_1) = 0$. Moving u to V_1 and x to V_2, we obtain an equitable partition $V_1 \cup \{u\} \setminus \{x\}$ and $V_2 \cup \{x\} \setminus \{u\}$ of $V(G)$, each of which induces a forest, a contradiction. $\square$

Proof of Theorem 4. Let G be a counterexample to Theorem 4 with $|E(G)| + |V(G)|$ being as small as possible. Since $|S| \geq a$, there are at least $\frac{a}{2}$ edges in $F_1 - xx_1$, otherwise there is at least one vertex in S that is incident with no edge in $F_1 - xx_1$, contradicting Lemma 9. However, this is impossible since $e(F_1 - xx_1) = \frac{a}{2} - 1$. This contradiction completes the proof. $\square$

References

[1] O.V. Borodin, On acyclic coloring of planar graphs, Discrete Math. 25 (1979) 211–236.
[2] G. Chartrand, H.V. Kronk, The point-arboricity of planar graphs, J. Lond. Math. Soc. 44 (1969) 612–616.
[3] L. Esperet, L. Lemoine, F. Maffray, Equitable partition of graphs into induced forests, Discrete Math. 338 (8) (2015) 1481–1483.
[4] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973) 390–408.
[5] J.-L. Wu, X. Zhang, H. Li, Equitable vertex arboricity of graphs, Discrete Math. 313 (2013) 2696–2701.
[6] X. Zhang, J.-L. Wu, A conjecture on equitable vertex arboricity of graphs, Filomat 28 (1) (2014) 217–219.