

On total colorings of 1-planar graphs

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Abstract A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, we confirm the total-coloring conjecture for 1-planar graphs with maximum degree at least 13.

Keywords 1-Planar graph · Total coloring · Discharging method

1 Introduction

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G , respectively. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices that are adjacent to v in G . By $d_G(v) := |N_G(v)|$, we denote the degree of v in G . For a plane graph G , $F(G)$ denotes its face set and $d_G(f)$ denotes the degree of a face f in G . Throughout this paper, a k -, k^+ - and k^- -vertex (resp. face) is a vertex (resp. face) of degree k , at least k and at most k . Any undefined notation follows that of [Bondy and Murty \(1976\)](#).

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Given a graph G and a positive integer k , a *total k -coloring* of G is a mapping f from $V(G) \cup E(G)$ to $\{1, 2, \dots, k\}$ such that $f(x) \neq f(y)$ for every pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. The *total chromatic number* $\chi''(G)$ of a graph G is the least number of colors needed in any total coloring of G . It is clear that $\chi''(G) \geq \Delta(G) + 1$. The next step is to look for any Brooks-typed or Vizing-typed upper bound on the total chromatic number in terms of maximum degree. However, to obtain such bounds turns out to be a difficult problem and has eluded mathematicians for nearly fifty years. The most well-known speculation is the *total-coloring conjecture*, independently raised by Behzad (1965) and Vizing (1968), which asserts that every graph of maximum degree Δ admits a total $(\Delta + 2)$ -coloring. The validity of this conjecture is known to be true for graphs in several wide families. Rosenfeld (1971) and Vijayaditya (1971) confirmed it for $\Delta \leq 3$, Kostochka solved it for $\Delta = 4$ Kostochka (1977) and $\Delta = 5$ Kostochka (1996). For $\Delta \geq 6$ it remains open even for planar graphs, but more is known. Borodin (1989) confirmed the total-coloring conjecture for planar graphs with $\Delta \geq 9$. Yap (1989) proved it for planar graphs with $\Delta \geq 8$. The $\Delta = 7$ case was solved for planar graphs by Sanders and Zhao (1999).

A graph is *1-planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planarity was introduced by Ringel (1965) while trying to simultaneously color the vertices and faces of a plane graph G such that any pair of adjacent/incident elements receive different colors. Various coloring problems including vertex coloring Ringel (1965), Borodin (1984), Borodin (1995) list vertex coloring Albertson and Mohar (2006); Wang and Lih (2008), Wang and Lih (2008), acyclic vertex coloring Borodin et al. (2001), edge coloring Zhang and Liu (2012a), Zhang and Liu (2012b), Zhang et al. (2010), Zhang and Wu (2011), acyclic edge coloring Zhang et al. (2012), list edge and list total coloring Zhang et al. (2012), $(p, 1)$ -total labelling Zhang et al. (2011) and the linear arboricity Zhang et al. (2011) of 1-planar graphs have been extensively studied in the literature. In particular, Zhang et al. (2012) proved that every 1-planar graph with maximum degree $\Delta \geq 16$ is totally $(\Delta + 2)$ -choosable, which implies that the total-coloring conjecture holds for 1-planar graphs with maximum degree at least 16. In this paper, we improve the lower bound for the maximum degree in the above corollary to 13 by the following theorem.

Theorem 1 *Let G be a 1-planar graph with maximum degree Δ and let r be an integer. If $\Delta \leq r$ and $r \geq 13$, then $\chi''(G) \leq r + 2$.*

During the proof Theorem 1, we use the discharging method, and in particular, we involve an unusual approach to estimate the final charges of big vertices. This can be seen in Sect. 3.

2 Structural properties of a minimal 1-planar graph

Let r -*minimal graph* be a connected graph G on the fewest edges that has no total $(r + 2)$ -colorings. In the following lemmas, we always assume that $r \geq 13$.

Lemma 2 *Let G be a r -minimal graph and let uv be an edge in G . If $d_G(u) \leq \lfloor \frac{r+1}{2} \rfloor$, then $d_G(u) + d_G(v) \geq r + 3$.*

Proof Suppose, to the contrary, that $d_G(u) + d_G(v) \leq r + 2$. Since G is r -minimal, the graph $G' = G - uv$ has a total $(r + 2)$ -coloring φ . First of all, erase the color of u from φ . Since $d_{G'}(u) + d_{G'}(v) \leq r + 2 - 2 = r$, the uncolored edge uv is incident with at most r colored edges and one colored vertex, thus we can properly color uv with a color involved in φ . At last, the vertex u can be easily colored since it is incident with at most $2d_G(u) \leq r + 1$ colors. \square

Lemma 3 *Let G be a r -minimal graph and let v be a vertex of G . If $d_G(v) = 3$, then v cannot be contained in a triangle.*

Proof Let $N_G(v) = \{v_1, v_2, v_3\}$. Suppose, to the contrary, that v is contained in a triangle vv_2v_3 . By the choice of G , the graph $G' = G - vv_3$ has a total $(r + 2)$ -coloring φ with $\varphi(vv_i) = i$ for $i = 1, 2$. Now erase the color of v from φ . For any color $i \geq 3$, i must appear on v_3 or on some edge incident with v_3 , since otherwise, we can color vv_3 with i , a contradiction. Thus, the colors 1 and 2 cannot appear on v_3 or the edges incident with v_3 . Now uncolor vv_2 and color vv_3 with 2. By the same argument, any color $i \geq 3$ must appear on v_2 or the edges incident with v_2 and the colors 1 and 2 cannot appear on there. Now recolor v_2v_3 with 1, color vv_3 with $\varphi(v_2v_3)$ and color vv_2 with 2. At last, the vertex v can be easily colored since it is incident with at most 6 colors. \square

Lemma 4 *Let G be a r -minimal graph and let v be a 4-vertex of G with $N_G(v) = \{v_1, v_2, v_3, v_4\}$. For any $1 \leq i \leq 4$, the edge vv_i cannot be contained in two triangles.*

Proof Suppose, to the contrary, that the edge vv_4 is contained in two triangles vv_1v_4 and vv_3v_4 . By the choice of G , the graph $G' = G - vv_4$ has a total $(r + 2)$ -coloring φ with $\varphi(vv_i) = i$ for $i = 1, 2, 3$. Now erase the color of v from φ . For any vertex v in G' , let $S_\varphi(v)$ denote the set of colors not appearing on v or the edges incident with v . First of all, we have $i \notin S_\varphi(v_4)$ for any color $i \geq 4$, since otherwise, we can color vv_4 with i and then the vertex v can be easily colored (in the following we would not mention the coloring of v for the last step). This implies that $S_\varphi(v_4) \subseteq \{1, 2, 3\}$. Note that $|S_\varphi(v_4)| \geq 2$.

Claim 1 $S_\varphi(v_4) \supseteq \{1, 3\}$

Proof Otherwise, assume that $1 \notin S_\varphi(v_4)$. This implies that $S_\varphi(v_4) = \{2, 3\}$. Since φ is a proper total coloring of G' , we may assume that $\varphi(v_1v_4) = 4$. If $i \in S_\varphi(v_1)$ for some $i \in \{2, 3\}$, then recolor v_1v_4 with i and color vv_4 with 4. Otherwise, there is a color $i_0 \geq 5$ such that $i_0 \in S_\varphi(v_1)$. Note that 1 must appear on v_2 (resp. v_3) or edges incident with v_2 (resp. v_3), since otherwise, we can recolor vv_2 (resp. vv_3) with 1, recolor vv_1 with i_0 , and color vv_4 with 2 (resp. 3). Moreover, for any $i \geq 4$, the color i must appear on v_2 (resp. v_3), since otherwise, we can color vv_2 (resp. vv_3) with i and color vv_4 with 2 (resp. 3). This implies that $3 \in S_\varphi(v_2)$ and $2 \in S_\varphi(v_3)$. Now we consider the color on v_3v_4 . If $\varphi(v_3v_4) \neq 1$, then recolor v_3v_4 with 2 and color vv_4 with $\varphi(v_3v_4)$. Otherwise, $\varphi(v_3v_4) = 1$. In this case, recolor vv_3, v_1v_4 with 1, v_3v_4 with 3, vv_1 with i_0 and color vv_4 with 4. \square

By the above claim, one can see that one of the edges v_1v_4 and v_3v_4 shall be colored with a color $i \geq 4$. Without loss of generality, assume that $\varphi(v_1v_4) = 4$. Note

that $3 \notin S_\varphi(v_1)$, since otherwise, we can recolor v_1v_4 with 3 and color vv_4 with 4. Moreover, $1 \notin S_\varphi(v_3)$, since otherwise, we can exchange the colors on v_1v_4 and v_1v , then recolor vv_3 with 1 and color vv_4 with 3. For any $i \geq 4$, the color $i \notin S_\varphi(v_j)$ for any $j = 1, 3$, since otherwise, we can recolor vv_j with i and color vv_4 with j . Thus $S_\varphi(v_1) = S_\varphi(v_3) = \{2\}$. If there is a color $i \geq 4$ such that $i \in S_\varphi(v_2)$, then we can recolor vv_2 with i , vv_1 with 2, and color vv_4 with 1. Otherwise, we have $S_\varphi(v_2) \subseteq \{1, 3\}$. Without loss of generality, let $1 \in S_\varphi(v_2)$. We then recolor vv_2 and v_1v_4 with 1, vv_1 with 2, and color vv_4 with 4. \square

Lemma 5 *Let G be a r -minimal graph and let V_i be the set of i -vertices in G . We have $|V_\Delta| > 2|V_3|$.*

Proof If $|V_3| = 0$, then it is trivial. If $|V_3| \neq 0$, then by Lemma 2, $r = \Delta$. Let E be the set of edges in G having one end-vertex in V_3 and let H be the bipartite subgraph with vertex set $V_3 \cup V_\Delta$ and edge set E . First of all, we prove that H is a forest. Suppose, to the contrary, that H contains a cycle C . Then this cycle is of even length in which alternate vertices have degree 3 in G . Since G is Δ -minimal, the graph $G' = G - E(C)$ has a total $(\Delta + 2)$ -coloring φ . Now erase the colors of the 3-vertices on C from φ . Let e be an arbitrary edge of C . One can see that e is now incident with at most $\Delta - 1$ colored edges and one colored vertex, hence there are at least $(\Delta + 2) - (\Delta - 1 + 1) = 2$ available colors for e . Therefore, the edges in $E(C)$ can be properly colored since every even cycle is edge 2-choosable. At last, the 3-vertices on C can be colored since each of them is now incident with at most six colored elements and no two of them are adjacent in G by Lemma 2. This contradiction implies that H is a forest and thus $|V(H)| = |V_3| + |V_\Delta| > |E(H)|$. Moreover, the neighbors of every vertex in V_3 belong to the vertex set V_Δ by Lemma 2. This implies that $|E(H)| = 3|V_3|$. Hence we conclude that $|V_\Delta| > 2|V_3|$. \square

In the following, we restrict the minimal graph G to be a 1-planar graph and assume that G has already been embedded on a plane so that every edge is crossed by at most one other edge and the number of crossings is as small as possible. The associated plane graph G^\times of G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in G^\times is false if it is not a vertex of G and true otherwise. By a false face, we mean a face f in G^\times that is incident with at least one false vertex; otherwise, we call f true.

Lemma 6 (Zhang and Wu 2011) *Let v be a 3-vertex in G . If v is incident with two false 3-faces vv_1v_2 and vv_1v_3 in G^\times , then v_2 and v_3 are both false and v is incident with a 5^+ -face in G^\times .*

Lemma 7 *Every 4-vertex in G is incident with at most three 3-faces in G^\times .*

Proof Let v be a 4-vertex in G and let v_1, v_2, v_3, v_4 be the neighbors in G^\times of v that occurs clockwise around v . Suppose that v is incident with four 3-faces in G^\times , that is, $v_1v_2, v_2v_3, v_3v_4, v_4v_1 \in E(G^\times)$. Since no two false vertices are adjacent in G^\times , there are at most two false vertices among v_1, v_2, v_3 and v_4 . If two of them, say v_1 and v_3 , are false, then we would find two edges in G that connect v_2 to v_4 : one goes through the point v_1 and the other goes through the point v_3 , contradicting the fact

that G is simple. Thus we shall assume that there are at least three true vertices, say v_1, v_2 and v_3 , among the four neighbors of v . However, this is impossible by Lemma 4 since vv_1v_2 and vv_2v_3 are two adjacent triangles in G with $d_G(v) = 4$. \square

Lemma 8 *Every 5-vertex in G is either incident with at least two 4^+ -faces in G^\times , or adjacent to at least three true vertices in G^\times , or incident with exactly one 4^+ -face and adjacent to two true vertices in G^\times .*

Proof Let v be a 5-vertex in G and let v_1, v_2, v_3, v_4, v_5 be the neighbors in G^\times of v that occurs clockwise around v . Suppose that v is incident with at most one 4^+ -face and adjacent to at most two true vertices in G^\times . Without loss of generality, assume that $v_1v_2, v_2v_3, v_3v_4, v_4v_5 \in E(G^\times)$. Since no two false vertices are adjacent in G^\times , there are at most three false vertices among v_1, v_2, v_3, v_4 and v_5 . This implies that v is adjacent to exactly two true vertices in G^\times . On the other hand, v is incident with exactly one 4^+ -face because otherwise $v_1v_2v_3v_4v_5$ would be a 5-cycle in G^\times , which implies that at least three of those five vertices are true, a contradiction to our assumption. \square

Lemma 9 *Every 5-face in G^\times is incident with at most four 4^- -vertices.*

Proof Suppose, to the contrary, that the 5-face f is incident only with 4^- -vertices in G^\times . Then f is incident with at least three false vertices, because otherwise we would find an edge uv on f such that u and v are both true 4^- -vertices, which is impossible by Lemma 2. On the other hand, f can be incident with at most two false vertices since no two false vertices are adjacent in G^\times . This contradiction completes the proof. \square

3 The proof of Theorem 1

We call a vertex v in G^\times *small* if $d_{G^\times}(v) \leq 5$. Note that the degree of a false vertex in G^\times is four, so every false vertex is small. We call u the *tri-neighbor* of v if uv is an edge of G with $d_G(v) = 4$ and uv is incident with a 3-face uvw in G^\times so that w is true. Note that in this situation u cannot be a tri-neighbor of w by Lemma 2. Now we start to prove Theorem 1.

Suppose that G is a minimum counterexample to it. We then conclude that G is 2-connected and moreover, $\delta(G) \geq 3$ by Lemma 2. In the following, we apply the discharging method to the associated plane graph G^\times of G and complete the proof by contradiction. Note that G^\times is also 2-connected.

We now assign an initial charge c to each element $x \in V(G^\times) \cup F(G^\times)$ as follows. If $x \in V(G^\times)$, then let $c(x) = d_{G^\times}(x) - 6$. If $x \in F(G^\times)$, then let $c(x) = 2d_{G^\times}(x) - 6$. Since G^\times is a plane graph, $\sum_{x \in V(G^\times) \cup F(G^\times)} c(x) = -12$ by the well-known Euler's formula. We redistribute the initial charges on $V(G^\times) \cup F(G^\times)$ by the discharging rules below. Let $c'(x)$ be the final charge of an element $x \in V(G^\times) \cup F(G^\times)$ after the discharging process. We still have $\sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) = -12 < 0$, since our rules only move charge around and do not affect the sum.

R1. Every 4^+ -face redistributes its initial charge uniformly among the small vertices that are incident with it in G^\times .

- R2. Every Δ -vertex gives $\frac{1}{2}$ to a common pot from which each 3-vertex receives 1, if $|V_3| > 0$.
- R3. Let u, v be true vertices of G^\times and let $uv \in E(G^\times)$. If v is small, then u sends $\frac{1}{3}$ to v ; moreover, if u is a tri-neighbor of v , then u sends an addition of $\frac{1}{12}$ to v . Note that in R2, the common pot can also be seen as a pseudo-point that has initial charge zero. In the next six rules, we assume that uv crosses xy at a false vertex w in G^\times .
- R4. If $d_{G^\times}(u) \geq 9, ux, uy \notin E(G^\times)$ and v is a small vertex, then u sends $\frac{1}{3}$ to v through w .
- R5. If $d_{G^\times}(u) \geq 9, ux \notin E(G^\times)$ and $uy \in E(G^\times)$, then u sends $\frac{1}{4}$ to w . Furthermore, if $d_{G^\times}(v) \leq 4$, then u sends $\frac{1}{3}$ to v through w .
- R6. If $d_{G^\times}(u) \geq 9, ux, uy, vx \in E(G^\times)$ and y is a small vertex, then u sends $\frac{3}{4}$ to w . Furthermore, if $d_{G^\times}(v) \leq 4$, then u sends $\frac{1}{24}$ to v through w .
- R7. If $d_{G^\times}(u) \geq 9, ux, uy \in E(G^\times)$ and either $vx \notin E(G^\times)$ or y is not a small vertex, then u sends $\frac{2}{3}$ to w . Furthermore, if $d_{G^\times}(v) \leq 4$, then u sends $\frac{1}{8}$ to w through w .
- R8. If $d_{G^\times}(u) = 8$ and $ux, uy \in E(G^\times)$, then u sends $\frac{1}{2}$ to w .
- R9. If $d_{G^\times}(u) = 8, ux \in E(G^\times)$ and $uy \notin E(G^\times)$, then u sends $\frac{1}{12}$ to w .

In the following, we check that the final charge c' on each vertex and face is nonnegative, and we also show that the final charge of the common pot is nonnegative. This implies that $\sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) \geq 0$, a contradiction.

First of all, since $|V_\Delta| > 2|V_3|$ by Lemma 5, the final charge of the common pot is at least $\frac{1}{2}|V_\Delta| - |V_3| > 0$ by R2. One can also check that the final charge of every face in $F(G^\times)$ is nonnegative by R1. Thus in the following we consider the vertices in G^\times .

Let v be a d -vertex in G^\times and let v_1, v_2, \dots, v_d be its neighbors in G^\times that occur around v in a clockwise order. By f_i denote the face incident with vv_i and vv_{i+1} in G^\times , where the addition on subscripts are taken modulo d .

Case 1 $d = 3$.

Case 1.1 If v is adjacent to at most one false vertex in G^\times , then without loss of generality assume that v_2 and v_3 are true. By Lemmas 2 and 3, neither v_2 nor v_3 is small and f_2 is a 4^+ -face. Thus by R1 and R3, v receives at least $2 \times \frac{1}{3} + \frac{2}{4-2} = \frac{5}{3}$ from v_2, v_3 and f_2 . By Lemmas 3 and 6, at least one of f_1 and f_3 , say f_1 , shall be a 4^+ -face. By R1, f_1 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v . Furthermore, v would receive 1 from the common pot by R2. Therefore, $c'(v) \geq -3 + \frac{5}{3} + \frac{2}{3} + 1 > 0$.

Case 1.2 If v is adjacent to two false vertices in G^\times , say v_1 and v_2 , then f_1 is a 4^+ -face since $v_1v_2 \notin E(G^\times)$. By R1 and R3, v receives a total of $1 + \frac{1}{3} = \frac{4}{3}$ from the common pot and v_3 . Now we consider three subcases.

First, assume that f_2 and f_3 are both 4^+ -faces. By R1, f_1, f_2 and f_3 sends at least $\frac{2}{4} = \frac{1}{2}, \frac{2}{4-1} = \frac{2}{3}$ and $\frac{2}{4-1} = \frac{2}{3}$ to v , respectively. Therefore, $c'(v) \geq -3 + \frac{4}{3} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} > 0$.

Second, assume that f_2 is a 4^+ -face and f_3 is a 3-face. Let v'_1 be a vertex such that vv'_1 is an edge in G that goes through the false vertex v_1 in G^\times . By Lemmas 2 and 3, v'_1 is a Δ -vertex and $v'_1v_3 \notin E(G^\times)$, because otherwise vv'_1v_3 would be a triangle in

G . Thus by R4 and R5, v receives $\frac{1}{3}$ from v'_1 . If f_2 is a 5^+ -face, then by R1, f_2 sends at least $\frac{4}{5-1} = 1$ to v (note that v_3 is not a small vertex). Since f_1 is a 4^+ -face, f_1 would send at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. Thus, $c'(v) \geq -3 + \frac{4}{3} + \frac{1}{3} + 1 + \frac{1}{2} > 0$. So we suppose that f_2 is a 4-face, from which v receives at least $\frac{2}{4-1} = \frac{2}{3}$ by R1. If f_1 is a 5^+ -face, then by R1, f_1 sends at least $\frac{4}{5}$ to v . Thus $c'(v) \geq -3 + \frac{4}{3} + \frac{1}{3} + \frac{2}{3} + \frac{4}{5} > 0$. So suppose that f_1 is a 4-face. Let v'_2 and v'_3 be the fourth (undefined) vertex on f_2 and f_1 , respectively. Since v_2 is false and $v_2v'_2, v_2v'_3 \in E(G^\times)$, $v'_2v'_3$ is an edge in G . By Lemma 2, one of v'_2 and v'_3 is not small. If v'_2 is not small, then by R1, f_1 and f_2 sends at least $\frac{2}{4} = \frac{1}{2}$ and $\frac{2}{4-2} = 1$ to v , respectively. It follows that $c'(v) \geq -3 + \frac{4}{3} + \frac{1}{3} + \frac{1}{2} + 1 > 0$. If v'_3 is not small, then by R1, f_1 and f_2 sends at least $\frac{2}{4-1} = \frac{2}{3}$ and $\frac{2}{4-1} = \frac{2}{3}$ to v , respectively. It follows that $c'(v) \geq -3 + \frac{4}{3} + \frac{1}{3} + \frac{2}{3} + \frac{2}{3} = 0$.

Third, assume that f_2 and f_3 are both 3-faces. By Lemma 6, f_1 is a 5^+ -face. Let v'_i ($i = 1, 2$) be a vertex such that vv'_i is an edge in G that goes through the false vertex v_i in G^\times . By a similar argument as the beginning of the second subcase above, one can prove that v receives $\frac{1}{3}$ from each of v'_1 and v'_2 . If f_1 is a 6^+ -face, then by R1, f_1 sends at least $\frac{6}{6} = 1$ to v . If f_1 is a 5-face, then assume that v_3x_1 crosses vv'_1 and v_3x_2 crosses vv'_2 in G . It follows that $x_1x_2 \in E(G)$. By Lemma 2, at least one of x_1 and x_2 is not small. Thus by R1, f_1 sends at least $\frac{4}{5-1} = 1$ to v . In each case we have $c'(v) \geq -3 + \frac{4}{3} + 2 \times \frac{1}{3} + 1 = 0$.

Case 1.3 If v is adjacent to three false vertices in G^\times , then f_1, f_2 and f_3 are 4^+ -faces. By R2, v receives 1 from the common pot. If two of f_1, f_2 and f_3 are of degree at least 5, then by R1 it is easy to calculate that v receives at least $\frac{4}{5} + \frac{4}{5} + \frac{2}{4} > 2$ from its incident faces and therefore $c'(v) \geq -3 + 1 + 2 = 0$. If exactly one of f_1, f_2 and f_3 , say f_3 , is a 5^+ -face, then let x_1 and x_2 be the fourth (undefined) vertices of the 4-faces f_1 and f_2 , respectively. One can easily see that $x_1x_2 \in E(G)$ and thus by Lemma 2, at least one of x_1 and x_2 is not small. Therefore, v receives at least $\frac{4}{5} + \frac{2}{4} + \frac{2}{4-1} = \frac{59}{30}$ from its incident faces by R1. Assume that vv'_2 crosses x_1x_2 in G , then by Lemma 2, v'_2 is a Δ -vertex. Thus, v'_2 sends at least $\frac{1}{8}$ to v by R4–R7. This implies that $c'(v) \geq -3 + 1 + \frac{59}{30} + \frac{1}{8} > 0$. If f_1, f_2 and f_3 are all 4-faces, then let x_i ($i = 1, 2, 3$) be the fourth (undefined) vertices of the 4-faces f_i . It is easy to check that $x_1x_2, x_2x_3, x_3x_1 \in E(G)$ by the drawing of G . Thus, at most one of x_1, x_2 and x_3 is small by Lemma 2. This implies that v receives at least $\frac{2}{4} + \frac{2}{4-1} + \frac{2}{4-1} = \frac{11}{6}$ from its incident faces by R1. Assume that vv'_i ($i = 1, 2, 3$) crosses $x_{i-1}x_i$ in G , where the subscripts are taken modulo 3, then by Lemma 2, v'_i is a Δ -vertex, from which v receives at least $\frac{1}{8}$ by R4–R7. Therefore, $c'(v) \geq -3 + 1 + \frac{11}{6} + 3 \times \frac{1}{8} > 0$.

Case 2 $d = 4$ and v is a true vertex.

By Lemma 7, v is incident with at least one 4^+ -face in G^\times . Thus we consider four subcases.

Case 2.1 If v is incident with four 4^+ -faces in G^\times , then v receives at least $\frac{2}{4} = \frac{1}{2}$ from each of its incident faces by R1. This implies that $c'(v) \geq -2 + 4 \times \frac{1}{2} = 0$.

Case 2.2 If v is incident with exactly three 4^+ -faces in G^\times , say f_2, f_3 and f_4 , then $v_1v_2 \in E(G^\times)$. Since no two false vertices are adjacent in G^\times , at least one of v_1 and v_2 , say v_1 , is true, and moreover, is a 12^+ -vertex by Lemma 2. By R3 and R1, v

receives $\frac{1}{3}$ from v_1 , at least $\frac{2}{4-1} = \frac{2}{3}$ from f_4 and at least $\frac{2}{4} = \frac{1}{2}$ from each of f_2 and f_3 . Therefore, $c'(v) \geq -2 + \frac{1}{3} + \frac{2}{3} + 2 \times \frac{1}{2} = 0$.

Case 2.3 If v is incident with exactly two 4^+ -faces in G^\times , then we consider two subcases.

Assume first that f_1 and f_3 are both 4^+ -faces. By a same argument as in Case 2.2, at least one of v_2 and v_3 and at least one of v_1 and v_4 are 12^+ -vertices. If v_1 and v_2 are both 12^+ -vertices, then by R3 and R1, v receives $\frac{1}{3}$ from each of v_1 and v_2 , at least $\frac{2}{4-2} = 1$ from f_1 and at least $\frac{2}{4} = \frac{1}{2}$ from f_3 . Thus, $c'(v) \geq -2 + 2 \times \frac{1}{3} + 1 + \frac{1}{2} > 0$. If v_1 and v_3 are both 12^+ -vertices, then by R3 and R1, v receives $\frac{1}{3}$ from each of v_1 and v_3 and at least $\frac{2}{4-1} = \frac{2}{3}$ from each of f_1 and f_3 . This implies that $c'(v) \geq -2 + 2 \times \frac{1}{3} + 2 \times \frac{2}{3} = 0$.

Second, assume that f_1 and f_2 are 4^+ -faces. If v_1 and v_3 are both true, then by Lemma 2 they are 12^+ -vertices. So by R3 and R1, v receives $\frac{1}{3}$ from each of v_1 and v_3 and at least $\frac{2}{4-1} = \frac{2}{3}$ from each of f_1 and f_2 . This implies that $c'(v) \geq -2 + 2 \times \frac{1}{3} + 2 \times \frac{2}{3} = 0$. So we assume that at least one of v_1 and v_3 is false, which implies that v_4 is true since no two false vertices are adjacent in G^\times .

If v_1 is false and v_3 is true, then let v'_1 be the vertex of G so that vv'_1 is a crossed edge in G with a crossing v_1 . By Lemma 4, $v'_1v_4 \notin E(G)$, because otherwise $vv_4v'_1$ and vv_3v_4 would be two adjacent triangles in G with a common 4-vertex. Note that v'_1 and v_3 are 12^+ -vertices by Lemma 2. So v receives $\frac{1}{3}$ from v'_1 by R4 and R5, $\frac{1}{3}$ from each of v_3 and v_4 by R3 and at least $\frac{2}{4} = \frac{1}{2}$ from each of f_1 and f_2 by R1. This implies that $c'(v) \geq -2 + \frac{1}{3} + 2 \times \frac{1}{3} + 2 \times \frac{1}{2} = 0$.

If v_1 and v_3 are both false, then let v'_i and x_i ($i = 1, 3$) be the vertices of G so that vv'_i crosses v_4x_i in G at the crossing v_i . Note that v'_1 and v'_3 are both 12^+ -vertices by Lemma 2. By Lemma 4, v'_1v_4 and v'_3v_4 cannot simultaneously be the edges of G , because otherwise $vv_4v'_1$ and $vv_4v'_3$ would be two adjacent triangles in G with a common 4-vertex. Without loss of generality, assume that $v'_1v_4 \notin E(G)$. By R3, R4 and R5, each of v'_1 and v_4 sends $\frac{1}{3}$ to v (recall that v_4 is true). If v_2 is true, then v receives $\frac{1}{3}$ from v_2 by R3. Moreover, each of f_1 and f_2 sends at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. Thus, $c'(v) \geq -2 + \frac{1}{3} + 2 \times \frac{1}{3} + 2 \times \frac{1}{2} = 0$. If v_2 is false, then let v'_2 be the vertex of G so that vv'_2 is a crossed edge in G with a crossing v_2 . By Lemma 2, v'_2 is a 12^+ -vertex. If at least one of f_1 and f_2 , say f_1 , is a 5^+ -face, then f_1 sends at least $\min\{\frac{6}{6}, \frac{4}{4}\} = 1$ to v by R1 and Lemma 9 and f_2 sends at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. Thus, $c'(v) \geq -2 + 2 \times \frac{1}{3} + 1 + \frac{1}{2} > 0$. So we assume that f_1 and f_2 are both 4-faces. This implies that x_1x_3 is a crossed edge in G with the crossing v_2 . By Lemma 2, at most one of x_1 and x_3 is small. So f_1 and f_2 totally sends at least $\frac{2}{4-1} + \frac{2}{4} = \frac{7}{6}$ to v by R1. Recall that v'_2 and v'_3 are 12^+ -vertices. By R4–R7, v'_2 sends at least $\frac{1}{8}$ and v'_3 sends at least $\frac{1}{24}$ to v . Therefore, $c'(v) \geq -2 + 2 \times \frac{1}{3} + \frac{7}{6} + \frac{1}{8} + \frac{1}{24} = 0$.

Case 2.4 If v is incident with exactly one 4^+ -faces in G^\times , say f_1 , then $v_2v_3, v_3v_4, v_4v_1 \in E(G^\times)$. Now we claim that at least one of v_1 and v_2 is false. Suppose, to the contrary, that v_1 and v_2 are true vertices. If v_3 is true, then either vv_3v_4 (when v_4 is true) or vv_1v_3 (when v_4 is false) is a triangle in G that is adjacent to another triangle vv_2v_3 , which is impossible by Lemma 4. Thus we shall assume that v_3 is false. By symmetry, v_4 is also false, but it contradicts the fact that $v_3v_4 \in E(G^\times)$.

Without loss of generality, assume that v_1 is false. It follows that v_4 is a true vertex. By Lemma 4, exactly one of v_2 and v_3 shall be false, because otherwise vv_2v_3 and vv_3v_4 would be two adjacent triangles in G with a common 4-vertex. Thus we consider two subcases.

Assume first that v_2 is false and v_3 is true. One can check that v_3 and v_4 are both tri-neighbors of v , which follows that each of v_3 and v_4 sends $\frac{1}{3} + \frac{1}{12} = \frac{5}{12}$ to v by R3. Let v'_i ($i = 1, 2$) be the vertex of G so that vv'_i is a crossed edge in G with the crossing v_i . It is easy to see that v'_1 and v'_2 are 12^+ -vertices by Lemma 2. One can also prove that $v'_1v_4, v'_2v_3 \notin E(G)$ by a similar argument as in Case 2.3. Thus by R4 and R5, each of v'_1 and v'_2 sends $\frac{1}{3}$ to v . Since f_1 is a 4^+ -face, f_1 sends at least $\frac{2}{4} = \frac{1}{2}$ to v by R1. Therefore, $c'(v) \geq -2 + 2 \times \frac{5}{12} + 2 \times \frac{1}{3} + \frac{1}{2} = 0$.

Now assume that v_2 is true and v_3 is false. It is easy to see that vv_2v_4 is a triangle in G by the drawing of G . Let v'_i ($i = 1, 3$) be the vertex of G so that vv'_i is a crossed edge in G with the crossing v_i . One can see that v'_1 and v'_3 are 12^+ -vertices by Lemma 2 and can prove that $v'_1v_4, v'_3v_4 \notin E(G)$ by a similar argument as in Case 2.3. So each of v'_1 and v'_3 sends $\frac{1}{3}$ to v by R3 and R4. Meanwhile, each of v_2 and v_4 sends $\frac{1}{3}$ to v by R3 and f_1 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v by R1 (note that v_2 is not small). Therefore, $c'(v) \geq -2 + 2 \times \frac{1}{3} + 2 \times \frac{1}{3} + \frac{2}{3} = 0$.

Case 3 $d = 4$ and v is a false vertex.

Case 3.1 If v is incident with no 3-faces in G^\times , then by R1, each of f_1, f_2, f_3 and f_4 sends at least $\frac{2}{4} = \frac{1}{2}$ to v . So $c'(v) \geq -2 + 4 \times \frac{1}{2} = 0$.

Case 3.2 If v is incident with exactly one 3-face, say f_1 , then $v_1v_2 \in E(G)$. This implies that at most one of v_1 and v_2 can be a 7^- -vertex by Lemma 2. Assume first that $\min\{d_{G^\times}(v_1), d_{G^\times}(v_2)\} \geq 8$. By R1, each of f_2 and f_4 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v and f_3 sends at least $\frac{2}{4} = \frac{1}{2}$ to v . Moreover, each of v_1 and v_2 sends at least $\frac{1}{12}$ to v by R5 and R9. Thus $c'(v) \geq -2 + 2 \times \frac{2}{3} + \frac{1}{2} + 2 \times \frac{1}{12} = 0$. Now assume that $d_{G^\times}(v_1) \leq 7$. It follows that $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} \geq 9$ by Lemma 2. Thus f_2, f_3 and f_4 sends at least $\frac{2}{4-2} = 1, \frac{2}{4-1} = \frac{2}{3}$ and $\frac{2}{4} = \frac{1}{2}$ to v by R1, respectively. Therefore, $c'(v) \geq -2 + 1 + \frac{2}{3} + \frac{1}{2} > 0$.

Case 3.3 If v is incident with exactly two 3-faces, then we consider two subcases.

Assume first that f_1 and f_2 are both 3-faces. Then $v_1v_2, v_2v_3 \in E(G)$. If $d_{G^\times}(v_2) \leq 8$, then by Lemma 2, $\min\{d_{G^\times}(v_1), d_{G^\times}(v_3), d_{G^\times}(v_4)\} \geq 7$. This implies that each of f_3 and f_4 sends at least $\frac{2}{4-2} = 1$ to v and thus $c'(v) \geq -2 + 2 \times 1 = 0$. So we assume that $d_{G^\times}(v_2) \geq 9$. It follows that v_2 sends $\frac{2}{3}$ to v by R7. If one of v_1 and v_3 , say v_1 , is small, then by R1, f_3 and f_4 sends at least $\frac{2}{4-1} = \frac{2}{3}$ and $\frac{2}{4} = \frac{1}{2}$ to v , respectively, since in this case we also have $d_{G^\times}(v_3) \geq 11$ by Lemma 2. Moreover, v_3 sends $\frac{1}{4}$ to v by R5. Therefore, $c'(v) \geq -2 + \frac{2}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{4} > 0$. On the other hand, if neither v_1 nor v_3 is small, then by R1, each of f_3 and f_4 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v . Thus $c'(v) \geq -2 + \frac{2}{3} + 2 \times \frac{2}{3} = 0$.

Now assume that f_1 and f_3 are both 3-faces. If none of v_1, v_2, v_3 and v_4 is small, then by R1, each of f_2 and f_4 sends at least $\frac{2}{4-2} = 1$ to v , which implies that $c'(v) \geq -2 + 2 \times 1 = 0$. If at least one of v_1, v_2, v_3 and v_4 , say v_1 , is small, then by Lemma 2, $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} \geq 11$. So f_2 and f_4 sends at least $\frac{2}{4-2} = 1$ and

$\frac{2}{4} = \frac{1}{2}$ to v by R1, respectively. Moreover, each of v_2 and v_3 sends $\frac{1}{4}$ to v by R5. Therefore, $c'(v) \geq -2 + 1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$.

Case 3.4 If v is incident with exactly three 3-faces, say f_1, f_2 and f_3 , then $v_1v_2, v_2v_3, v_3v_4 \in E(G)$. If $d_{G^\times}(v_2) \leq 7$, then by Lemma 2, $\min\{d_{G^\times}(v_1), d_{G^\times}(v_3), d_{G^\times}(v_4)\} \geq 9$. So f_4 sends at least $\frac{2}{4-2} = 1$ to v by R1, each of v_1 and v_4 sends $\frac{1}{4}$ to v by R5 and v_3 sends at least $\frac{2}{3}$ to v by R6 and R7. Thus $c'(v) \geq -2 + 1 + 2 \times \frac{1}{4} + \frac{2}{3} > 0$. So we shall assume that $d_{G^\times}(v_2) \geq 8$. Similarly, we shall assume that $d_{G^\times}(v_3) \geq 8$. If both v_1 and v_4 are small, then by Lemma 2, $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} \geq 11$. It follows that each of v_2 and v_3 sends $\frac{3}{4}$ to v by R6. Moreover, f_4 sends at least $\frac{2}{4} = \frac{1}{2}$ to v . Thus $c'(v) \geq -2 + 2 \times \frac{3}{4} + \frac{1}{2} = 0$. So we assume that at least one of v_1 and v_4 is not small. It follows that f_4 sends at least $\frac{2}{4-1} = \frac{2}{3}$ to v by R1. If $d_{G^\times}(v_1) \leq 7$ or $d_{G^\times}(v_4) \leq 7$, then by Lemma 2, $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} \geq 9$. So by R6 and R7, each of v_2 and v_3 sends at least $\frac{2}{3}$ to v . Thus $c'(v) \geq -2 + \frac{2}{3} + 2 \times \frac{2}{3} = 0$. So we shall assume that $\min\{d_{G^\times}(v_1), d_{G^\times}(v_4)\} \geq 8$. It follows that f_4 sends at least $\frac{2}{4-2} = 1$ to v by R1. Moreover, each of v_2 and v_3 sends at least $\frac{1}{2}$ to v by R6, R7 and R8. Therefore, $c'(v) \geq -2 + 1 + 2 \times \frac{1}{2} = 0$.

Case 3.5 If v is incident with four 3-faces, then $v_1v_2, v_2v_3, v_3v_4, v_4v_1 \in E(G)$ and thus at most one of v_1, v_2, v_3 and v_4 is a 7^- -vertex by Lemma 2. Assume first that $d_{G^\times}(v_1) \leq 7$. Then all of v_2, v_3 and v_4 are 9^+ -vertices by Lemma 2. So by R6 and R7, each of v_2, v_3 and v_4 sends at least $\frac{2}{3}$ to v , which implies that $c'(v) \geq -2 + 3 \times \frac{2}{3} = 0$. Now assume that all of v_1, v_2, v_3 and v_4 are 8^+ -vertices. By R6, R7 and R8, each of those four vertices sends at least $\frac{1}{2}$ to v . This implies that $c'(v) \geq -2 + 4 \times \frac{1}{2} = 0$.

Case 4 $d = 5$.

By R1 and R3, v receives at least $\frac{2}{4} = \frac{1}{2}$ from each of its incident 4^+ -faces and $\frac{1}{3}$ from each of its adjacent true vertices in G^\times . We consider three sub-cases according to Lemma 8. If v is incident with at least two 4^+ -faces, then $c'(v) \geq -1 + 2 \times \frac{1}{2} = 0$. If v is adjacent to at least three true vertices in G^\times , then $c'(v) \geq -1 + 3 \times \frac{1}{3} = 0$. If v is incident with one 4^+ -face and adjacent to two true vertices in G^\times , then $c'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{3} > 0$.

Case 5 $d \geq 6$.

If $d \leq 7$, then it is trivial that $c'(v) = c(v) \geq 0$, so we assume that $d \geq 8$.

Let $S_f(v)$ denote the subgraph induced by the faces that are incident with v in G^\times . Then $S_f(v)$ can be decomposed into many parts, each of which is one of the five clusters in Figure 1, and any two parts of which are adjacent only if they have a coJPGmmon edge vw such that w is a true vertex. The hollow vertices in Figure 1 are false vertices and the solid ones are true vertices; all the marked faces are 4^+ -faces and there is at least one 4^+ -face contained in the clusters of type 2, 4 and 5.

Let a_i denote the largest possible value of the charges sent by v to or through its adjacent false vertices in a cluster of type i .

If $d = 8$, then by R8 and R9 we have $a_1 = \frac{1}{2}, a_2 = \frac{1}{12}, a_3 = 0, a_4 = 2 \times \frac{1}{12} = \frac{1}{6}$ and $a_5 = 0$.

If $9 \leq d \leq 11$, then by Lemma 2, v is adjacent to no 4^- -vertices in G . Thus by R4, R5, R6 and R7 we have $a_1 = \frac{3}{4}, a_2 = \frac{1}{4}, a_3 = 0, a_4 = 2 \times \frac{1}{4} = \frac{1}{2}$ and $a_5 = 0$.

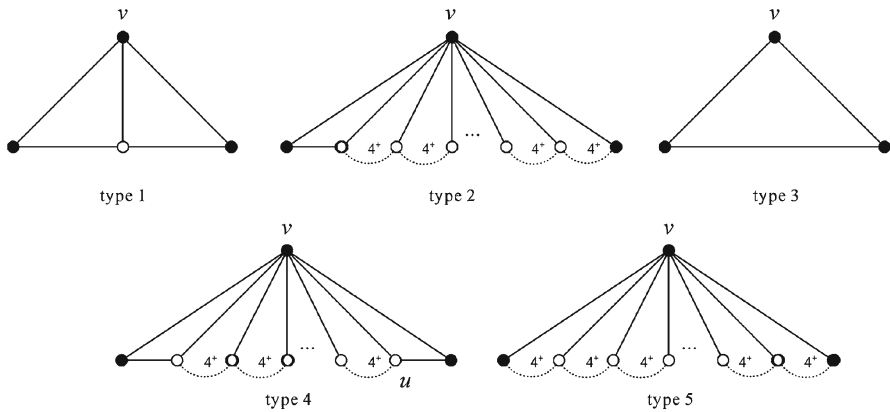


Fig. 1 Five types of cluster

If $d \geq 12$, then v may be adjacent to 4^- -vertices in G , to which v can send charges through the false vertices that are adjacent to v in G^\times . First of all, $a_1 = \max\{\frac{3}{4} + \frac{1}{24}, \frac{2}{3} + \frac{1}{8}\} = \frac{19}{24}$ by R6 and R7 and $a_3 = 0$. Let H_i ($i = 2, 4, 5$) be a cluster of type i . Suppose that there are s_i false vertices that are adjacent to v in H_i . By R4 and R5, we have $a_2 = \frac{1}{4} + \frac{1}{3}s_2$, $a_4 = 2 \times \frac{1}{4} + \frac{1}{3}s_4 = \frac{1}{2} + \frac{1}{3}s_4$ and $a_5 = \frac{1}{3}s_5$.

Denote by n_i the number of clusters of type i contained in $S_f(v)$. Let m be the total number of false vertices that are adjacent to v in the clusters of type 2, 4 and 5. One can easy to see that the following facts hold.

- (1) v is adjacent to $n_1 + n_2 + n_3 + n_4 + n_5$ true vertices in G^\times .
- (2) v is adjacent to $n_1 + m$ false vertices in G^\times .
- (3) $2n_1 + 2n_2 + n_3 + 3n_4 + n_5 \leq d$.

By (1) and (2), it is easy to see that $m = d - 2n_1 - n_2 - n_3 - n_4 - n_5$.

First of all, we calculate the largest possible value of the charges sent by v to or through its adjacent false vertices in G^\times , that is, the value of $n_1a_1 + n_2a_2 + n_3a_3 + n_4a_4 + n_5a_5$. Recall the values of a_i we have obtained in each of the above cases. One can deduce that

$$n_1a_1 + n_2a_2 + n_3a_3 + n_4a_4 + n_5a_5 = \frac{1}{2}n_1 + \frac{1}{12}n_2 + \frac{1}{6}n_4$$

if $d = 8$,

$$n_1a_1 + n_2a_2 + n_3a_3 + n_4a_4 + n_5a_5 = \frac{3}{4}n_1 + \frac{1}{4}n_2 + \frac{1}{2}n_4$$

if $9 \leq d \leq 11$, and

$$\begin{aligned} n_1a_1 + n_2a_2 + n_3a_3 + n_4a_4 + n_5a_5 &= \frac{19}{24}n_1 + \frac{1}{4}n_2 + \frac{1}{2}n_4 + \frac{1}{3}m \\ &= \frac{19}{24}n_1 + \frac{1}{4}n_2 + \frac{1}{2}n_4 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{3}(d - 2n_1 - n_2 - n_3 - n_4 - n_5) \\
 & = \frac{1}{3}d + \frac{1}{8}n_1 - \frac{1}{12}n_2 - \frac{1}{3}n_3 + \frac{1}{6}n_4 - \frac{1}{3}n_5.
 \end{aligned}$$

if $d \geq 12$.

Now, we calculate the largest possible value of the charges sent by v to its adjacent true small vertices in G^\times . Note that we should only consider the case $d \geq 11$ by Lemma 2. Since no two true small vertices are adjacent in G , in each cluster of type 1 or 3 v is adjacent to at most one true small vertex in G^\times . This implies that v is adjacent to at most $n_1 + n_2 + n_3 + n_4 + n_5 - \frac{1}{2}(n_1 + n_3) = \frac{1}{2}(n_1 + n_3) + n_2 + n_4 + n_5$ true small vertices in G^\times . Recall the definition of tri-neighbors at the beginning of this section. One can see that v can be tri-neighbors of at most n_3 vertices. Therefore, v sends at most

$$\frac{1}{6}(n_1 + n_3) + \frac{1}{3}(n_2 + n_4 + n_5) + \frac{1}{12}n_3$$

to its adjacent true small vertices in G^\times by R3. Note that R2 cannot be applied to v if $6 \leq d \leq 12$, since the application of R2 implies $\Delta = r \geq 13$ by Lemma 2, and that v may send $\frac{1}{2}$ to a common pot by R2 if $d \geq 13$.

We combine those lines of calculations. Let γ_d be the largest possible value of the charges sent by v if $d_G(v) = d$. We have

$$\begin{aligned}
 \gamma_8 &= \frac{1}{2}n_1 + \frac{1}{12}n_2 + \frac{1}{6}n_4 \\
 \gamma_9 = \gamma_{10} &= \frac{3}{4}n_1 + \frac{1}{4}n_2 + \frac{1}{2}n_4, \\
 \gamma_{11} &= \frac{3}{4}n_1 + \frac{1}{4}n_2 + \frac{1}{2}n_4 + \frac{1}{6}(n_1 + n_3) + \frac{1}{3}(n_2 + n_4 + n_5) + \frac{1}{12}n_3 \\
 &= \frac{11}{12}n_1 + \frac{7}{12}n_2 + \frac{1}{4}n_3 + \frac{5}{6}n_4 + \frac{1}{3}n_5, \\
 \gamma_{12} &= \frac{1}{3}d + \frac{1}{8}n_1 - \frac{1}{12}n_2 - \frac{1}{3}n_3 + \frac{1}{6}n_4 - \frac{1}{3}n_5 + \frac{1}{6}(n_1 + n_3) \\
 &\quad + \frac{1}{3}(n_2 + n_4 + n_5) + \frac{1}{12}n_3 \\
 &= 4 + \frac{7}{24}n_1 + \frac{1}{4}n_2 - \frac{1}{12}n_3 + \frac{1}{2}n_4, \quad \text{and} \\
 \gamma_d &= \frac{1}{3}d + \frac{1}{8}n_1 - \frac{1}{12}n_2 - \frac{1}{3}n_3 + \frac{1}{6}n_4 - \frac{1}{3}n_5 + \frac{1}{6}(n_1 + n_3) \\
 &\quad + \frac{1}{3}(n_2 + n_4 + n_5) + \frac{1}{12}n_3 + \frac{1}{2} \\
 &= \frac{1}{3}d + \frac{7}{24}n_1 + \frac{1}{4}n_2 - \frac{1}{12}n_3 + \frac{1}{2}n_4 + \frac{1}{2}
 \end{aligned}$$

if $d \geq 13$.

For each $8 \leq d \leq 12$, we consider the following program \mathcal{P}_d :

$$\begin{aligned} & \max && \gamma_d \\ & \text{s.t.} && 2n_1 + 2n_2 + n_3 + 3n_4 + n_5 \leq d \\ & && n_1, n_2, n_3, n_4, n_5, d \in \mathbb{Z}^+. \end{aligned}$$

Let q_d be the optimal value of the program \mathcal{P}_d .

Since $\gamma_8 \leq \frac{1}{4}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) \leq 2$, $q_8 \leq 2$.

Since $\gamma_9 \leq \frac{3}{8}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) - \frac{3}{8}(n_3 + n_4 + n_5) \leq 3$, $q_9 \leq 3$. Note that if $2n_1 + 2n_2 + n_3 + 3n_4 + n_5 = 9$, then $n_3 + n_4 + n_5 \geq 1$.

Since $\gamma_{10} \leq \frac{3}{8}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) \leq \frac{15}{4}$, $q_{10} \leq \frac{15}{4}$.

Since $\gamma_{11} \leq \frac{11}{24}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) - \frac{1}{8}(n_2 + n_3 + n_4 + n_5) \leq \frac{59}{12}$, $q_{11} \leq \frac{59}{12}$. Note that if $2n_1 + 2n_2 + n_3 + 3n_4 + n_5 = 11$, then $n_2 + n_3 + n_4 + n_5 \geq 1$.

Since $\gamma_{12} \leq 4 + \frac{1}{6}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) \leq 6$, $q_{12} \leq 6$.

Therefore, $c'(v) \geq d - 6 - q_d \geq 0$ for each $8 \leq d \leq 12$.

If $d \geq 13$, then $2n_1 + 2n_2 + n_3 + 3n_4 + n_5 \leq d$ implies $\gamma_d - (d - 6) \leq \frac{1}{6}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) - \frac{2}{3}d + \frac{13}{2} \leq \frac{13-d}{2} \leq 0$. Therefore, $c'(v) \geq d - 6 - \gamma_d \geq 0$ for $d \geq 13$.

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