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On Equitable Colorings of Sparse Graphs

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Abstract A graph is equitably k-colorable if G has a proper vertex k-coloring such that the sizes of any two color classes differ by at most one. Chen, Lih and Wu conjectured that any connected graph G with maximum degree Δ distinct from the odd cycle, the complete graph $K_{\Delta+1}$ and the complete bipartite graph $K_{\Delta,\Delta}$ are equitably m-colorable for every $m \geq \Delta$. Let $\mathcal{G}_k$ be the class of graphs G such that $e(G') \leq k(v(G')-2)$ for every subgraph G' of G with order at least 3. In this paper, it is proved that any graph in $\mathcal{G}_4$ with maximum degree $\Delta \geq 17$ is equitably m-colorable for every $m \geq \Delta$. As corollaries, we confirm Chen–Lih–Wu Conjecture for 1-planar graphs, 3-degenerate graphs and graphs with maximum average degree less than 6, provided that $\Delta \geq 17$.

Keywords Equitable coloring · 3-Degenerate graph · Average degree · 1-Planar graph

Mathematics Subject Classification 05C15

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1 Introduction

All graphs considered in this paper are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. Set $e(G) = |E(G)|$ and $v(G) = |V(G)|$. For $X, Y \subseteq V(G)$, let $e(X, Y) = \{xy \in E(G)|x \in X, y \in Y\}$. For a set $S \subseteq V(G)$, by $G[S]$ we denote the graph induced by S. We say a graph G is *d-degenerate* if $\delta(G') \le d$ for each $G' \subseteq G$. The *maximum average degree* of a graph G is $\mathrm{mad}(G) = \max\{2e(G')/v(G')|G' \subseteq G\}$. For other undefined concepts, we refer the readers to [1].

A graph is *1-planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planarity was introduced by Ringel [19] while trying to simultaneously color the vertices and faces of a plane graph G such that any pair of adjacent/incident elements receives different colors. One can trivially see that the class of 1-planar graphs is a larger class than the one of planar graphs.

A *k-coloring* of G is a function f from $V(G)$ to $\{1, 2, \dots, k\}$ such that $f(u) \neq f(v)$ if $uv \in E(G)$. We say a k-coloring of G is *equitable* if the size of any two color classes differ by at most one. The smallest integer k such that G is equitably k-colorable is the *equitable chromatic number* of G, denoted by $\chi_{\mathrm{eq}}(G)$. Note that a graph may have an equitable k-coloring but no equitable $k+1$ colorings. Turán graph $T_{n,k}$ (the balanced complete k-partite graph with n-vertices) is such an example. In view of this, another interesting parameter in the area of equitable coloring is presented. The *equitable chromatic threshold*, denoted by $\chi^*_{\mathrm{eq}}(G)$, is the smallest integer k such that G is equitably k'-colorable for every $k' \ge k$. It is clear that $\chi_{\mathrm{eq}}(G) \le \chi^*_{\mathrm{eq}}(G)$. Here, one should be careful that these two parameters may vary a lot. Take the complete bipartite graph $K_{7,7}$ for example, one can calculate that $\chi_{\mathrm{eq}}(K_{7,7}) = 2$ but $\chi^*_{\mathrm{eq}}(K_{7,7}) = 8$.

In 1970, Hajnal and Szemerédi [8] answered a question of Erdős by proving that every graph G with $\Delta(G) \le r$ has an equitable $(r+1)$-coloring. Actually, Hajnal–Szemerédi Theorem states that $\chi^*_{\mathrm{eq}}(G) \le \Delta(G)+1$ for any graph G. In 2008, Kierstead and Kostochka [10] simplified the proof of Hajnal–Szemerédi Theorem by a very technical method and provided a polynomial-time algorithm for such a coloring. Kierstead and Kostochka [9] also proved that every graph G with $d(x)+d(y) \le 2r+1$ for every edge xy has an equitable $(r+1)$-coloring, which strengthened the Hajnal–Szemerédi Theorem by weakening the degree constraint.

Here one should pay attention to the sharpness of the upper bound on $\chi^*_{\mathrm{eq}}(G)$ in Hajnal–Szemerédi Theorem, since the complete graphs and the odd cycles admit no proper colorings with the number of involved colors being equal to their maximum degree, and the complete bipartite graph $K_{2m+1,2m+1}$ has an equitable 2-coloring but no equitable $(2m+1)$-colorings.

In view of this, a natural question is: can the upper bound for $\chi^*_{\mathrm{eq}}(G)$ in Hajnal–Szemerédi Theorem be reduced if we add some restrictions on the graph G? To answer this question, we first ask about an equitable version of Brook's Theorem. Actually, many researches conducted in the area of equitable colorings have been focusing either on *Equitable Coloring Conjecture* formulated by Meyer [17] in 1973 or on *Equitable Δ-Coloring Conjecture* (also known as *Chen–Lih–Wu Conjecture*) made by Chen et al. [3] in 1994. Note that Equitable Δ-Coloring Conjecture is stronger than Equitable Coloring Conjecture, as one can see below.

Conjecture 1 (Equitable Coloring Conjecture) *For any connected graph G, except the complete graph and the odd cycle, $\chi_{\rm eq}(G) \le \Delta(G)$.*

Conjecture 2 (Equitable Δ- Coloring Conjecture) *For any connected graph G, except the complete graph, the odd cycle and the complete bipartite graph $K_{2m+1,2m+1}$, $\chi^*_{\rm eq}(G) \le \Delta(G)$.*

As far as we know, Equitable Δ-Coloring Conjecture has already been confirmed for many classes of graphs such as graphs with $\Delta \le 3$ [3,4] or $\Delta = 4$ [11] or $\Delta \ge (|G| + 1)/3$ [2,3], bipartite graphs [16], interval graphs [6], outerplanar graphs [23], series–parallel graphs [25], pseudo-outerplanar graphs [20], planar graphs with $\Delta \ge 9$ [18], d-degenerate graphs with $d \le (\Delta - 1)/14$ [13] or $d \le \Delta/10$ and $\Delta \ge 46$ [12], and graphs with $\Delta \ge 46$ and maximum average degree at most $\Delta/5$ [12]. In general, Equitable Coloring Conjecture and Equitable Δ-Coloring Conjecture are still wide open. Recently in 2013, Lih [15] gives a nice survey on the recent progresses on equitable colorings of graphs. For other colorings of sparse graphs such as planar graphs, one can refer to [5,21,22] for an extending reading.

Let $\mathcal{G}_k$ be the class of graphs G such that $e(G') \le k(v(G') - 2)$ for every subgraph G' of G with order at least 3. Yap and Zhang [24] proved that $\chi^*_{\rm eq}(G) \le \Delta(G)$ for every planar graph G with maximum degree at least 13. Indeed, their proof in that reference implies the following more general result.

Theorem 3 [24] *If $G \in \mathcal{G}_3$ is a graph with maximum degree $\Delta \ge 13$, then G is equitably m-colorable for every $m \ge \Delta$.*

In this paper, we consider the class $\mathcal{G}_4$ and prove that Equitable Δ-Coloring Conjecture holds for the class $\mathcal{G}_4$, provided that Δ is large enough. The following theorem is the main result of this paper.

Theorem 4 *If $G \in \mathcal{G}_4$ is a graph with maximum degree $\Delta \ge 17$, then G is equitably m-colorable for every $m \ge \Delta$.*

It is known that the class of 1-planar graphs is a subclass of $\mathcal{G}_4$ [7]. We immediately have the following corollary.

Corollary 5 *If G is a 1-planar graph with maximum degree $\Delta \ge 17$, then G is equitably m-colorable for every $m \ge \Delta$.*

Note that for any d-degenerate graph G it trivially holds that $e(G) \le dv(G)$ since we can destroy G by successively deleting vertices of degree at most d. Let G be a 3-degenerate graph and let G' be an arbitrary subgraph of G with $v(G') \ge 3$. If $v(G') \ge 8$, then $e(G') \le 3v(G') \le 4v(G') - 8$. If $v(G') = 7$, then $G' \ne K_7$ since the complete graph K_7 is not 3-degenerate, and thus, we shall have $e(G') \le \big(v(G')(v(G') - 1)\big)/2 - 1 = 4v(G') - 8$. If $3 \le v(G') \le 6$, then one can also easily check that $e(G') \le \big(v(G')(v(G') - 1)\big)/2 \le 4v(G') - 8$. Hence, the class of 3-degenerate graphs is exactly a subclass of $\mathcal{G}_4$, and we obtain another immediate consequence of Theorem 4.

Corollary 6 *If G is a 3-degenerate graph with maximum degree $\Delta \geq 17$, then G is equitably m-colorable for every $m \geq \Delta$.*

Note that it was proved in [13] by Kostochka and Nakprasitthat that the equitable chromatic threshold of a 3-degenerate graph G is at most $\Delta(G)$ if $\Delta(G) \geq 43$, which is the best known result in terms of $\Delta(G)$ to our knowledge, so Corollary 6 can be viewed as an improvement of the corresponding result in [13].

Another immediate corollary of Theorem 4 is a partial improvement of Kostochka and Nakprasitthat's result on the equitable Δ-coloring of graphs with low average degree in [12]. Let G be a graph with $\mathrm{mad}(G) < 6$ and let G' be an arbitrary subgraph of G with $v(G') \geq 3$. If $v(G') \geq 7$, then $e(G') \leq 3v(G') - 1 \leq 4v(G') - 8$. If $3 \leq v(G') \leq 6$, then we also have $e(G') \leq \big(v(G')(v(G') - 1)\big)/2 \leq 4v(G') - 8$. Hence, the class of graphs with maximum average degree less than 6 is also contained in $\mathcal{G}_4$. Thus, the following corollary seems natural.

Corollary 7 *If G is a graph with* $\mathrm{mad}(G) <6$ *and maximum degree $\Delta \geq 17$, then G is equitably m-colorable for every $m \geq \Delta$.*

2 Some Useful Lemmas

In this section, we first introduce some notions that may be used in next arguments and give some useful lemmas. Let $\mathcal{G}$ be a class of graphs and let G be a graph belonging to $\mathcal{G}$. By $\delta(\mathcal{G})$, we denote the maximum minimum degree of the class $\mathcal{G}$, that is, the value of $\max_{G\in\mathcal{G}} \delta(G)$.

Lemma 8 $\delta(\mathcal{G}_3) = 5$, $\delta(\mathcal{G}_4) = 7$ *and* $\delta(\mathcal{G}_k) \leq 2k - 1$ *for every* $k \geq 5$.

Lemma 9 *Every graph $G \in \mathcal{G}_k$ is $2k$-colorable, K_{2k-1}-free and $(2k-1)$-degenerate.*

The proofs of above two lemmas are directly followed from the definition of $\mathcal{G}_k$. Here note that there exists a 5-regular planar graph and a 7-regular 1-planar graph (see Figure 1 in [7]); hence, $\delta(\mathcal{G}_3) = 5$ and $\delta(\mathcal{G}_4) = 7$.

Lemma 10 *Let $m \geq 1$ and $k \geq 3$ be two fixed integers. If any graph $G \in \mathcal{G}_k$ of order mt is equitably m-colorable for any integer $t \geq 1$, then any graph in $\mathcal{G}_k$ is also equitably m-colorable.*

Proof Let G be a graph in $\mathcal{G}_k$. If $v(G)$ is not divisible by m, then we can assume, without loss of generality, that $\delta(G) = 2k - 1$ with $k \geq 3$ and $|G| = mt - j$ with $0 < j < m$. If $m \leq 2k - 1$, then it is trivial that $0 < j < 2k - 1$. Now we claim that $0 < j < 2k - 1$ holds for $m \geq 2k$. Suppose this does not hold. Let u be a vertex in G with $d(u) = \delta(G)$. Using induction on $|G|$, the graph $G - u$ admits an equitably m-coloring with color classes $V_1, \ldots, V_m$, where $|V_i| = t - 1$ or t for all $i \geq 1$. Assume that $N(u) \in \bigcup_{i=1}^{2k-1} V_i$. If there exists a class V_i with $i \geq 2k$ such that $|V_i| = t - 1$, then move u to V_i, and thus, we can get an equitably m-coloring of G. If $|V_i| = t$ for all $i \geq 2k$, then we also have $|G| = mt - j$ with $0 < j < 2k - 1$. Let $G' = G \cup K_j$. One can easily see that $|G'| = mt$ and $G' \in \mathcal{G}_k$. Hence, G' is equitably m-colorable by the assumption, and so is G by restricting the coloring of G' to G. □

Lemma 11 [24] *Let $m \geq 4$, $t \geq 1$ be two fixed integers and let G be an m-colorable graph of order mt. If $e(G) \leq (m-1)t$, then G is equitably m-colorable.*

Lemma 12 *Let $m \geq 1$, $s \geq 1$ and $t \geq 3$ be integers. Suppose that G is a K_t-free graph with $\Delta(G) \leq m$. If G has an independent s-set I and there exists $A \subseteq V(G)\backslash I$ such that $|A| > \frac{s(m+t-2)}{2}$ and $e(v, I) \geq 1$ for all $v \in A$, then A contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in I$.*

Lemma 13 *Let $m \geq 3$, $t \geq 1$ be integers and let $\mathcal{G}$ be a class of graphs with $\delta(\mathcal{G}) \leq m-2$. Suppose that $G \in \mathcal{G}$ is a graph with order mt and maximum degree Δ. If $e(G) \leq (2m-3)t - \max\{\Delta - 3, t\}$, then G is equitably m-colorable.*

The proofs of Lemmas 12 and 13 are highly similar to the proofs of Lemmas 4 and 5 in the reference [24]. Hence, we omit them here.

Lemma 14 *Let $\mathcal{G}$ be a class of graphs with $\delta(\mathcal{G}) := \delta$ and let $G \in \mathcal{G}$ be an m-colorable graph with order mt, where $m \geq \delta + 1$. Suppose that uv is an edge in G with $d(u) \leq \delta$. If G has no equitable m-colorings but $G - uv$ admits an equitable m-coloring having color classes $V_1, \ldots, V_m$, where $|V_i| = t$ for all $1 \leq i \leq m$, such that $u, v \in V_1$ and $N(u) \subseteq \bigcup_{i=1}^{\delta} V_i$, then the following results hold.*

(a) If $V_1' = V_1\backslash\{u\}$, then

$$e\left(\bigcup_{i=\delta+1}^{m} V_i, V_1'\right) \geq (m-\delta)t. \tag{1}$$

(b) If there exists $w \in V_j$ for some $2 \leq j \leq \delta$ such that $e(w, V_1') = 0$, then

$$e\left(\bigcup_{i=\delta+1}^{m} V_i, V_j'\right) \geq (m-\delta)t, \tag{2}$$

where $V_j' = V_j\backslash\{w\}$.

(c) If there exists $x \in V_k$ for some $2 \leq k \leq \delta$ and $k \neq j$ [here j is defined by (b)] such that $e(x, V_j') = 0$, then

$$e\left(\bigcup_{i=\delta+1}^{m} V_i, V_k'\right) \geq (m-\delta)t, \tag{3}$$

where $V_k' = V_k\backslash\{x\}$.

(d) If there exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \leq j \leq \delta$, then

$$e\left(\bigcup_{i=\delta+1}^{m} V_i, \bigcup_{j=1}^{\delta} V_j'\right) \geq \delta(m-\delta)t, \tag{4}$$

where $V_j' = V_j\backslash\{v_j\}$ for each $1 \leq j \leq \delta$ and $u = v_1$.

(e) *Let μ be a fixed integer between 2 and $\delta - 1$. If there exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \le j \le \mu$ and $e(v, V_k') \ge 1$ for any $v \in \bigcup_{i=\mu+1}^{\delta} V_i$ and $1 \le k \le \mu$, then*

$$e\left(\bigcup_{i=\mu+1}^{m} V_i \cup \{u\}, \bigcup_{j=1}^{\mu} V_j'\right) \ge \mu(m-\mu)t + 1, \tag{5}$$

where $V_k' = V_k \backslash \{v_k\}$ for each $1 \le k \le \mu$ and $u = v_1$.

(f) *If $e(w, V_1') \ge 1$ for any $w \in \bigcup_{i=2}^{\delta} V_i$, then*

$$e\left(\bigcup_{i=2}^{m} V_i \cup \{u\}, V_1'\right) \ge (m-1)t + 1. \tag{6}$$

Proof (a) Here, we only need to prove that $e(v, V_1') \ge 1$ for any $v \in \bigcup_{i=\delta+1}^{m} V_i$. Suppose, to the contrary, that w is a vertex in $V_{\delta+1}$ with no neighbors in V_1'. Redefine $V_1 := (V_1 \backslash \{u\}) \cup \{v\}$ and $V_{\delta+1} := (V_{\delta+1} \backslash \{v\}) \cup \{u\}$. We then get an equitably m-coloring of G with color classes $V_1, \ldots, V_m$.

(b) We only need to prove that $e(v, V_j') \ge 1$ for any $v \in \bigcup_{i=\delta+1}^{m} V_i$. Suppose, to the contrary, that v is a vertex in $V_{\delta+1}$ with no neighbors in V_j'. We can get an equitably m-coloring of G by redefining $V_1 := (V_1 \backslash \{u\}) \cup \{w\}$, $V_j := (V_1 \backslash \{w\}) \cup \{v\}$ and $V_{\delta+1} := (V_{\delta+1} \backslash \{v\}) \cup \{u\}$.

(c) Similarly, we just need to prove that $e(v, V_k') \ge 1$ for any $v \in \bigcup_{i=\delta+1}^{m} V_i$. Suppose, to the contrary, that v is a vertex in $V_{\delta+1}$ with no neighbors in V_k'. We can again get an equitably m-coloring of G by redefining $V_1 := (V_1 \backslash \{u\}) \cup \{w\}$, $V_j := (V_1 \backslash \{w\}) \cup \{x\}$, $V_k := (V_k \backslash \{x\}) \cup \{v\}$ and $V_{\delta+1} := (V_{\delta+1} \backslash \{v\}) \cup \{u\}$.

(d) The inequalities (4) directly follow from (a).

(e) The inequalities (5) directly follow (b).

(f) This is trivial since $|\bigcup_{i=2}^{m} V_i| = (m-1)t$. □

Lemma 15 *Let $\mathcal{G}$ be a class of graphs with $\delta(\mathcal{G}) := \delta$ and let $G \in \mathcal{G}$ be an m-colorable, K_δ-free graph with order mt. Suppose that uv is an edge in G with $d(u) = \delta(G) \le \delta$. If G has no equitable m-colorings but $G - uv$ admits an equitable m-coloring having color classes $V_1, \ldots, V_m$, where $|V_i| = t$ for all $1 \le i \le m$, such that $u, v \in V_1$ and $N(u) \subseteq \bigcup_{i=1}^{\delta} V_i$, then the following results hold.*

(a) *If there exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \le j \le \delta$, then (4) holds and*

$$e(G) \ge \delta(m-\delta)t + 1, \tag{7}$$

where $V_j' = V_j \backslash \{v_j\}$ for each $1 \le j \le \delta$ and $v_1 = u$.

(b) *Let ρ be a fixed integer between 2 and $\delta - 1$. Suppose that $m \ge \Delta(G)$ and there exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \le j \le \rho$ and $e(v, V_1') \ge 1$ for any $v \in \bigcup_{i=\rho+1}^{\delta} V_i$. If there exists $v_j \in V_j$, for some $\rho + 1 \le j \le \delta$, such that $e(v_j, V_k') = 0$ for some $2 \le k \le \rho$, where $V_j' = V_j \backslash \{v_j\}$ for each $1 \le j \le \rho$ and $v_1 = u$, then (4) holds while $\rho = \delta - 1$, and at least one of the inequalities among*

(4) and the $\delta-\rho-1$ ones represented by (5) via fixing μ from $\rho+1$ to $\delta-1$ holds while $2 \le \rho \le \delta-2$. If $e(w, V_k') \ge 1$ for any $w \in \bigcup_{i=\rho+1}^{\delta} V_i$ and all $2 \le k \le \rho$, then (5) holds while $\mu = \rho$. Set $\xi = \delta - \rho + 1$ and $S_\xi = \bigcup_{i=\rho+1}^{m} V_i \cup \{u\}$ (note that $2 \le \xi \le \delta - 1$). We have

$$e(G) \ge \rho(m-\rho)t + 1, \tag{8}$$

$$t \ge \frac{m\rho + 1}{\rho^2} \tag{9}$$

and

$$|S_\xi| > \frac{(t-1)(m+\delta-2)}{2} \tag{10}$$

if and only if $\xi \ge \frac{3\delta - m}{2}$. Moreover, if (10) holds, then S_ξ contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Set

$$G_1^\xi = G\left[((V_1' \backslash \{\gamma\} \cup \{\alpha, \beta\})) \cup \bigcup_{i=2}^{\rho} V_i\right]$$

and

$$G_\xi = G[(S_\xi \backslash \{\alpha, \beta\} \cup \{\gamma\})].$$

We have

$$e(G_\xi) \le e(G) - (\delta - \xi + 1)mt + (\delta - \xi + 1)^2 t + m - 3. \tag{11}$$

(c) *Let $S_\delta = \bigcup_{i=2}^{m} V_i \cup \{u\}$. If $m \ge \Delta(G)$ and $e(w, V_1') \ge 1$ for any $w \in \bigcup_{i=2}^{\delta} V_i$, then*

$$t \ge m + 1 \tag{12}$$

and

$$|S_\delta| > \frac{(t-1)(m+\delta-2)}{2} \tag{13}$$

if and only if $\delta \le m$. Moreover, if (13) holds, then S_δ contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Set

$$G_1^\delta = G[(V_1' \backslash \{\gamma\} \cup \{\alpha, \beta\})]$$

and

$$G_\delta = G[(S_\delta \backslash \{\alpha, \beta\} \cup \{\gamma\})].$$

We have

$$e(G_\delta) \le e(G) - mt + t + m - 3. \tag{14}$$

Proof (a) This is trivial since $e(G) \geq e(\bigcup_{i=\delta+1}^{m} V_i \cup \{u\}, \bigcup_{j=1}^{\delta} V_j')$.

(b) Since there exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \leq j \leq \rho$, by Lemma 14(a) and Lemma 14(b) we have $e(\bigcup_{i=\delta+1}^{m} V_i, \bigcup_{j=1}^{\rho} V_j') \geq \rho(m-\delta)t$.

If $\rho = \delta - 1$ and there exists $v_\delta \in V_\delta$ such that $e(v_\delta, V_k') = 0$ for some $2 \leq k \leq \rho$, then by Lemma 14(c) we also have $e(\bigcup_{i=\delta+1}^{m} V_i, V_\delta') \geq (m-\delta)t$. Hence, $e(\bigcup_{i=\delta+1}^{m} V_i, \bigcup_{j=1}^{\delta} V_j') \geq (\rho+1)(m-\delta)t = \delta(m-\delta)t$ and (4) holds.

If $2 \leq \rho \leq \delta - 2$ and there exists $v_j \in V_j$, for some $\rho + 1 \leq j \leq \delta$, such that $e(v_j, V_k') = 0$ for some $2 \leq k \leq \rho$, then by Lemma 14(c) we have $e(\bigcup_{i=\delta+1}^{m} V_i, V_j') \geq (m-\delta)t$. If this inequality holds for all $\rho+1 \leq j \leq \delta$, then by a similar argument as above we can deduce that $e(\bigcup_{i=\delta+1}^{m} V_i, \bigcup_{j=1}^{\delta} V_j') \geq (\rho+\delta-\rho)(m-\delta)t = \delta(m-\delta)t$, and then, (4) holds. On the other hand, suppose, without loss of generality, that this inequality holds for all $\rho+1 \leq j \leq \mu \leq \delta-1$ but does not hold for any $\mu+1 \leq j \leq \delta$. By Lemma 14(c), $e(\bigcup_{i=\delta+1}^{m} V_i, \bigcup_{j=\rho+1}^{\mu} V_j') \geq (\mu-\rho)(m-\delta)t$ and $e(v_k, V_j') \geq 1$ for all $\mu+1 \leq k \leq \delta$ and $2 \leq j \leq \mu$ (otherwise there exists $\mu+1 \leq k \leq \delta$ such that $e(\bigcup_{i=\delta+1}^{m} V_i, V_k') \geq (m-\delta)t$, a contradiction to our assumption). Hence, $e(\bigcup_{k=\mu+1}^{\delta} V_k, \bigcup_{j=2}^{\mu} V_j') \geq (\delta-\mu)(\mu-1)t$. Finally, note that $e(\bigcup_{k=\mu+1}^{\delta} V_k, V_1') \geq (\delta-\mu)t$ by the assumption of (b). Combine all these arguments, one can easily deduce that $e(\bigcup_{i=\mu+1}^{m} V_i, \bigcup_{j=1}^{\mu} V_j') \geq (\delta-\mu)(\mu-1)t + (\delta-\mu)t + \rho(m-\delta)t + (\mu-\rho)(m-\delta)t = \mu(m-\mu)t$. Hence, at least one of the $\delta-\rho-1$ inequalities represented by (5) via fixing μ from $\rho+1$ to $\delta-1$ holds.

If $e(w, V_k') \geq 1$ for any $w \in \bigcup_{i=\rho+1}^{\delta} V_i$ and all $2 \leq k \leq \rho$, then the inequality (5) holds while $\mu = \rho$ by Lemma 14(e), so we have $\rho(m-\rho)t + 1 \leq e(S_\xi, \bigcup_{j=1}^{\rho} V_j') \leq \rho(t-1)\Delta(G) \leq m\rho(t-1)$. This implies (8) and (9). Since $|S_\xi| = (m-\rho)t + 1$, the following four properties are equivalent:

1. $|S_\xi| > \frac{(t-1)(m+\delta-2)}{2}$;
2. $mt - (\delta+2\rho-2)t + m + \delta > 0$ for all t;
3. $m \geq \delta + 2\rho - 2$;
4. $\xi \geq \frac{3\delta-m}{2}$.

Hence, (10) holds if and only if $\xi \geq (3\delta-m)/2$. Therefore, by Lemma 12, S_ξ with $\xi \geq (3\delta-m)/2$ contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Recall the definitions of G_1^ξ and G_ξ in this lemma. One can deduce that $e(G_\xi) \leq e(G[S_\xi]) + m - 2 \leq e(G) - e(S_\xi, \bigcup_{j=1}^{\rho} V_j') + m - 2 \leq e(G) - [\rho(m-\rho)t + 1] + m - 2 = e(G) - \rho m t + \rho^2 t + m - 3$. Hence, (11) holds.

(c) This proof is just similar to the one in (b), thus we leave it to the readers. □

3 The Proof of Theorem 4

This section is devoted to proving the following general theorem in this paper. Whereafter, Theorem 4 would be a direct corollary of Lemmas 8–10 and Theorem 16(4).

Theorem 16 *Let $\mathcal{G}$ be a class of graphs with $\delta(\mathcal{G}) = 7$ and let $G \in \mathcal{G}$ be a K_7-free and 7-degenerate graph with order mt and maximum degree Δ. If*

1. $e(G) \leq \max\{mt + m + 12t - 8, mt + 2m + 10t - 10, mt + 3m + 9t - 11\}$ *and* $\Delta \geq 14$, *or*

2. $e(G) \leq \max\{2mt + m + 8t - 9, 2mt + 2m + 7t - 11\}$ *and* $\Delta \geq 15$*, or*
3. $e(G) \leq 3mt + m + 4t - 10$ *and* $\Delta \geq 16$*, or*
4. $e(G) \leq 4mt - 8$ *and* $\Delta \geq 17$*,*

then G *is equitably* m*-colorable for every* $m \geq \Delta$*.*

Before proving Theorem 16, it is better to make everyone clear with the strategy we used in this proof. Indeed, we want to prove the equitable chromatic threshold of every K_7-free and 7-degenerate graph G with order mt and maximum degree Δ is at most Δ if we are given a function $f(m,t)$ and a constant C such that $e(G) \leq f(m,t)$ and $\Delta \geq C$. We proceed by induction on the size of G. Since G is 7-degenerate, G has an edge uv with $d(u) = \delta(G) \leq 7$. By induction hypothesis, $G - uv$ admits an equitable m-coloring having color classes $V_1, \ldots, V_m$, where $|V_i| = t$ for all $1 \leq i \leq m$, such that $u, v \in V_1$ and $N(u) \subseteq \bigcup_{i=1}^{\delta} V_i$. Now we aim to extend the coloring of $G - uv$ to G. In the following, we assume that this extension is impossible. Since every 7-degenerate graph can be 8-colorable, Lemma 15 is valid during our proof of Theorem 16. Notations such as ξ and G_ξ are followed from Lemma 15.

During each proof of the four results in Theorem 16, one can see that at least one of the seven cases would occur.

Case 1: There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \leq j \leq 7$.
Case ξ with $2 \leq \xi \leq 6$: There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \leq j \leq 8 - \xi$ and $e(v, V_1') \geq 1$ for any $v \in \bigcup_{i=9-\xi}^{7} V_i$.
Case 7: For any $w \in \bigcup_{i=2}^{7} V_i$, $e(w, V_1') \geq 1$.

We then split Case ξ with $2 \leq \xi \leq 6$ into two subcases.

Subcase ξ(1): There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \leq j \leq 8 - \xi$ and $e(v, V_1') \geq 1$ for any $v \in \bigcup_{i=9-\xi}^{7} V_i$. Meanwhile, there exists $v_j \in V_j$ for some $9 - \xi \leq j \leq 7$ such that $e(v_j, V_k') = 0$ for some $2 \leq k \leq 8 - \xi$.
Subcase ξ(2): There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $2 \leq j \leq 8 - \xi$ and $e(v, V_k') \geq 1$ for any $v \in \bigcup_{i=9-\xi}^{7} V_i$ and $1 \leq k \leq 8 - \xi$.

Our effort is to show orderly that if some of the seven cases occurs, then we can extend the coloring of $G - uv$ to G. By Lemma 15, we have the following observation.

Observation 1 *If Case 1 occurs, then the inequality (4) holds. For a fixed* $2 \leq \xi \leq 6$*, if Subcase* $\xi(1)$ *occurs, then the inequality (4) holds while* $\xi = 2$*, and at least one of the inequalities among (4) and the* $\xi - 2$ *ones represented by (5) via fixing* μ *from* $9 - \xi$ *to 6 holds while* $3 \leq \xi \leq 6$*; if Subcase* $\xi(2)$ *occurs, then the inequality represented by (5) via fixing* $\mu = 8 - \xi$ *holds.*

Our strategy is to consider first Case 1 and show that if the inequality (4) holds then we can obtain a contradiction. We then consider Case 2 [and thus consider Subcase 2(1) and Subcase 2(2) separately]. By Observation 1, if Subcase 2(1) occurs, then (4) holds, which has been proved false, so we just need to consider Subcase 2(2) and prove that if the inequality represented by (5) via fixing $\mu = 6$ holds then we can also obtain a contradiction. Afterward, we consider Cases 3–7 one by one similarly. Indeed, while considering Case ξ with $2 \leq \xi \leq 6$, we only need to investigate Subcase $\xi(2)$

carefully, since if Subcase $\xi(1)$ occurs, then any of the $\xi - 1$ inequalities as described in Observation 1 that may hold has already been proved false while considering Cases from 1 to $\xi - 1$.

As one can see in the following detailed proof of Theorem 16, when trying to prove the i-th with $2 \le i \le 4$ part of this theorem, we need to use the first $i - 1$ parts as inductive Lemmas. However, the proof of any one part is highly similar to another, only with minor differences in the computations. Hence, we only show the proof of Theorem 16(4) in detail below.

Proof of Theorem 16 As we have mentioned above, we only prove (4) here. Recall the strategy described as above, we consider Cases 1–7 orderly.

If Case 1 occurs, then by Lemma 15(a) we have $4mt - 8 \ge e(G) \ge 7(m-7)t + 1$, that is false for $m \ge 17$.

If Case ξ with $2 \le \xi \le 6$ occurs, then by the above arguments we only need consider Subcase $\xi(2)$. By Lemma 15(b), $t \ge 3$ and the inequality (10) holds since $\xi \ge 2 \ge (21-m)/2$; thus, the two graphs G_1^{ξ} and G_{ξ} in Lemma 15(b) are well defined and can be used here. Moreover, G_1^{ξ} is a graph with order $(8-\xi)t$ that is equitably $8-\xi$ colorable by its definition. By (11), we deduce that $e(G_{\xi}) \le (\xi - 4)mt + (8-\xi)^2 t + m - 11$. Note that $V(G) = V(G_1^{\xi}) \cup V(G_{\xi})$ and $|G_{\xi}| = (m + \xi - 8)t$. We just need to prove that G_{ξ} is equitably $m + \xi - 8$ colorable.

If Case 7 occurs, then by Lemma 15(c), the inequality (13) holds since $m \ge 17 > 7$. Hence, we only need to prove that the graph G_7 as defined in Lemma 15(c) is equitably $m - 1$ colorable, where $e(G_7) \le 3mt + t + m - 11$ by (14).

Now we shall involve Lemmas 11 and 13 as technical tools. The following many formulas are dedicated to some computations.

$$e(G_2) \le -2mt + 36t + m - 11 \le [(m-6) - 1]t \quad \text{for } m - 6 \ge 11 \text{ and } t \ge 1$$
$$e(G_3) \le -mt + 25t + m - 11 \le [(m-5) - 1]t \quad \text{for } m - 5 \ge 12 \text{ and } t \ge 2$$
$$e(G_4) \le 16t + m - 11 \le [2(m-4) - 3]t - \max\{\Delta - 3, t\} \quad \text{for } m - 4 \ge 13 \text{ and } t \ge 3$$
$$e(G_5) \le mt + 9t + m - 11 = (m-3)t + (m-3) + 12t - 8 \quad \text{for } m - 3 \ge 14$$
$$e(G_6) \le 2mt + 4t + m - 11 = 2(m-2)t + (m-2) + 8t - 9 \quad \text{for } m - 2 \ge 15$$
$$e(G_7) \le 3mt + t + m - 11 = 3(m-1)t + (m-1) + 4t - 10 \quad \text{for } m - 1 \ge 16$$

By Lemma 11, G_2 and G_3 are, respectively, equitably $m - 6$ and $m - 5$ colorable; by Lemma 13, G_4 is equitably $m - 4$ colorable; by Theorem 16(1), G_5 is equitably $m - 3$ colorable; by Theorem 16(2), G_6 is equitably $m - 2$ colorable; and by Theorem 16(3), G_7 is equitably $m - 1$ colorable. Hence, we complete the proof of Theorem 16(4). □

4 Open Problems

Note that all of the Lemmas in Sect. 2 are valid for any graph $G \in \mathcal{G}_k$ with $k \ge 3$. We believe that Theorems 3 and 4 can be extended to a general version for every $k \ge 3$, although this extension is still a partial case of the Equitable Δ-Coloring Conjecture.

Conjecture 17 *If $G \in \mathcal{G}_k$ with $k \geq 3$ is a graph with maximum degree $\Delta \geq \frac{(2k-1)^2}{k-1}$, then G is equitably m-colorable for every $m \geq \Delta$.*

On the other hand, Kostochka, Nakprasit and Pemmaraju [14] proved that if G is a d-degenerate graph with $|V(G)| \geq 15\Delta(G)$, then G is equitably m-colorable for every $m \geq 16d$. Lemma 9 tells us that every graph in $\mathcal{G}_k$ is $(2k-1)$-degenerate, so the above result of Kostochka et al. implies the following theorem.

Theorem 18 *If $G \in \mathcal{G}_k$ with $k \geq 3$ is a graph with $|V(G)| \geq 15\Delta(G)$, then G is equitably m-colorable for every $m \geq 32k - 16$.*

Although Theorem 18 is weaker in some sense, it gives us a new and interesting direction for further research. After looking at Theorem 18, one can find that m is not really relative to $\Delta(G)$, that is to say, for large $\Delta(G)$ (larger than $32k - 16$) one may equitably color each graph G in $\mathcal{G}_k$ by fewer colors. We end this paper by an open problem.

Problem 19 *Fix an integer $\Delta_0(k, m)$ such that every graph $G \in \mathcal{G}_k$ with maximum degree at least Δ_0 is equitably m-colorable, where k is a given integer and m is a fixed integer less than $\Delta(G)$.*

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