J Comb Optim (2016) 32:188–197
DOI 10.1007/s10878-015-9870-1

List edge and list total coloring of planar graphs with maximum degree 8

Huijuan Wang[1] · Bin Liu[2] · Xin Zhang[3] · Lidong Wu[4] · Weili Wu[5,6] · Hongwei Gao[1]

Published online: 2 April 2015

Abstract Let G be a planar graph with maximum degree $\Delta \geq 8$ and without chordal 5-cycles. Then $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$.

Keywords Choosability · Planar graph · List edge coloring · List total coloring

✉ Huijuan Wang
sduwhj@163.com

Bin Liu
sduliubin@163.com

Xin Zhang
xzhang@xidian.edu.cn

Lidong Wu
wu-ld@hotmail.com

Weili Wu
weiliwu@utdallas.edu

Hongwei Gao
gaohongwei@qdu.edu.cn

1 College of Mathematics, Qingdao University, Qingdao 266071, China

2 Department of Mathematics, Ocean University of China, Qingdao 266100, China

3 School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

4 Department of Computer Science, University of Texas at Tyler, Tyler, TX 75799, USA

5 College of Computer Science and Technology, TaiYuan University of Technology, Taiyuan 030024, China

6 Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA

1 Introduction

The problem of minimum number of choosability of graphs appears in some practical problems when concerning frequency assignment. List edge coloring and list total coloring are two important list colorings. In this paper, we study these two coloring problems on planar graph. Here are some other interesting colorings, we refer the readers to Angelini and Frati (2012); Du et al. (2004); Garg et al. (1996); Li et al. (2013); Wang et al. (2014a, c) etc.

Firstly, all graphs considered in this paper are simple. Let G be a planar graph and has embedded in the plane. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$(or simply V, E, Δ and δ) to denote the vertex set, the edge set, the face set, the maximum degree and the minimum degree of G, respectively. A *k-total-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. If the choice of a color for each vertex or each edge, from its list of allowed colors, then this is a list total coloring. Strictly, a mapping L is called to be a *total assignment* for a graph G if it assigns a list $L(x)$ of possible colors for each element $x \in V \cup E$. If G has a total coloring φ such that $\varphi(x) \in L(x)$ for all $x \in V \cup E$, and no two adjacent or incident elements receive the same color, then we say that φ is a *total-L-coloring* of G or G is *total-L-colorable*. A graph G is called *total-k-choosable* if it is total-L-colorable for every total assignment L satisfying $|L(x)| \geq k$ for each element $x \in V \cup E$. The *list total chromatic number* $\chi_l''(G)$ of G is the smallest integer k such that G is total-k-choosable. Similarly, the *list edge chromatic number* $\chi_l'(G)$ of G can be defined in terms of coloring the edges alone. The ordinary edge chromatic number of G is denoted by $\chi'(G)$. Obviously, it holds that $\chi_l'(G) \geq \chi'(G) \geq \Delta$ and $\chi_l''(G) \geq \chi''(G) \geq \Delta + 1$.

For a long time, list edge coloring and list total coloring are widely studied, quite a few interesting results have been obtained in recent years. Firstly, we introduce a famous conjecture as follows.

Conjecture 1 (List coloring conjecture) *For any graph G,*

(i) $\chi_l'(G) = \chi'(G)$*;*
(ii) $\chi_l''(G) = \chi''(G)$.

Part (i) of Conjecture 1 is well known as the *list edge coloring conjecture* and it was formulated independently by Vizing, Gupta, Albertson and Collins, Bollobás, Harris (see Hägkvist and Chetwynd (1992) or Jensen and Toft (1995)). Part (ii) is well known as the *list total coloring conjecture* and it was formulated by Borodin et al. (1997). Although this conjecture has been proved for a few special cases, for example, outerplanar graphs, planar graphs with $\Delta \geq 12$ Borodin et al. (1997), list coloring conjecture still opens. Moreover, by Vizing's Theorem and total coloring conjecture, the later is for any graph G, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$, there is another natural but interesting conjecture as follows.

Conjecture 2 *For any graph G,*

(i) $\chi_l'(G) \leq \Delta + 1$*;*
(ii) $\chi_l''(G) \leq \Delta + 2$.

There are some results about this conjecture. Interestingly, the list edge chromatic number and the list total chromatic number of planar graphs with large maximum degree equals a lower bound. Hou et al. (2006) proved $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta+1$ for planar graphs with $\Delta \geq 8$ and without 4-cycles. Li and Xu (2011) proved this result for planar graphs with $\Delta \geq 8$ and no 3-cycles adjacent to 4-cycles. Liu et al. (2009) proved $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$ holds for planar graphs with $\Delta \geq 8$ and without intersecting 4-cycles. In our another paper (Wang et al. 2014b), we strength this result for planar graphs with $\Delta \geq 8$ and without chordal 5-cycles. Some other results about list coloring the readers can see Borodin et al. (1997); Hägkvist and Chetwynd (1992); Wang and Liu (2005). In this paper, we also consider the planar graph with $\Delta \geq 8$ and get the following theorem.

Theorem 1 *Suppose G is a planar graph without chordal 5-cycles. If $\Delta \geq 8$, then $\chi_l'(G) = \Delta$ and $\chi_l''(G) = \Delta + 1$.*

2 Reducible configurations

Now, we introduce some more notations here. For a vertex v of G, the *degree* $d(v)$ denote the number of edges incident with v, and for a face f of G, the *degree* $d(f)$ denote the length of the boundary walk of f. Denote by *k-vertex* the vertex of degree k, by *k^--vertex* the vertex of degree at most k, by *k^+-vertex* the vertex of degree at least k. If two cycles share at least one edge, we call them *adjacent*. We use $n_k(v)$, $f_k(v)$ and $n_k(f)$ to denote the number of k-vertices adjacent to the vertex v, the number of k-faces incident with the vertex v, and the number of k-vertices incident with the face f, respectively. In Borodin et al. (1997), Theorem 1 was proved for $\Delta \geq 12$. Henceforth, to prove Theorem 1, it suffices to prove the result as follows.

- Let r be a positive integer and $8 \leq r \leq 11$. If G is a planar graph with $8 \leq \Delta \leq r$ and without chordal 5-cycles, then $\chi_l'(G) \leq \max\{8, r\}$ and $\chi_l''(G) \leq \max\{9, r + 1\}$.

Let an r-minimal graph be a connected graph $G = (V, E, F)$ with $|V| + |E|$ as small as possible. If G is not total-($\max\{9, r + 1\}$)-choosable, then there is a total assignment L for G with $|L(x)| = \max\{9, r + 1\}$ for any $x \in V \cup E$, such that G is not total-L-colorable. If G is not edge-$\max\{8, r\}$-choosable, then there is an edge assignment L for G with $|L(e)| = \max\{8, r\}$ for any edge $e \in E$ such that G is not edge-L-colorable.

By the minimality of G, we first show some known properties (see Borodin et al. (1997)).

(a) G is a connected graph;
(b) G contains no 2-alternating cycle;
(c) G contains no edge uv with $\min\{d(u), d(v)\} \leq \lfloor \frac{\max\{8,r\}}{2} \rfloor$ and $d(u) + d(v) \leq \max\{9, r + 1\}$.

Since G has properties (b) and (c), and G has no chordal 5-cycles, we can get the following observations easily:

(O_1). Let f be a face of G. Then f is not incident with too many 3^--vertices. By (b) and (c), the number of 3^--vertices incident with f is at most $\lfloor \frac{d(f)}{2} \rfloor$. Moreover, if f is incident with exactly $\frac{d(f)}{2}$ 3^--vertices (note that $d(f)$ is even), then f is incident with at least one 3-vertex.
(O_2). Each 5^+-vertex v is incident with at most $\lceil \frac{d(v)+1}{2} \rceil$ 3-faces, since G has no chordal 5-cycles.

Let G_2 be the induced subgraph by the edges incident with the 2-vertices of G. By (b) and (c), we have G_2 is a forest and the vertices of G_2 are all 2-vertices and $\max\{8, r\}$-vertices. We root G_2 at a $\max\{8, r\}$-vertex, then every 2-vertex has exactly one parent and exactly one child, which are all $\max\{8, r\}$-vertices. Moreover, if a $\max\{8, r\}$-vertex is adjacent to at least two 2-vertices, this $\max\{8, r\}$-vertex may be the child of exactly one 2-vertex and the parent of the remaining 2-vertices.

3 Discharging

Discharging method is an important and interesting tool during the proof of the coloring problems of planar graphs. In this paper, we shall use discharging method to complete the proof of above result.

Case 1 $r = 8$.

We define $c(x)$ to be the initial charge. Let $c(x) = d(x) - 4$ for each $x \in V \cup F$. By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -8 < 0$$

So $\sum_{x \in V \cup F} c(x) = -8 < 0$. Then we apply the following rules to redistribute the initial charge that leads to a new charge $c'(x)$.

(R1) From each face f to each of its incident vertices v, transfer
 (R1-1) $\frac{1}{2}$, if $d(f) = 5$ and $d(v) = 2$;
 (R1-2) $\frac{7}{12}$, if $d(f) = 6$, $d(v) = 2$ and v is incident with no 3-cycle;
 (R1-3) $\frac{5}{6}$, if $d(f) = 6$, $d(v) = 2$ and v is incident with one 3-cycle;
 (R1-4) 1, if $d(f) \geq 7$ and $d(v) = 2$;
 (R1-5) $\frac{1}{2}$, if $d(f) \geq 5$ and $d(v) = 3$;
 (R1-6) $\frac{1}{8}$, if $d(f) \geq 5$, $d(v) = 8$ and $n_{3^-}(f) < \lfloor \frac{d(f)}{2} \rfloor$.
(R2) From each vertex v to each of its incident 3-faces, transfer
 (R2-1) $\frac{1}{3}$, if $d(v) = 5$;
 (R2-2) $\frac{1}{2}$, if $d(v) \geq 6$.
(R3) From each 7^+-vertex v to each of its adjacent 3-vertices u, transfer
 (R3-1) $\frac{1}{3}$, if $f_{5^+}(u) = 0$ or Fig. 1(1), but except Fig. 1(2);
 (R3-2) $\frac{1}{6}$, if $f_{5^+}(u) = 1$ but except Fig. 1(1, 2);
 (R3-3) $\frac{1}{12}$, if u is incident with two 3-cycles (see Fig. 1(2));
 (R3-4) 0, if $f_{5^+}(v) \geq 2$.
(R4) From each 8^+-vertex v to its parent u, transfer

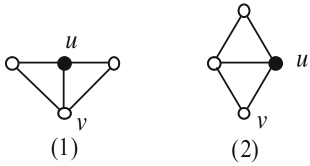

Fig. 1 The figures of discharging rules

(R4-1) $\frac{5}{3}$, if $f_3(u) = 1$ and $f_4(u) = 1$;
(R4-2) $\frac{5}{3}$, if $f_4(u) = 1$ and $f_4(u) = 1$;
(R4-3) $\frac{4}{3}$, if $f_3(u) = 1$ and $f_5(u) = 1$;
(R4-4) $\frac{7}{6}$, if $f_4(u) = 1$ and $f_5(u) = 1$;
(R4-5) $\frac{5}{4}$, if $f_{4^-}(u) = 1$, $f_6(u) = 1$ and u is incident with no 3-cycle;
(R4-6) 1, if $f_{4^-}(u) = 1$, $f_6(u) = 1$ and u is incident with one 3-cycle;
(R4-7) 1, if $f_{4^-}(u) = 1$ and $f_{7^+}(u) = 1$;
(R4-8) 1, if $f_{5^+}(u) = 2$.

(R5) From each 8^+-vertex v to each of its children u, transfer
(R5-1) $\frac{1}{3}$, if $f_3(u) = 1$ and $f_4(u) = 1$;
(R5-2) $\frac{1}{3}$, if $f_4(u) = 2$;
(R5-3) $\frac{1}{3}$, if $f_4(u) = 1$ and $f_5(u) = 1$;
(R5-4) $\frac{1}{6}$, if $f_3(u) = 1$ and $f_5(u) = 1$;
(R5-5) $\frac{1}{6}$, if $f_3(u) = 1$ and $f_6(u) = 1$.

In the following, we shall check $c'(x) \geq 0$ for all $x \in V \cup F$ which will be the desired contradiction.

Final charge of faces. Let $f \in F$. Suppose $d(f) = 3$. Then $c(f) = d(f) - 4 = -1$. If $n_{4^-}(f) = 1$, then $n_{6^+}(f) = 2$ by (c), and $c'(f) = -1 + \frac{1}{2} \times 2 = 0$ by (R2). Otherwise, $n_{5^+}(f) = 3$ and $c'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by (R2). Suppose $d(f) = 4$. Then f does not send out any charge and $c'(f) = c(f) = 0$. Recall that, $n_{3^-}(f) \leq \lfloor \frac{d(f)}{2} \rfloor$ by (O_1). When $d(f) \geq 5$, we consider two cases, $n_{3^-}(f) = \lfloor \frac{d(f)}{2} \rfloor$ and $n_{3^-}(f) < \lfloor \frac{d(f)}{2} \rfloor$, since just in the latter case f sends charge to its incident 8-vertices by (R1-6). Suppose $d(f) = 5$. Then $c(f) = 1$ and $c'(f) \geq 1 - \max\{\frac{1}{2} \times 2, \frac{1}{2} + \frac{1}{8} \times 4, \frac{1}{8} \times 5\} = 0$ by (R1). Suppose $d(f) = 6$. Then $c(f) = 2$. Note that, if $n_{3^-}(f) = \lfloor \frac{d(f)}{2} \rfloor = 3$, then $n_3(f) \geq 1$ by (O_1). And if $n_2(f) = 2$, then there is at most one 2-vertex which is incident with a 3-face, because G has no chordal 5-cycles. Therefore, $c'(f) \geq 2 - \max\{\frac{5}{6} + \frac{7}{12} + \frac{1}{2}, \frac{5}{6} + \frac{7}{12} + \frac{1}{8} \times 4\} = \frac{1}{12}$ by (R1). Suppose $d(f) \geq 7$. If $n_{3^-}(f) = \lfloor \frac{d(f)}{2} \rfloor$, then $c'(f) = c(f) - 1 \times n_2(f) - \frac{1}{2} \times n_3(f) \geq (d(f) - 4) - 1 \times \lfloor \frac{d(f)}{2} \rfloor \geq 0$. Otherwise, $n_{3^-}(f) < \lfloor \frac{d(f)}{2} \rfloor$. And therefore, $c'(f) = c(f) - 1 \times n_2(f) - \frac{1}{2} \times n_3(f) - \frac{1}{8} \times n_8(f) \geq (d(f) - 4) - 1 \times (\lfloor \frac{d(f)}{2} \rfloor - 1) - \frac{1}{8} \times (d(f) - \lfloor \frac{d(f)}{2} \rfloor + 1) \geq 0$ by (R1).

Let v be a 2-vertex of G. Suppose that u, w are the two vertices adjacent to v, and that v is the child of w. We call v a *bad parent* of u if $f_3(u) = 1$ and $f_4(u) = 1$, or $f_4(u) = 1$ and $f_4(u) = 1$. Moreover, we call the former parent *A-parent* and the later parent *B-parent* (see Fig. 21, 2). We call v a *bad child* of w if w needs send charge to v by (R5). Similarly, we call u in (R5-1) *A-child*, in (R5-2) *B-child*, in (R5-3) *C-child*,

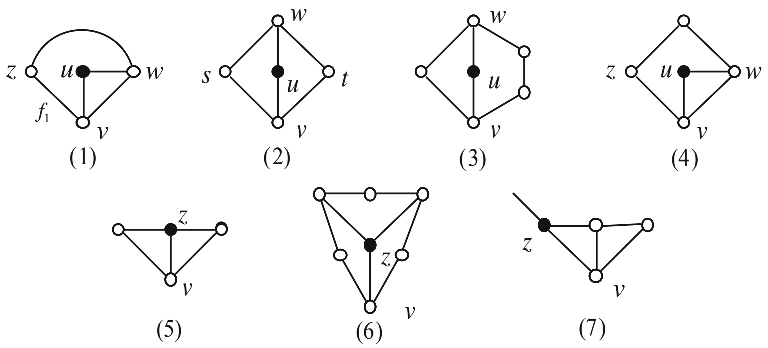

Fig. 2 The figures of (O_3) and (O_4)

in (R5-4) *D-child* (see Fig. 21–4), in (R5-5) *E-child* . Then each 8-vertex sends $\frac{5}{3}$ to each of its bad parents, and sends $\frac{1}{3}$ to each of A,B,C-children, and sends $\frac{1}{6}$ to each of D,E-children. We call 3-vertex v is *bad* of u if it needs receive charge from u, and is *good* otherwise. Moreover, a 3-vertex u is called *A-vertex* of v if Fig. 2(5, 6), and is *B-vertex* of v if v sends $\frac{1}{6}$ to u, and is *C-vertex* of v if v sends $\frac{1}{12}$ to u. So each 7^+-vertex transfers charge only to its bad 3-vertices and sends at most $\frac{1}{3}$ to each of its adjacent 3-vertices by (R3). Moreover, we have the following observations:

(O_3). Let u be a 2-vertex in Fig. 2(1). Then $d(f_1) \geq 5$. Moreover, if $d(z) = 3$, then z is a good 3-vertex of v.
(O_4). Let z be a 3-vertex in Fig. 2(7). Then z is a good 3-vertex of v.

Moreover, we have the other following observations:

(O_5). Let v be a 8^+-vertex. If v is adjacent to no parent, then v sends at most $\frac{1}{3}$ to each of its adjacent 2-vertices.
(O_6). Let v be a 8^+-vertex. If v is adjacent a B-parent (see Fig. 2(2)), then the vertices s and t are not 2-vertices and v sends at most $(\frac{5}{3} + \frac{1}{6} \times 2) = 2$ to u, s and t.

Final charge of vertices. Let $v \in V$. Note that G has no vertex of degree one. Suppose $d(v) = 2$. Then $c(v) = d(v) - 4 = -2$ and $n_8(v) = 2$ by (c). First, suppose $f_{4^-}(v) = 2$. Then $f_3(v) = 1$ and $f_4(v) = 1$ or $f_4(v) = 2$ since G has no chordal 5-cycles. So $c'(v) \geq -2 + \frac{5}{3} + \frac{1}{3} = 0$. Second, suppose $f_{4^-}(v) = 1$. If $f_5(v) = 1$, then $c'(v) = -2 + \min\{\frac{7}{6} + \frac{1}{3} + \frac{1}{2}, \frac{4}{3} + \frac{1}{6} + \frac{1}{2}\} = 0$. If $f_6(v) = 1$, then $c'(v) = -2 + \min\{\frac{7}{12} + \frac{5}{4} + \frac{1}{6}, \frac{5}{6} + 1 + \frac{1}{6}\} = 0$ by (R1-2,3),(R4-5,6) and (R5-5). If $f_{7^+}(v) = 1$, then $c'(v) = -2 + 1 + 1 = 0$ by (R1-4) and (R4-7). Third, suppose $f_{5^+}(v) = 2$. Then $c'(v) \geq -2 + \frac{1}{2} \times 2 + 1 = 0$ by (R1-1) and (R4-8).

Suppose $d(v) = 3$. Then $c(v) = -1$, and $n_{7^+}(v) = 3$ by (c). If $f_{5^+}(v) = 0$, then by (R3-1), $c'(v) \geq -1 + \frac{1}{3} \times 3 = 0$. If $f_{5^+}(v) = 1$, then we can clearly see $c'(v) = -1 + \min\{\frac{1}{2} + \frac{1}{6} \times 3, \frac{1}{2} + \frac{1}{3} + \frac{1}{12} \times 2\} = 0$. If $f_{5^+}(v) \geq 2$, then $c'(v) \geq -1 + \frac{1}{2} \times 2 = 0$ by (R1-5). Suppose $d(v) = 4$. Then clearly $c'(v) = c(v) = 0$. Suppose $d(v) = 5$. Then by (c), $c(v) = 1$, and $n_{4^-}(v) = 0$. Note that by our discharging rules, v sends charge only to its incident 3-faces. By (O_2), $f_3(v) \leq 3$.

Hence, $c'(v) \geq 1 - \frac{1}{3} \times 3 = 0$ by (R2). Suppose $d(v) = 6$. This case is similar to that of $d(v) = 5$. We have $c(v) = 2, n_{3^-}(v) = 0$ by (c), and $f_3(v) \leq 4$ by (O_2). So $c'(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by (R2). Suppose $d(v) = 7$. Then $c(v) = 3, n_2(v) = 0$ by (c), and $f_3(v) \leq 4$. If $f_3(v) = 4$, then v is adjacent to at most three bad 3-vertices, and $c'(v) \geq 3 - \frac{1}{2} \times 4 - \frac{1}{3} \times 3 = 0$. If $f_3(v) = 3$, then v is adjacent to at most four bad 3-vertices, and $c'(v) \geq 3 - \frac{1}{2} \times 3 - \frac{1}{3} \times 4 = \frac{1}{6} > 0$. If $f_3(v) \leq 2$, then $c'(v) > 3 - \frac{1}{2} \times 2 - \frac{1}{3} \times 5 = \frac{1}{3} > 0$.

Suppose $d(v) = 8$. Then by (O_2), $c(v) = 4$ and $f_3(v) \leq 5$. In the following, we will consider the parent of v.

Firstly, v is adjacent to no parent. Then v sends at most $\frac{1}{3}$ to each of its adjacent children and 3-vertices, and sends at most $\frac{1}{2}$ to each of its incident 3-faces. Suppose v is incident with five 3-faces. Then v is adjacent to at most three children and bad 3-vertices, so $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$. Suppose v is incident with four 3-faces. Then v sends at most $\frac{1}{3} \times 4$ to its adjacent vertices. So $c'(v) \geq 4 - \frac{1}{2} \times 4 - \frac{1}{3} \times 4 = \frac{2}{3} > 0$. Suppose v is incident with three 3-faces. Then v is adjacent to at most five children and bad 3-vertices, so $c'(v) \geq 4 - \frac{1}{2} \times 3 - \frac{1}{3} \times 5 = \frac{5}{6} > 0$. Suppose v is incident with at most two 3-faces. Then $c'(v) \geq 4 - \frac{1}{2} \times 2 - \frac{1}{3} \times 8 = \frac{1}{3} > 0$.

Secondly, v is adjacent to one parent u and u is not a bad parent. Then v sends at most $\frac{4}{3}$ to u and v is adjacent to at most two A-children. In the following, we will consider the number of A-children.

Suppose v is adjacent to two A-children. Then v is incident with at least two 3-faces and at most three 3-faces. If v is incident with three 3-faces, then v is adjacent to no bad 3-vertex, so $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 3 - \frac{1}{3} \times 2 = \frac{1}{2} > 0$. If v is incident with two 3-faces, then v is adjacent to at most one bad 3-vertex, so $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 2 - \frac{1}{3} \times 2 - \frac{1}{3} = \frac{2}{3} > 0$.

Suppose v is adjacent to one A-child. Then v is incident with at least one 3-face and at most four 3-faces. If v is incident with four 3-faces, then v is adjacent to no B-child and C-child. Moreover, v is adjacent to at most one bad 3-vertex and D,E-child. But v only needs send $\frac{1}{6}$ to D,E-child. So $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{3} = 0$. If v is incident with three 3-faces, then v is adjacent to at most two bad 3-vertices and B,C,D,E-children, so $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 3 - \frac{1}{3} - \frac{1}{3} \times 2 = \frac{1}{6} > 0$. If v is incident with two 3-faces, then v is adjacent to at most three bad 3-vertices and B,C,D,E-children, so $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 2 - \frac{1}{3} - \frac{1}{3} \times 3 = \frac{1}{3} > 0$. If v is incident with one 3-face, then v is adjacent to at most four bad 3-vertices and B,C,D,E-children, so $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{3} \times 4 = \frac{1}{2} > 0$.

Suppose v is adjacent to no A-child. Then v is incident with at most five 3-faces. Suppose v is incident with five 3-faces. Then v is adjacent to at most two bad 3-vertices and adjacent to no B-child and C-child. Moreover, u, the parent of v, is incident with a 6^+-face, since G has no chordal 5-cycle. So v sends at most 1 to u by (R4). If v is adjacent to no A-vertex, then $c'(v) \geq 4 - 1 - \frac{1}{2} \times 5 - \frac{1}{6} \times 2 = \frac{1}{6} > 0$. If v is adjacent to at most one A-vertex, then $c'(v) \geq 4 - 1 - \frac{1}{2} \times 5 - \frac{1}{3} - \frac{1}{6} = 0$. If v is adjacent to two A-vertices, then v is incident with two 5^+-faces, each of which will send $\frac{1}{8}$ to v. So $c'(v) \geq 4 - 1 - \frac{1}{2} \times 5 - \frac{1}{3} \times 2 + \frac{1}{8} \times 2 = \frac{1}{12} > 0$. Suppose v is incident with four 3-faces. Then v is adjacent to at most one C-child and no B-child. Suppose v is adjacent to one C-child. Then v is adjacent to at most one A-vertex and u is incident with one 6^+-face.

So $c'(v) \geq 4 - 1 - \frac{1}{2} \times 4 - \frac{1}{3} \times 2 - \frac{1}{6} = \frac{1}{6} > 0$. Suppose v is adjacent to no C-child. Then v is adjacent to at most two A-vertices. If v is adjacent to at most one A-vertex, then $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{6} \times 2 = 0$. If v is adjacent to two A-vertices, then u receives at most $\frac{7}{6}$ from v. So, $c'(v) \geq 4 - \frac{7}{6} - \frac{1}{2} \times 4 - \frac{1}{3} \times 2 - \frac{1}{6} = 0$. Suppose v is incident with three 3-faces. Then v is adjacent to at most one B-child. In the following, we will consider the number of B-children. If v is adjacent to one B-child, then v is adjacent to at most one A-vertex. So $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 3 - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 = \frac{1}{6} > 0$. If v is adjacent to no B-child, then we will consider the number of C-children. We see the number of C-children and bad 3-vertices adjacent to v is at most three. So we have $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 3 - \frac{1}{3} \times 3 - \frac{1}{6} = \frac{1}{6} > 0$. Suppose v is incident with two 3-faces. Then v is adjacent to at most five children and bad 3-vertices. So $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} \times 2 - \frac{1}{3} \times 5 = 0$. Suppose v is incident with one 3-face. Then v is adjacent to at most six children and bad 3-vertices. So $c'(v) \geq 4 - \frac{4}{3} - \frac{1}{2} - \frac{1}{3} \times 6 = \frac{1}{6} > 0$. Suppose v is incident with no 3-face. Then v sends at most $\frac{7}{6}$ to u and $c'(v) \geq 4 - \frac{7}{6} - \frac{1}{3} \times 7 = \frac{1}{2} > 0$.

Thirdly, v is adjacent to one parent u and u is a A-parent. Then v sends at most $\frac{5}{3}$ to u and v is adjacent to at most one A-child. In the following, we will consider the number of A-children. Note, v is incident with at least one 3-face.

Suppose v is adjacent to one A-child. Then v is incident with at most four 3-faces. If v is incident with four 3-faces, then v sends no charge to vertex except u and A-child. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{3} = 0$. If v is incident with three 3-faces, then v sends at most $\frac{1}{3}$ to its adjacent vertices except u and A-child. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 3 - \frac{1}{3} \times 2 = \frac{1}{6} > 0$. If v is incident with two 3-faces, then v sends at most $\frac{1}{3} \times 2$ to vertices except u and A-child. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 2 - \frac{1}{3} \times 3 = \frac{1}{3} > 0$.

Suppose v is adjacent to no A-child. Then v is incident with at most four 3-faces. Suppose v is incident with four 3-faces. Then v is adjacent to no B,C,D-children. We also known v is adjacent to at most one E-child, since A-vertex of v needs more charge from v than E-child. Suppose v is adjacent to one E-child. If v is adjacent to no A-vertex, then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{6} \times 2 = 0$. Otherwise, v is adjacent to one A-vertex. Then v is incident with at least two 5^+-faces each of which $n_{3^-}(f_1) < \lfloor \frac{d(f_1)}{2} \rfloor$. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{6} + \frac{1}{8} \times 2 = \frac{1}{12} > 0$. In the following, we suppose v is adjacent to no E-child and we will consider the number of bad 3-vertices adjacent to v. By O_3 and O_4, v is adjacent to at most two bad 3-vertices, moreover v is adjacent to at most one A-vertex. If v is adjacent to no A-vertex, then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{6} \times 2 = 0$. If v is adjacent to one bad 3-vertex and this bad 3-vertex is an A-vertex, then Fig. 3.(3) does not appear. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{3} = 0$. If v is adjacent to one A-vertex and one B- or C-vertex, then only consider the case of Fig. 3.(4) and (5). Then v is incident with at least one 5^+-face, denote this face f_1 and $n_{3^-}(f_1) < \lfloor \frac{d(f_1)}{2} \rfloor$. If the number of this face is at least two, then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{6} + \frac{1}{8} \times 2 = \frac{1}{12} > 0$. If the number of this face is only one, then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 4 - \frac{1}{3} - \frac{1}{12} + \frac{1}{8} = \frac{1}{24} > 0$. Suppose v is incident with three 3-faces. Then v is adjacent to at most three bad 3-vertices and children. Moreover, v sends at most $\frac{1}{6}$ to one vertex which is adjacent to v and needs receive charge from v. So $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 3 - \frac{1}{3} \times 2 - \frac{1}{6} = 0$.

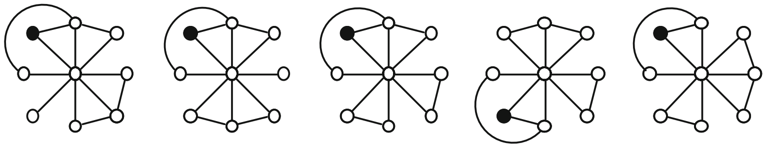

Fig. 3 The figures of v is adjacent one A-parent and incident with four 3-faces

Suppose v is incident with two 3-faces. Then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} \times 2 - \frac{1}{3} \times 4 = 0$. Suppose v is incident with one 3-face. Then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{2} - \frac{1}{3} \times 5 = \frac{1}{6} > 0$.

Fourthly, v is adjacent to one parent u and u is a B-parent. Then v is incident with at most three 3-faces. Suppose v is incident with three 3-faces. Then v is adjacent to at most one A-vertex, and v is adjacent to one 3-vertex, u, one 3-vertex, consequently. Moreover, v sends at most 2 to these three vertices by O_6. So $c'(v) \geq 4 - 2 - \frac{1}{2} \times 3 - \frac{1}{3} - \frac{1}{6} = 0$. Suppose v is incident with two 3-faces. Then similarly we have $c'(v) \geq 4 - 2 - \frac{1}{2} \times 2 - \frac{1}{3} \times 3 = 0$. Suppose v is incident with one 3-face. Then $c'(v) \geq 4 - 2 - \frac{1}{2} - \frac{1}{3} \times 4 = \frac{1}{6} > 0$. Suppose v is incident with no 3-face. Then $c'(v) \geq 4 - \frac{5}{3} - \frac{1}{3} \times 7 = 0$.

Case 2 $9 \leq r \leq 11$.

We also rewrite Euler's formula as

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -8 < 0$$

(R1) From each face f to each of its incident vertices v, transfer
  (R1-1) $\frac{1}{2}$, if $d(f) = 5$ and $d(v) = 2$;
  (R1-2) $\frac{3}{4}$, if $d(f) \geq 6$ and $d(v) = 2$;
  (R1-3) $\frac{1}{2}$, if $d(f) \geq 5$ and $d(v) = 3$;
  (R1-4) $\frac{1}{8}$, if $d(f) \geq 5, d(v) = 9$ and $n_{3^-}(f) < \lfloor \frac{d(f)}{2} \rfloor$.

(R2) From each vertex v to each of its incident 3-faces, transfer $\frac{1}{2}$, if $d(v) \geq 6$.

(R3) From each 8^+-vertex v to each of its adjacent 3-vertices u, transfers
  (R3-1) $\frac{1}{3}$, if $f_{5^+}(v) = 1$;
  (R3-2) $\frac{1}{6}$, otherwise.

(R4) From each 9^+-vertex v to its parent u, transfer
  (R4-1) 2, if $f_3(u) = 1$ and $f_4(u) = 1$ or $f_4(u) = 2$;
  (R4-2) $\frac{3}{2}$, if $f_{4^-}(u) = 1$, $f_5(u) = 1$;
  (R4-3) $\frac{5}{4}$, otherwise.

Clearly, for all faces f, $c'(f) \geq 0$ and for all vertices v and $c'(v) \geq 0$, if $d(v) \leq 5$. If $d(v) = 6$, then $c(v) = 2, n_{3^-}(v) = 0$ by (c), and $f_3(v) \leq 4$. So $c'(v) \geq 2 - \frac{1}{2} \times 4 = 0$. If $d(v) = 7$, then $c(v) = 3, n_{3^-}(v) = 0$ by (c), and $f_3(v) \leq 4$. So $c'(v) \geq 3 - \frac{1}{2} \times 4 = 1 > 0$. If $d(v) = 8$, then $c(v) = 4, n_{2^-}(v) = 0$ by (c), and $f_3(v) \leq 5$. So $c'(v) \geq 4 - \frac{1}{2} \times 5 - \frac{1}{6} \times 8 = \frac{1}{6} > 0$.

Suppose $d(v) = 9$. Then $c(v) = 5$, $f_3(v) \leq 6$. Suppose v is adjacent to no parent. Then $c'(v) \geq 5 - \frac{1}{2} \times 6 - \frac{1}{3} \times 5 = \frac{1}{3} > 0$. So v is adjacent to one parent.

Suppose $f_3(v) = 6$. Then the parent of v (if exist) is incident with a 6^+-face and v is incident with at most two 3-vertices, each of which receives charge from v. So $c'(v) \geq 5 - \frac{5}{4} - \frac{1}{2} \times 6 - \frac{1}{3} \times 2 = \frac{1}{12} > 0$. Suppose $f_3(v) = 5$. If v does not need to send 2 to the parent of v, then $c'(v) \geq 5 - \frac{3}{2} - \frac{1}{2} \times 5 - \frac{1}{3} \times 2 - \frac{1}{6} = \frac{1}{6} > 0$. Otherwise, $c'(v) \geq 5 - 2 - \frac{1}{2} \times 5 - \frac{1}{3} \times 2 + \frac{1}{8} \times 2 = \frac{1}{12} > 0$. Suppose $f_3(v) = 4$. Then $c'(v) \geq 5 - 2 - \frac{1}{2} \times 4 - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 = 0$. Suppose $f_3(v) \leq 3$. Then $c'(v) \geq 5 - 2 - \frac{1}{2} \times 3 - \frac{1}{6} \times 9 = 0$.

Suppose $d(v) = 10$. Then $c(v) = 6$, $f_3(v) \leq 6$. If $f_3(v) = 6$, then $c'(v) \geq 6 - 2 - \frac{1}{2} \times 6 - \frac{1}{3} \times 3 = 0$. Otherwise, $f_3(v) \leq 5$ and $c'(v) \geq 6 - 2 - \frac{1}{2} \times 5 - \frac{1}{6} \times 9 = 0$. Suppose $d(v) = 11$. Then $c(v) = 7$, $f_3(v) \leq 7$. If $f_3(v) = 7$, then v is adjacent to at most four 3-vertices, each of which needs receive charge from v, so $c'(v) \geq 7 - 2 - \frac{1}{2} \times 7 - \frac{1}{3} \times 3 - \frac{1}{6} = \frac{1}{3} > 0$. Otherwise, $f_3(v) \leq 6$ and $c'(v) \geq 7 - 2 - \frac{1}{2} \times 6 - \frac{1}{6} \times 10 = \frac{1}{3} > 0$.

Hence we complete the proof of Theorem 1.

Acknowledgments This work was supported in part by the National Natural Science Foundation of China under Grants 11201440, 11271006, 11301410, 11401386, 71171120, the Projects of International (Regional) Cooperation and Exchanges of NSFC (71411130215), the Specialized Research Fund for the Doctoral Program of Higher Education (20133706110002), the Shandong Provincial Natural Science Foundation of China under Grant ZR2014AQ001, the Natural Science Basic Research Plan in Shanxi Province of China under Grant 2013JQ1002, and the Scientific Research Foundation for the Excellent Young and Middle-Aged Scientists of Shandong Province of China under Grant BS2013-DX002.

References

Angelini P, Frati F (2012) Acyclically 3-colorable planar graphs. J Comb Optim 24:116–130

Borodin OV, Kostochka AV, Woodall DR (1997) List edge and list total colourings of multigraphs. J Comb Theory Ser B 71:184–204

Du HW, Jia XH, Li DY, Wu WL (2004) Coloring of double disk graphs. J Global Optim 28:115–119

Garg N, Papatriantafilou M, Tsigas P (1996) Distributed list coloring: how to dynamically allocate frequencies to mobile base stations. In: Eighth IEEE symposium on parallel and distributed processing, pp 18–25. doi:10.1109/SPDP.1996.570312

Hägkvist R, Chetwynd A (1992) Some upper bounds on the total and list chromatic numbers of multigraphs. J Graph Theory 16:503–516

Hou JF, Liu GZ, Cai JS (2006) List edge and list total colorings of planar graphs without 4-cycles. Theor Comput Sci 369:250–255

Jensen T, Toft B (1995) Graph coloring problems. Wiley-Interscience, New York

Li XW, Mak-Hau V, Zhou SM (2013) The $L(2, 1)$-labelling problem for cubic Cayley graphs on dihedral groups. J Comb Optim 25:716–736

Li R, Xu BG (2011) Edge choosability and total choosability of planar graphs with no 3-cycles adjacent 4-cycles. Discret Math 311:2158–2163

Liu B, Hou JF, Wu JL, Liu GZ (2009) Total colorings and list total colorings of planar graphs without intersecting 4-cycles. Discret Math 309:6035–6043

Wang HJ, Wu LD, Wu WL, Wu JL (2014) Minimum number of disjoint linear forests covering a planar graph. J Comb Optim 28:274–287

Wang HJ, Wu LD, Zhang X, Wu WL, Liu B (2014) A note on the minimum number of choosability of planar graphs. J Comb Optim doi:10.1007/s10878-014-9805-2

Wang HJ, Wu LD, Wu WL, Pardalos PM, Wu JL (2014) Minimum total coloring of planar graph. J Global Optim 60:777–791

Wang W, Liu X (2005) List coloring based channel allocation for open-spectrum wireless networks. In: IEEE 62nd vehicular technology conference (VTC 2005-Fall) (1):690–694, doi:10.1109/VETECF.2005.1558001