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A note on the minimum number of choosability of planar graphs

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Abstract The problem of minimum number of choosability of graphs was first introduced by Vizing. It appears in some practical problems when concerning frequency assignment. In this paper, we study two important list coloring, list edge coloring and list total coloring. We prove that $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$ for planar graphs with $\Delta \geq 8$ and without adjacent 4-cycles.

Keywords Choosability · Planar graph · Cycle · List edge coloring

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1 Introduction

The theory of graph coloring has important application in combinatorial optimization, web design, computer science and so on. Particularly, in files transmission of a computer network, channel assignment, pattern matching, computation of Hessians matrix. In this paper, we consider two important coloring, list edge coloring and list total coloring, which arise in some practical problems concerning frequency assignment. Here are some other interesting colorings, we refer the readers to Angelini and Frati (2012); Bessy and Havet (2013); Du et al. (2004); Garg et al. (1996); Li et al. (2013); Wang et al. (2014b, a) etc.

All graphs considered in this paper are simple. Let G be a planar graph and has embedded in the plane. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, the edge set, the face set, the maximum degree and the minimum degree of G, respectively. For a vertex v of G, the *degree* $d(v)$ denote the number of edges incident with v, and for a face f of G, the *degree* $d(f)$ denote the length of the boundary walk of f. Denote by *k-vertex* the vertex of degree k, by *k^--vertex* the vertex of degree at most k, by *k^+-vertex* the vertex of degree at least k. If two cycles share at least one edge, we call them *adjacent*. We use $n_k(v)$, $f_k(v)$ and $n_k(f)$ to denote the number of k-vertices adjacent to the vertex v, the number of k-faces incident with the vertex v, and the number of k-vertices incident with the face f, respectively. In the following, we shall introduce the notations of list total coloring and list edge coloring.

The problem of minimum number of choosability of graphs was first introduced by Vizing (1976). Firstly, list total coloring is a type of coloring that combines list coloring with total coloring. It is a choice of a color for each vertex or each edge, from its list of allowed colors; the coloring is proper if no two adjacent or incident elements receive the same color. Specifically, a mapping L is called to be a *total assignment* for a graph G if it assigns a list $L(x)$ of possible colors for each element $x \in V \cup E$. If G has a total coloring φ such that $\varphi(x) \in L(x)$ for all $x \in V \cup E$, and no two adjacent or incident elements receive the same color, then we say that φ is a *total-L -coloring* of G or G is *total-L -colorable*. A graph G is called *total-k -choosable* if it is total-L-colorable for every total assignment L satisfying $|L(x)| \geq k$ for each element $x \in V \cup E$. The *list total chromatic number* $\chi_l''(G)$ of G is the smallest integer k such that G is total-k-choosable. Similarly, the *list edge chromatic number* $\chi_l'(G)$ of G can be defined in terms of coloring the edges alone. The ordinary edge chromatic number of G are denoted by $\chi'(G)$. Obviously, it holds that $\chi_l'(G) \geq \chi'(G) \geq \Delta$ and $\chi_l''(G) \geq \chi''(G) \geq \Delta + 1$.

As far as list edge colorings and list total colorings are widely studied, quite a few interesting results have been obtained in recent years. Firstly, we introduce a famous conjecture.

Conjecture 1 (List Coloring Conjecture) *For any graph G,*

(a) $\chi_l'(G) = \chi'(G)$*;*
(b) $\chi_l''(G) = \chi''(G)$.

Part (a) of Conjecture 1 was formulated independently by a number of people, including Vizing, Gupta, Albertson and Collins, Bollobás and Harris (see Hägkvist and Chetwynd 1992 or Jensen and Toft 1995), and it is well known as the *List Edge Coloring Conjecture*. Part (b) was formulated by Borodin et al. (1997), and it is well known as the *List Total Coloring Conjecture*. Although this conjecture has been proved for a few special cases, for example outerplanar graphs and planar graphs with $\Delta \geq 12$ Borodin et al. (1997), List Coloring Conjecture remains open.

In graph coloring, we known famous Vizing's Theorem and Total Coloring Conjecture, the later is for any graph G, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$. Combining these two results with List Coloring Conjecture, Conjecture 2 as follows is natural but remain interesting.

Conjecture 2 *For any graph G,*

(a) $\chi_l'(G) \leq \Delta + 1$*;*
(b) $\chi_l''(G) \leq \Delta + 2$.

There are some interesting results about Conjecture 2. Interestingly, the list edge chromatic number and the list total chromatic number of planar graphs with large maximum degree equals a lower bound. Hou et al. (2006) proved $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$ for planar graphs with $\Delta \geq 7$ and without 4-cycles. Li and Xu (2011) proved this result for planar graphs with $\Delta \geq 6$ and no 3-cycles adjacent to 4-cycles. Liu et al. (2009) proved $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$ holds for planar graphs with $\Delta \geq 8$ and without intersecting 4-cycles. In this paper, we strength this result and get the following theorem.

Theorem 1 *Suppose G is a planar graph without adjacent 4-cycles. If $\Delta \geq 8$, then $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$.*

2 Reducible configurations

In Borodin et al. (1997), Theorem 1 was proved for $\Delta \geq 12$. Henceforth, to prove Theorem 1, it suffices to prove the result as follows.

- Let r be a positive integer and $8 \leq r \leq 11$. Suppose G is a planar graph with $\Delta \leq r$ and without adjacent 4-cycles, then $\chi_l'(G) \leq \max\{8, r\}$ and $\chi_l''(G) \leq \max\{9, r + 1\}$.

Let an $r - minimalgraph$ be a connected graph $G = (V, E, F)$ with $|V| + |E|$ as small as possible. If G is not edge-$\max\{8, r\}$-choosable, then there is an edge assignment L for G with $|L(e)| = \max\{8, r\}$ for any edge $e \in E$ such that G is not edge-L-colorable. If G is not total-($\max\{9, r + 1\}$)-choosable, then there is a total assignment L for G with $|L(x)| = \max\{9, r + 1\}$ for any $x \in V \cup E$, such that G is not total-L-colorable.

By the minimality of G, we first show some known properties (see Borodin et al. 1997).

(a) G is connected;
(b) G contains no 2-alternating cycle;
(c) G contains no edge uv with $\min\{d(u), d(v)\} \leq \lfloor \frac{\max\{8,r\}}{2} \rfloor$ and $d(u) + d(v) \leq \max\{9, r + 1\}$.

Since G has properties (b) and (c), and any 4-cycles are not adjacent in G, we can get the following observations easily:

(O_1) Each face f is incident with at most $\lfloor \frac{d(f)}{2} \rfloor$ 3^--vertices. And furthermore, if f is incident with exactly $\frac{d(f)}{2}$ 3^--vertices (note that $d(f)$ is even), then f is incident with at least one 3-vertex.

(O_2) Each 5^+-vertex v is incident with at most $\lceil \frac{d(v)+1}{2} \rceil$ 3-faces.

Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. By (b) and (c), we have G_2 is a forest. We root G_2 at a $\max\{8, r\}$-vertex, then every 2-vertex has exactly one parent and exactly one child, which are all $\max\{8, r\}$-vertices. Moreover, if a $\max\{8, r\}$-vertex is adjacent to at least two 2-vertices, this $\max\{8, r\}$-vertex may be the child of exactly one 2-vertex and the parent of the remaining 2-vertices.

3 Discharging

We shall complete the proof of above result by using the *discharging* method. This is an important and interesting tool during the proof of the colorings of planar graphs.

Case 1 $r = 8$.

Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -8 < 0$$

We define $c(x)$ to be the initial charge. Let $c(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} c(x) = -8 < 0$. Then we apply the following rules to redistribute the initial charge that leads to a new charge $c'(x)$.

(R1) From each face f to each of its incident vertices v, transfer

(R1-1) $\frac{1}{2}$, if $d(f) = 5$ and $d(v) = 2$;

(R1-2) $\frac{3}{4}$, if $d(f) = 6$ and $d(v) = 2$;

(R1-3) 1, if $d(f) \geq 7$ and $d(v) = 2$;

(R1-4) $\frac{1}{2}$, if $d(f) \geq 5$ and $d(v) = 3$;

(R1-5) $\frac{1}{8}$, if $d(f) \geq 5, d(v) = 8$ and $n_{3^-}(f) < \lfloor \frac{d(f)}{2} \rfloor$.

(R2) From each vertex v to each of its incident 3-faces, transfer

(R2-1) $\frac{1}{3}$, if $d(v) = 5$;

(R2-2) $\frac{1}{2}$, if $d(v) \geq 6$.

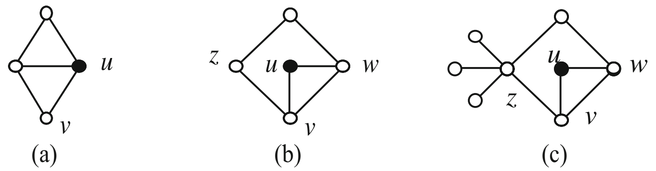

Fig. 1 $2 \leq d(z) \leq 3$ in (**b**) and $d(z) \geq 4$ in (**c**)

(R3) From each 7^+-vertex v to each of its incident 3-vertices u, transfer

(R3-1) $\frac{1}{4}$, if $d(v) \geq 7$ and Fig. 1a;

(R3-2) $\frac{1}{6}$, if $d(v) \geq 7$ and $f_4(u) = 1$.

(R4) From each 8^+-vertex v to its parent u, transfer

(R4-1) $\frac{5}{3}$, if $f_3(u) = 1$ and $f_4(u) = 1$;

(R4-2) $\frac{3}{2}$, if $f_{4^-}(u) = 1$, $f_5(u) = 1$ and except Fig. 1b, c;

(R4-3) $\frac{17}{12}$, if Fig. 1b;

(R4-4) $\frac{5}{4}$, if Fig. 1c;

(R4-5) $\frac{5}{4}$, if $f_{4^-}(u) = 1$ and $f_6(u) = 1$;

(R4-6) 1, if $f_{4^-}(u) = 1$ and $f_{7^+}(u) = 1$;

(R4-7) 1, if $f_{5^+}(u) = 2$.

(R5) From each 8^+-vertex v to each of its children u, transfer

(R5-1) $\frac{1}{3}$, if $f_3(u) = 1$ and $f_4(u) = 1$;

(R5-2) $\frac{1}{4}$, if Fig. 1b;

(R5-3) $\frac{1}{12}$, if Fig. 1c.

In the following we shall check $c'(x) \geq 0$ for all $x \in V \cup F$ which will be the desired contradiction.

Final charge of faces. Let $f \in F$. Suppose $d(f) = 3$. Then $c(f) = d(f) - 4 = -1$. If $n_{4^-}(f) = 1$, then $n_{6^+}(f) = 2$ by (c), and $c'(f) = -1 + \frac{1}{2} \times 2 = 0$ by (R2). Otherwise, $n_{5^+}(f) = 3$ and $c'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by (R2). Suppose $d(f) = 4$. Then f does not send out any charge and $c'(f) = c(f) = 0$. Recall that $n_{3^-}(f) \leq \lfloor \frac{d(f)}{2} \rfloor$ by (O_1). When $d(f) \geq 5$, we consider two cases, $n_{3^-}(f) = \lfloor \frac{d(f)}{2} \rfloor$ and $n_{3^-}(f) < \lfloor \frac{d(f)}{2} \rfloor$,

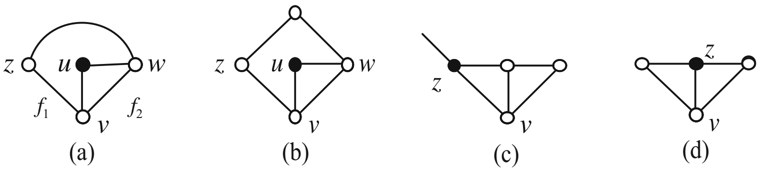

Fig. 2 (O_3) and (O_4)

since just in the latter case f sends charge to its incident 8-vertices by (R1-5). Suppose $d(f)=5$. Then $c(f)=1$ and $c'(f)\geq 1-\max\{\frac{1}{2}\times 2, \frac{1}{2}+\frac{1}{8}\times 4, \frac{1}{8}\times 8\}=0$ by (R1). Suppose $d(f)=6$. Then $c(f)=2$. Note that, if $n_{3^-}(f)=\lfloor\frac{d(f)}{2}\rfloor=3$, then $n_3(f)\geq 1$ by (O_1). Therefore, $c'(f)\geq 2-\max\{\frac{3}{4}\times 2+\frac{1}{2}, \frac{3}{4}\times 2+\frac{1}{8}\times 4\}=0$ by (R1). Suppose $d(f)\geq 7$. If $n_{3^-}(f)=\lfloor\frac{d(f)}{2}\rfloor$, then $c'(f)=c(f)-1\times n_2(f)-\frac{1}{2}\times n_3(f)\geq (d(f)-4)-1\times\lfloor\frac{d(f)}{2}\rfloor\geq 0$. Otherwise, $n_{3^-}(f)<\lfloor\frac{d(f)}{2}\rfloor$. And therefore, $c'(f)=c(f)-1\times n_2(f)-\frac{1}{2}\times n_3(f)-\frac{1}{8}\times n_8(f)\geq (d(f)-4)-1\times(\lfloor\frac{d(f)}{2}\rfloor-1)-\frac{1}{8}\times(d(f)-\lfloor\frac{d(f)}{2}\rfloor+1)\geq 0$ by (R1).

Let u be a 2-vertex of G. Suppose that v, w are the two vertices adjacent to u, and that u is the child of w. We call u a *bad parent* of v if $f_3(u)=1$ and $f_4(u)=1$ (i.e., $vw\in E$ and uv is incident with a 4-face). We call u a *bad child* of w if $f_3(u)=1$ and $f_{5^-}(u)=1$ (see Fig. 2a and b). Then each 8-vertex sends $\frac{5}{3}$ to its bad parent, and sends at most $\frac{1}{3}$ to each of its bad children by (R4-1) and (R5). A 3-vertex u is *bad* if $f_{5^+}(u)=1$ (i.e., u is incident with exactly one 5^+-face), and is *good* otherwise. Moreover, a 3-vertex u is *worst* for v if Fig. 1a, and is *worse* if $f_4(u)=1$. So each 7^+-vertex transfers charge only to bad 3-vertices adjacent to it by (R3). Moreover, we have the following observations:

- (O_3) Let u be a 2-vertex in Fig. 2a. Then $d(f_1)\geq 5$ and $d(f_2)\geq 4$. Moreover, if $d(z)=3$, then z is a good 3-vertex.
- (O_4) Let z be a 3-vertex in Fig. 2b–d. Then z is a good 3-vertex.

Final charge of vertices. Let $v\in V$. Note that G has no vertex of degree one. Suppose $d(v)=2$. Then $c(v)=d(v)-4=-2$ and $n_8(v)=2$ by (c). First, suppose $f_{4^-}(v)=2$. Then $f_3(v)=1$ and $f_4(v)=1$ since G has no adjacent 4-cycles. So $c'(v)\geq -2+\frac{5}{3}+\frac{1}{3}=0$ by (R4-1) and (R5-1). Second, suppose $f_{4^-}(v)=1$. If $f_5(v)=1$, then $c'(v)=-2+\min\{\frac{1}{2}+\frac{3}{2}, \frac{1}{2}+\frac{17}{12}+\frac{1}{12}, \frac{1}{2}+\frac{5}{4}+\frac{1}{4}\}=0$. If $f_6(v)=1$, then $c'(v)=-2+\frac{3}{4}+\frac{5}{4}=0$ by (R1-2) and (R4-5). If $f_{7^+}(v)=1$, then $c'(v)=-2+1+1=0$ by (R1-3) and (R4-6). Third, suppose $f_{5^+}(v)=2$. Then $c'(v)\geq -2+\frac{1}{2}\times 2+1=0$ by (R1-1) and (R4-7).

Suppose $d(v)=3$. Then $c(v)=-1$, and $n_{7^+}(v)=3$ by (c). Note that $f_{5^+}(v)\geq 1$ since G has no adjacent 4-cycles. If $f_{5^+}(v)=1$, then clearly $c'(v)=-1+\min\{\frac{1}{2}+\frac{1}{6}\times 3, \frac{1}{2}+\frac{1}{4}\times 2\}=0$ by (R1-4) and (R3). Otherwise, $f_{5^+}(v)\geq 2$ and $c'(v)\geq -1+\frac{1}{2}\times 2=0$ by (R1-4). Suppose $d(v)=4$. Then clearly $c'(v)=c(v)=0$. Suppose $d(v)=5$. Then $c(v)=1$, and $n_{4^-}(v)=0$ by (c). Note that by our discharging rules, v sends charge only to its incident 3-faces. By (O_2), $f_3(v)\leq 3$. Hence, $c'(v)\geq 1-\frac{1}{3}\times 3=0$ by (R2). Suppose $d(v)=6$. This case is similar to that of $d(v) = 5$. We have $c(v) = 2$, $n_{3^-}(v) = 0$ by (c), and $f_3(v) \leq 4$ by (O_2). So $c'(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by (R2). Suppose $d(v) = 7$. Then $c(v) = 3$, $n_2(v) = 0$ by (c), and $f_3(v) \leq 4$. If $f_3(v) = 4$, then v is adjacent to at most two bad 3-vertices, and $c'(v) \geq 3 - \frac{1}{2} \times 4 - \frac{1}{4} \times 2 = \frac{1}{2} > 0$ by (R2) and (R3). If $f_3(v) = 3$, then v is adjacent to at most four bad 3-vertices, and $c'(v) \geq 3 - \frac{1}{2} \times 3 - \frac{1}{4} \times 4 = \frac{1}{2} > 0$. If $f_3(v) \leq 3$, then $c'(v) > 3 - \frac{1}{2} \times 3 - \frac{1}{4} \times 3 - \frac{1}{6} \times 4 = \frac{1}{12} > 0$.

Suppose $d(v) = 8$. Then $c(v) = 4$, and $f_3(v) \leq 5$ by (O_2). Suppose $n_2(v) = 0$. If $f_3(v) = 5$, then v is adjacent to at most one bad 3-vertex. So $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{1}{4} = \frac{5}{4} > 0$. If $f_3(v) \leq 4$, then $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{1}{4} \times 4 - \frac{1}{6} \times 4 = \frac{1}{3} > 0$ by (R2) and (R3).

Suppose $n_2(v) = 1$. Let u be the 2-vertex adjacent to v. Suppose u is a child of v. Then v sends at most $\frac{1}{3}$ to u by R5. If $f_3(v) = 5$. then v is adjacent to at most one bad 3-vertex, so $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{1}{3} - \frac{1}{4} = \frac{11}{12} > 0$. If $f_3(v) \leq 4$, then $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{4} \times 4 - \frac{1}{6} \times 3 = \frac{1}{6} > 0$. Suppose u is the parent of v. Then v sends at most $\frac{5}{3}$ to u by R4. First, suppose u is a bad parent (also see Fig. 2a). Then $f_3(v) \leq 4$ since G has no adjacent 4-cycles. If $f_3(v) = 4$, then v is adjacent to at most one bad 3-vertex by (O_3) and (O_4). So $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{5}{3} - \frac{1}{4} = \frac{1}{12} > 0$. If $f_3(v) = 3$, then v is adjacent to at most two worst 3-vertices and at most one worse 3-vertex. So $c'(v) \geq 4 - \frac{1}{2} \times 3 - \frac{5}{3} - \frac{1}{4} \times 2 - \frac{1}{6} = \frac{1}{6} > 0$. If $f_3(v) \leq 2$, then $c'(v) \geq 4 - \frac{1}{2} \times 2 - \frac{5}{3} - \frac{1}{4} - \frac{1}{6} \times 3 = \frac{7}{12} > 0$. Second, suppose u is not a bad parent. Then v transfers at most $\frac{3}{2}$ to u by R4. If $f_3(v) \leq 2$, then $c'(v) \geq 4 - \frac{1}{2} \times 2 - \frac{3}{2} - \frac{1}{4} - \frac{1}{6} \times 6 = \frac{1}{4} > 0$. If $f_3(v) = 3$, then v is adjacent to at most three worst 3-vertices and at most one worse 3-vertex. So $c'(v) \geq 4 - \frac{1}{2} \times 3 - \frac{3}{2} - \frac{1}{4} \times 3 - \frac{1}{6} = \frac{1}{12} > 0$. Suppose $f_3(v) = 4$. Then v is adjacent to at most three bad 3-vertices. If v is adjacent to at most two bad 3-vertices, then $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{3}{2} - \frac{1}{4} \times 2 = 0$. Otherwise, v is adjacent to three bad 3-vertices. If the parent of v is incident with a 6^+-face, then $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{5}{4} - \frac{1}{4} \times 3 = 0$. Otherwise, u receives at most $\frac{5}{4}$ from v (see Fig. 1c), so $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{5}{4} - \frac{1}{4} \times 2 - \frac{1}{6} = \frac{1}{12} > 0$. Suppose $f_3(v) = 5$. Then v is adjacent to at most one bad 3-vertex by (O_4). If uv is incident with a 5-face, then $c'(v) \geq 4 - \frac{1}{2} \times 5 - \max\{\frac{17}{12}, (\frac{5}{4} + \frac{1}{6})\} = \frac{1}{12} > 0$. Otherwise, uv is incident with a 6^+-face and v transfers to u at most $\frac{5}{4}$ by (R4). So $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{5}{4} - \frac{1}{4} = 0$.

In the following, we assume $n_2(v) \geq 2$. Then v is adjacent to at most three bad children since v is incident with no adjacent 4-cycle.

First, suppose v is adjacent to three bad children. Then $f_3(v) = 3$ and v is adjacent to on bad 3-vertices and no bad parent. So $c'(v) \geq 4 - \frac{1}{2} \times 3 - \frac{3}{2} - \frac{1}{3} \times 3 = 0$.

Second, suppose v is adjacent to two bad children. Then v is adjacent to at most one bad parent and $f_3(v) \leq 5$. If v is adjacent to one bad parent, then $f_3(v) = 3$ and v is adjacent to on bad 3-vertices. So $c'(v) \geq 4 - \frac{1}{2} \times 3 - \frac{5}{3} - \frac{1}{3} \times 2 = \frac{1}{6} > 0$. Otherwise, v is adjacent to no bad parent. Suppose $f_3(v) = 5$. Then v is adjacent to no parent and v is adjacent to no bad 3-vertex. So $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{1}{3} \times 2 = \frac{5}{6} > 0$. Suppose $f_3(v) = 4$. If v is adjacent to no bad vertices, then the parent of v receives at most $\frac{5}{4}$ from v, so $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{5}{4} - \frac{1}{3} \times 2 = \frac{1}{12} > 0$. Otherwise, v is adjacent to at most bad 3-vertex. If the parent of v receives at most 1 from v, then $c'(v) \geq 4 - 1 - \frac{1}{2} \times 4 - \frac{1}{3} \times 2 - \frac{1}{4} = \frac{1}{12} > 0$. Otherwise, the parent of v receives at most $\frac{5}{4}$ from v, then the bad 3-vertex of v is a worse 3-vertex and v is incident with a 5^+-face which sends $\frac{1}{8}$ to v. So $c'(v) \geq 4 + \frac{1}{8} - \frac{5}{4} - \frac{1}{2} \times 4 - \frac{1}{3} \times 2 - \frac{1}{6} = \frac{1}{24} > 0$. Suppose $f_3(v) = 3$. Then v is adjacent to at most one bad 3-vertex by O_2 and $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{3} \times 2 - \frac{1}{2} \times 3 - \frac{1}{4} = 0$. Suppose $f_3(v) \leq 2$. Then $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \times 2 - \frac{1}{3} \times 2 - \frac{1}{6} \times 4 = \frac{1}{6} > 0$.

Third, suppose v is adjacent to exactly one bad child, say u. Then v is adjacent to at most one bad parent. Suppose v is adjacent to exactly one bad parent. Then $f_3(v) \leq 4$. Suppose $f_3(v) = 4$. Then v is adjacent to at most one bad 3-vertex. If $f_3(v) = 4$ and v is adjacent to no bad 3-vertex, then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{3} = 0$. If $f_3(v) = 4$ and v is adjacent to one bad 3-vertex, then the bad child of v receives at most $\frac{1}{12}$ from v by R(5-3). So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{4} - \frac{1}{12} = 0$. If $f_3(v) = 3$, then v is adjacent to at most two bad 3-vertices by O_2. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 3 - \frac{1}{3} - \frac{1}{4} \times 2 = 0$. If $f_3(v) \leq 2$, then v is adjacent to at most three bad 3-vertices and $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 2 - \frac{1}{3} - \frac{1}{4} \times 3 = \frac{1}{4} > 0$. Suppose v is adjacent to no bad parent. Then clearly, $f_3(v) \leq 5$. If $f_3(v) \leq 2$, then $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \times 2 - \frac{1}{3} - \frac{1}{6} \times 7 = 0$. If $f_3(v) = 3$, then v is adjacent to at most three bad 3-vertices by O_3 and $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \times 3 - \frac{1}{3} - \frac{1}{4} \times 2 - \frac{1}{6} = 0$. Suppose $f_3(v) = 4$. Then v is adjacent to at most two bad 3-vertices by O_3. If v is adjacent to no bad 3-vertices, then $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \times 4 - \frac{1}{3} = \frac{1}{6} > 0$. Suppose v is adjacent to exactly one bad 3-vertex. If u receives at most $\frac{1}{4}$ from v, then $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \times 4 - \frac{1}{4} - \frac{1}{4} = 0$. Otherwise, u receives $\frac{1}{3}$ from v and the parent of v receives at most $\frac{5}{4}$ from v. So $c'(v) \geq 4 - \frac{5}{4} - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{4} = \frac{1}{6} > 0$. Suppose v is adjacent to two bad 3-vertices. Then u receives at most $\frac{1}{12}$ from v. If the parent is incident with a 6^+-face, then $c'(v) \geq 4 - \frac{5}{4} - \frac{1}{2} \times 4 - \frac{1}{4} - \frac{1}{4} \times 2 = 0$. If the parent is incident with two 5^+-faces, then $c'(v) \geq 4 - 1 - \frac{1}{2} \times 4 - \frac{1}{4} - \frac{1}{4} \times 2 = \frac{1}{4} > 0$. So the parent is incident with exactly one 5-face and v is incident with at most one worse 3-vertex and no worst 3-vertices. So $c'(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \times 4 - \frac{1}{12} - \frac{1}{4} - \frac{1}{6} = 0$. Suppose $f_3(v) = 5$. Then u receives at most $\frac{1}{4}$ from v and v is adjacent to no bad 3-vertices. If the parent of v is incident with a 6^+-face f, then $c'(v) \geq 4 - \frac{5}{4} - \frac{1}{2} \times 5 - \frac{1}{4} = 0$. If the parent of v is incident with a 5-face and u receives $\frac{17}{12}$ from v, then v is adjacent to no bad 3-vertices and $c'(v) \geq 4 - \frac{17}{12} - \frac{1}{2} \times 5 = \frac{1}{12} > 0$. If the parent of v is incident with a 5-face and u receives $\frac{5}{4}$ from v, then $c'(v) \geq 4 - \frac{5}{4} - \frac{1}{2} \times 5 - \frac{1}{4} = 0$.

Fourth, we suppose that v is adjacent to no bad child. If v is adjacent to no parent, then $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{1}{6} \times 8 = \frac{1}{6} > 0$. Otherwise, v is adjacent to a parent, say u. Suppose $f_3(v) = 5$. Then v is adjacent to at most one bad 3-vertex. If u is incident with a 6^+-face, then $c'(v) \geq 4 - \frac{5}{4} - \frac{1}{2} \times 5 - \frac{1}{4} = 0$. If u is incident with a 5-face and u receives $\frac{17}{12}$ from v, then v is adjacent to no bad 3-vertices and $c'(v) \geq 4 - \frac{17}{12} - \frac{1}{2} \times 5 = \frac{1}{12} > 0$. If u is incident with a 5-face and u receives $\frac{5}{4}$ from v, then $c'(v) \geq 4 - \frac{5}{4} - \frac{1}{2} \times 5 - \frac{1}{4} = 0$. Suppose $f_3(v) = 4$. Then v is adjacent to at most two bad 3-vertices. If v is adjacent to two bad 3-vertices, then u receives at most $\frac{5}{4}$ from v. Otherwise, $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{4} = \frac{1}{12}0$. Suppose $f_3(v) = 3$. Then v is adjacent to at most three bad 3-vertices and $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 3 - \frac{1}{4} \times 3 = \frac{1}{12} > 0$. Suppose $f_3(v) \leq 2$. Then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 2 - \frac{1}{4} \times 2 - \frac{1}{6} \times 5 = 0$.

Case 2 $9 \leq r \leq 11$.
We also rewritten Euler's formula as

$$\sum_{v \in V}(d(v)-4)+\sum_{f \in F}(d(f)-4)=-8<0$$

(r1) From each face f to each of its incident vertices v, transfer

(r1-1) $\frac{1}{2}$, if $d(f)=5$ and $d(v)=2$;

(r1-2) $\frac{3}{4}$, if $d(f)\geq 6$ and $d(v)=2$;

(r1-3) $\frac{1}{2}$, if $d(f)\geq 5$ and $d(v)=3$.

(r2) From each vertex v to each of its incident 3-faces, transfer $\frac{1}{2}$, if $d(v)\geq 6$.

(r3) From each 8^+-vertex v to each of its incident 3-vertices u, transfers $\frac{1}{6}$, if $f_{5^+}(u)=1$.

(r4) From each 9^+-vertex v to its parent u, transfer

(r4-1) 2, if $f_3(u)=1$ and $f_4(u)=1$;

(r4-2) $\frac{3}{2}$, if $f_{4^-}(u)=1$, $f_5(u)=1$;

(r4-3) $\frac{5}{4}$, otherwise.

Clearly, for all faces f, $c'(f)\geq 0$ and for all vertices v and $c'(v)\geq 0$, if $d(v)\leq 5$. If $d(v)=6$, then $c(v)=2$, $n_{3^-}(v)=0$ by (c), and $f_3(v)\leq 4$. So $c'(v)\geq 2-\frac{1}{2}\times 4=0$. If $d(v)=7$, then $c(v)=3$, $n_{3^-}(v)=0$ by (c), and $f_3(v)\leq 4$. So $c'(v)\geq 3-\frac{1}{2}\times 4=1>0$. If $d(v)=8$, then $c(v)=4$, $n_{2^-}(v)=0$ by (c), and $f_3(v)\leq 5$. So $c'(v)\geq 4-\frac{1}{2}\times 5-\frac{1}{6}\times 8=\frac{1}{6}>0$. Suppose $d(v)=9$. Then $c(v)=5$, $f_3(v)\leq 6$. If $f_3(v)=6$, then the parent of v (if exists) is incident with a 6^+-face and v is incident with at most three 3-vertices each of which receives charge from v. So $c'(v)\geq 5-\frac{5}{4}-\frac{1}{2}\times 6-\frac{1}{6}\times 3=\frac{1}{4}>0$. If $f_3(v)=5$, then $c'(v)\geq 5-\max\{(2+\frac{1}{2}\times 5+\frac{1}{6}\times 2),(\frac{3}{2}+\frac{1}{2}\times 5+\frac{1}{6}\times 4)\}=\frac{1}{6}>0$. If $f_3(v)=4$, then $c'(v)\geq 5-2-\frac{1}{2}\times 4-\frac{1}{6}\times 6=0$. If $f_3(v)\leq 3$, then $c'(v)\geq 5-2-\frac{1}{2}\times 3-\frac{1}{6}\times 9=0$. Suppose $d(v)=10$. Then $c(v)=6$, $f_3(v)\leq 6$. If $f_3(v)=6$, then v is adjacent to at most four 3-vertices each of which needs receive charge from v, so $c'(v)\geq 6-2-\frac{1}{2}\times 6-\frac{1}{6}\times 4=\frac{1}{3}>0$. Otherwise, $f_3(v)\leq 5$ and $c'(v)\geq 6-2-\frac{1}{2}\times 5-\frac{1}{6}\times 9=0$. Suppose $d(v)=11$. Then $c(v)=7$, $f_3(v)\leq 7$. If $f_3(v)=7$, then v is adjacent to at most four 3-vertices each of which needs receive charge from v, so $c'(v)\geq 7-2-\frac{1}{2}\times 7-\frac{1}{6}\times 4=\frac{5}{6}>0$. Otherwise, $f_3(v)\leq 6$ and $c'(v)\geq 7-2-\frac{1}{2}\times 6-\frac{1}{6}\times 10=\frac{1}{3}>0$.

Hence we complete the proof of Theorem 1.

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