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On the lightness of chordal 4-cycle in 1-planar graphs with high minimum degree*

Xin Zhang †
Department of Mathematics, Xidian University, Xi'an 710071, P. R. China

Guizhen Liu ‡
School of Mathematics, Shandong Univeristy, Jinan 250100, P. R. China

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Abstract

A graph G is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. The family of 1-planar graphs with minimum vertex degree at least δ and minimum edge degree at least ε is denoted by $\mathcal{P}_\delta^1(\varepsilon)$. In this paper, it is proved that every graph in $\mathcal{P}_7^1(14)$ (resp. $\mathcal{P}_6^1(13)$) contains a copy of chordal 4-cycle with all vertices of degree at most 10 (resp. 12).

Keywords: 1-planar graph, light graph, cycle, discharging.

Math. Subj. Class.: 05C75, 05C10

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of G, respectively. By $F(G)$, we denote the face set of G when G is a plane graph. If $uv \in E(G)$, then u is said to be the *neighbor* of v. We use $N_G(v)$ to denote the set of neighbors of a vertex v. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the value of $|N_G(v)|$, and the degree of an edge $uv \in E(G)$, denoted by $d_G(uv)$, is the value of $d_G(u) + d_G(v)$. A k-, k^+- and k^--*vertex* is a vertex of degree k, at least k and at most k, respectively. In this paper, C_k and P_k denotes a cycle and a path with k vertices and K_4^- denotes a chordal 4-cycle, which is a

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‡Supported in part by the National Natural Science Foundation of China (Grants No. 61070230).
E-mail addresses: xzhang@xidian.edu.cn (Xin Zhang), gzliu@sdu.edu.cn (Guizhen Liu)

graph obtained by removing an edge from a complete graph K_4. A cycle $C = [x_1 \cdots x_k]$ of a graph G is of the type $(d_1, \cdots, d_k)$ if $d_G(x_i) = d_i$ for $1 \leq i \leq k$. Similarly we can define cycles of the type $(\geq d_1, \cdots, \geq d_k)$, etc. For other undefined concepts we refer the readers to [2].

A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planarity was introduced by Ringel [16] while trying to simultaneously color the vertices and faces of a plane graph G such that any pair of adjacent/incident elements receive different colors. Note that we can construct, for given plane graph, a 1-planar graph G^1 whose vertex set is $V(G) \cup F(G)$ and any two vertices of G^1 are adjacent if and only if their corresponding elements in G are adjacent or incident.

Borodin proved that each 1-planar graph is 6-colorable (the bound 6 being sharp) [3, 4] which positively answered a conjecture raised by Ringel in [16], and that each 1-planar graph is acyclically 20-colorable [7]. The list analogue of vertex coloring of 1-planar graphs was investigated by Albertson and Mohar [1], and by Wang and Lih [18]. Zhang et al. showed that each 1-planar graph G with maximum degree Δ is Δ-edge-colorable if $\Delta \geq 10$ [25], or $\Delta \geq 9$ and G contains no chordal 5-cycles [19], or $\Delta \geq 8$ and G contains no chordal 4-cycles [20], or $\Delta \geq 7$ and G contains no 3-cycles [21], is $(\Delta+1)$-edge-choosable and $(\Delta+2)$-total-choosable if $\Delta \geq 16$ [27], is Δ-edge-choosable and $(\Delta+1)$-total-choosable if $\Delta \geq 21$ [27]. Zhang et al. also showed that the $(p, 1)$-total labelling number of each 1-planar graph G is at most $\Delta(G) + 2p - 2$ if $\Delta(G) \geq 8p + 4$ [28], and the linear arboricity of each 1-planar graph G is exactly $\lceil \Delta(G)/2 \rceil$ if $\Delta(G) \geq 33$ [24].

Another topic concerning 1-planar graphs is to investigate their global and local structures. In [9], it is shown that each 1-planar graph with n vertices has at most $4n - 8$ edges and this upper bound is tight, which implies that the minimum vertex degree of any 1-planar graph is at most 7.

Let H be a connected graph and $\mathcal{G}$ be a family of graphs. If for any graph $G \in \mathcal{G}$, G contains a subgraph $K \simeq H$ such that

$$\max_{x \in V(K)} \{d_G(x)\} \leq t_h < +\infty \text{ and } \sum_{x \in V(K)} d_G(x) \leq t_w < +\infty, \tag{1.1}$$

then we say that H is *light* in $\mathcal{G}$, and otherwise say that H is *heavy* in $\mathcal{G}$. The smallest integers t_h and t_w satisfying (1.1) are called the *height* and the *weight* of H in the family $\mathcal{G}$, denoted by $h(H, \mathcal{G})$ and $w(H, \mathcal{G})$, respectively. By $\mathcal{L}(\mathcal{G})$, we denote the set of light graphs in the family $\mathcal{G}$. Throughout this paper, $\mathcal{P}_\delta^1(\varepsilon)$ (resp. $\mathcal{P}_\delta(\varepsilon)$) denotes the family of 1-planar graphs (resp. planar graphs) with minimum vertex degree at least δ and minimum edge degree at least ε. If $\varepsilon = 2\delta$, we use the natation $\mathcal{P}_\delta^1$ (resp. $\mathcal{P}_\delta$) for short to represent $\mathcal{P}_\delta^1(\varepsilon)$ (resp. $\mathcal{P}_\delta(\varepsilon)$). Note that for the parameter $\mathcal{P}_\delta^1(\varepsilon)$, we need to assume that $\delta \leq 7$ and $\varepsilon \geq 2\delta$.

The first complete description of the set of light graphs in the family of 1-planar graphs with high minimum degree was given in [9, 22]; there was proved that $\mathcal{L}(\mathcal{P}_4^1) = \{P_1, P_2, P_3\}$. Fabrici and Madaras [9], Zhang, Liu and Wu [22], and Dong [8] together proved that $\mathcal{L}(\mathcal{P}_5^1) = \{P_1, P_2, P_3, P_4, S_3\}$, where S_3 is a 3-star. For the lightness of some graphs in the family $\mathcal{P}_\delta^1$ where $6 \leq \delta \leq 7$, the readers can refer to [6, 9, 11, 10, 12, 13, 17, 22, 26, 23].

In this paper, we investigate the lightness of some graphs in $\mathcal{P}_\delta^1(\varepsilon)$ with not only $\varepsilon = 2\delta$ but also $\varepsilon > 2\delta$, the later case of which has not been considered for the family of 1-planar graphs even before. Our motivation comes from the analogical results for planar graphs with minimum degree δ and minimum edge degree $2\delta+1$ where $\delta \in \{3,4\}$. For example, Borodin proved in [5] that $\mathcal{L}(\mathcal{P}_3(7)) = \{P_1, P_2, P_3\}$ (also proved in [14] by Madaras and Škrekovski), and Mohar et al. [15] presented some light subgraphs in the class $\mathcal{P}_4(9)$.

In what follows, we show in Section 2.3 that K_4^- is light in the family $\mathcal{P}_7^1$ as well as in its superfamily $\mathcal{P}_6^1(13)$, and its height is at most 10 and at most 12, respectively.

2 The lightness of chordal 4-cycle

2.1 Basic terms

In the following, we always assume that G is a 1-planar graph that has been drawn on a plane so that every edge is crossed by at most one another edge and the number of crossings is as small as possible. The *associated plane graph* $G^\times$ of G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is *false* if it is not a vertex of G; otherwise, it is *true*. Similarly, by false (resp. true) face, we mean a face in $G^\times$ that is incident with at least one false (resp. no false) vertices. Let v and f be a vertex and a face in $G^\times$. The function $\zeta(v)$ (resp. $\zeta(f)$) denotes the number of false vertices that are adjacent to v (resp. incident with f) in $G^\times$.

For convenience, we introduce some specialized notations. Let v be a false vertex in $G^\times$ and let v_1, v_2, v_3, v_4 be its neighbors in a clockwise order. Define f_i to be the face incident with vv_i and vv_{i+1}, where subscripts are taken modulo 4. Note that if $d(f_i)=3$, then $v_iv_{i+1} \in E(G)$. In this case, let f_i' be the other face incident with the edge v_iv_{i+1}. If $d(f_i')=3$, then its third vertex will be denoted by v_i'. Thus v_i' is a false vertex if and only if f_i' is false, in which case we denote a neighbor of v_i (resp. v_{i+1}) in G to be v_i'' (resp. v_{i+1}''), so that v_iv_i'' and $v_{i+1}v_{i+1}''$ are two edges in G that crossed by each other at the point v_i'. Denote the face that is incident with the path $v_iv_i'v_{i+1}''$ (resp. $v_{i+1}v_i'v_i''$) in $G^\times$ by f_i^L (resp. f_i^R).

While proving the lightness of a graph in a given family of graphs, usually, the discharging method is used. In the proof of this paper, based on this method we consider a hypothetical counterexample G (a 1-planar graph) and then construct its associated plane graph $G^\times$. We first assign an initial charge c to each element $x \in V(G^\times)\bigcup F(G^\times)$ as follows:

$$c(x) = \begin{cases} \alpha d_{G^\times}(x) - 2(\alpha+\beta), & \text{if } x \in V(G^\times); \\ \beta d_{G^\times}(x) - 2(\alpha+\beta), & \text{if } x \in F(G^\times), \end{cases} \tag{2.1}$$

where α and β are some prescribed positive numbers. By combining the Euler formula $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 2$ on $G^\times$ and the relation $\sum_{v\in V(G^\times)} d_{G^\times}(v) = \sum_{f\in F(G^\times)} d_{G^\times}(f) = 2|E(G^\times)|$, we have $\sum_{x\in V(G^\times)\bigcup F(G^\times)} c(x) = -4(\alpha+\beta) < 0$. We then redistribute the charge of the vertices and the faces of $G^\times$ according to some discharging rules, which only move charge around but do not affect the total charges so that, after discharging, the final charge c' of each element in $V(G^\times)\bigcup F(G^\times)$ is nonnegative. This leads to a contradiction that $\sum_{x\in V(G^\times)\bigcup F(G^\times)} c(x) = \sum_{x\in V(G^\times)\bigcup F(G^\times)} c'(x) \geq 0$ and completes the proof.

2.2 A key discharging lemma

Let G be a 1-planar graph and let v be a true vertex in its associated plane graph $G^\times$. Denote $F(v)$ to be the subgraph induced by the faces that are incident with v. Note that $F(v)$ can be decomposed into many parts, each of which is one of the five clusters in Figure

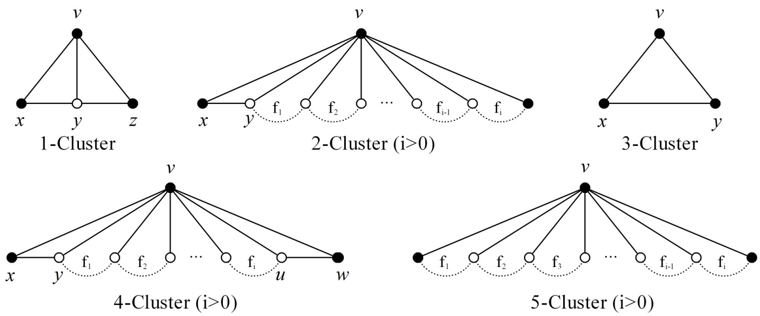

Figure 1: $F(v)$ can be decomposed into the combination of the above five clusters

1, and any two parts of which are adjacent only if they have a common edge vw such that w is a true vertex. The hollow vertices in Figure 1 are false and the solid ones are true, and all the faces marked by f_i are 4^+-faces.

Lemma 2.1. *Let v be a 6^+-vertex in the associated plane graph $G^\times$ of a 1-planar graph G. Assign v an initial charge $c(v) = \alpha d_{G^\times}(v) - 2(\alpha + \beta)$, where α and β are some prescribed positive numbers satisfying $2\alpha \geq \beta$. Suppose that v sends out charges only by the following three discharging rules:*

Rule A *v transfers a charge of $\frac{2\alpha-\beta}{3}$ to each incident 3-face in $G^\times$;*

Rule B *If v is incident with two adjacent 3-faces $f_1 = [xvz]$ and $f_2 = [yvz]$ so that z is a false vertex in $G^\times$, then v sends the charge λ to z;*

Rule C *If v is incident with a 3-face $f_1 = [xvz]$ sharing a common edge vz with a 4^+-face f_2 so that z is a false vertex in $G^\times$, then v sends the charge μ to z.*

Denote $c'(v)$ to be the final charge of v after applying the above rules. If $d_{G^\times}(v)$ is even with

$$\lambda \leq \left(\frac{2}{3} - \frac{4}{d_{G^\times}(v)}\right)(\alpha + \beta), \tag{2.2}$$

$$\mu = \frac{3}{4}\lambda, \tag{2.3}$$

or $d_{G^\times}(v) \geq 9$ is odd with

$$\lambda \leq \left(\frac{2d_{G^\times}(v) - 12}{3d_{G^\times}(v) - 3}\right)(\alpha + \beta), \tag{2.4}$$

$$\mu = \frac{1}{2}\lambda, \tag{2.5}$$

or $d_{G^\times}(v) = 7$ with

$$\lambda = \mu = \frac{\alpha + \beta}{12}, \tag{2.6}$$

then $c'(v) \geq 0$.

Proof. Denote n_i to be the number of i-clusters contained in $F(v)$ and m_i to be the charges sent out from v through an i-cluster. By their definitions, one can easily observe that

$$2n_1 + 2n_2 + n_3 + 3n_4 + n_5 \leq d_{G^\times}(v). \tag{2.7}$$

For the case when $d_{G^\times}(v)$ is even, by (2.7) and the choices of λ, μ as in (2.2) and (2.3), we have

$$\begin{aligned}
c'(v) &= c(v) - \frac{2\alpha - \beta}{3} d_{G^\times}(v) - \sum_{i=1}^{5} n_i m_i \\
&= \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \lambda n_1 - \mu n_2 - 2\mu n_4 \\
&\geq \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \frac{\lambda}{2}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) \\
&\geq \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \frac{\lambda}{2} d_{G^\times}(v) \\
&\geq \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \left(\frac{1}{3} - \frac{2}{d_{G^\times}(v)}\right)(\alpha + \beta) d_{G^\times}(v) \\
&= 0.
\end{aligned}$$

Now, we consider the case when $d_{G^\times}(v)$ is odd. Here, note that

$$n_2 + n_3 + n_4 + n_5 \geq 1 \tag{2.8}$$

since any copy of a 1-cluster consists even number of faces incident with v. By (2.7), (2.8) and the choices of λ, μ as in (2.4) and (2.5), we have

$$\begin{aligned}
c'(v) &= c(v) - \frac{2\alpha - \beta}{3} d_{G^\times}(v) - \sum_{i=1}^{5} n_i m_i \\
&= \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \lambda n_1 - \mu n_2 - 2\mu n_4 \\
&= \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \frac{\lambda}{2}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) + \\
&\quad \frac{\lambda}{2}(n_2 + n_3 + n_4 + n_5) \\
&\geq \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \frac{\lambda}{2} d_{G^\times}(v) + \frac{\lambda}{2} \\
&\geq \frac{\alpha + \beta}{3} d_{G^\times}(v) - 2(\alpha + \beta) - \left(\frac{d_{G^\times}(v) - 6}{3d_{G^\times}(v) - 3}\right)(\alpha + \beta)(d_{G^\times}(v) - 1) \\
&= 0.
\end{aligned}$$

For the particular case when $d_{G^\times}(v) = 7$, we can deduce from (2.7) that

$$n_1 + n_2 + 2n_4 = \left\lfloor \frac{2n_1 + 2n_2 + 3n_4}{2} \right\rfloor + \left\lceil \frac{1}{2} n_4 \right\rceil \leq \left\lfloor \frac{7}{2} \right\rfloor + \left\lceil \frac{1}{2} \left\lfloor \frac{7}{3} \right\rfloor \right\rceil = 4. \tag{2.9}$$

Thus by (2.9) along with the choices of λ and μ as in the equation (2.6), we have

$$\begin{aligned} c'(v) &= c(v) - \frac{2\alpha-\beta}{3} d_{G^\times}(v) - \sum_{i=1}^{5} n_i m_i \\ &= \frac{\alpha+\beta}{3} - \lambda n_1 - \mu n_2 - 2\mu n_4 \\ &= \frac{\alpha+\beta}{3} - \lambda(n_1 + n_2 + 2n_4) \\ &\geq \frac{\alpha+\beta}{3} - 4\lambda \\ &= 0. \end{aligned}$$

Consequently, we complete the proof of this lemma. □

2.3 The height of chordal 4-cycle in $\mathcal{P}_6^1(13)$ and $\mathcal{P}_7^1$

Theorem 2.2. *Each 1-planar graph with minimum degree at least 6 contains at least one of the following configurations:*

(a) *a pair of adjacent vertices of degree 6;*

(b) *a 4-cycle* $C = [x_1x_2x_3x_4]$ *of the type* $(6, \leq 12, \leq 8, \leq 12)$ *with a chord* x_1x_3*;*

(c) *a 4-cycle* $C = [x_1x_2x_3x_4]$ *of the type* $(7, \leq 10, \leq 8, \leq 10)$ *with a chord* x_1x_3.

Proof. The proof of the theorem is carried out by the discharging method as described in Section 2.1. Suppose G is a counterexample to the theorem. Consider the associated plane graph $G^\times$ of G. Assign the charges to each element $x \in V(G^\times) \cup F(G^\times)$ as mentioned in the inequation (2.1) of Section 2.1 by choosing $\alpha = 2$ and $\beta = 3$. If v is a true vertex in G, then $d_{G^\times}(v) = d_G(v)$, so in the following we use $d(v)$ for short to represent both of the two notions. A *big* vertex, *semi-big* vertex, *intermediate* vertex and *semi-intermediate* vertex refer to a vertex $v \in V(G^\times)$ with $d(v) \geq 13$, $d(v) \geq 11$, $6 \leq d(v) \leq 12$ and $6 \leq d(v) \leq 10$, respectively. Therefore, a true vertex in $G^\times$ is either big or intermediate, and an intermediate vertex in $G^\times$ is either semi-big or semi-intermediate. By *big* face, we denote a face $f \in F(G^\times)$ with degree at least 4. Now, we define the discharging rules as follows.

Rule 1 Each 6^+-vertex sends $\frac{1}{3}$ to each incident face;

Rule 2 Each 4-vertex sends $\frac{1}{3}$ to each incident false 3-face;

Rule 3 Each big face sends $\frac{11}{12}$ to each incident 4-vertex;

Rule 4 Let f be a big face having a common edge xy with a false 3-face $g = [xyz]$. If z is a 4-vertex, then f sends $\frac{5}{12}$ to z through xy;

Rule 5 Let f be a big face having a common edge xy with a false 3-face $g = [xyz]$. If x is a 4-vertex and yz is incident with another false 3-face $h = [yzu]$, then f sends $\frac{5}{24}$ to u through xy and yz;

Rule 6 Let $f = [xyz]$ be a true 3-face having a common edge yz with a false 3-face $g = [uyz]$. If $d(x) \geq 11$, then x sends $\frac{5}{12}$ to u through yz;

Rule 7 Let $f = [xyz]$ and $g = [uyz]$ be two adjacent false 3-faces, and let z be a 4-vertex. Suppose yu is incident with another false 3-face $h = [yuw]$ so that yy' crosses uu' in G at w. If at least one of the following four occasions appears in $G^\times$

- $d(x) \geq 13$, $\min\{d(u), d(y)\} = 6$, $\max\{d(u), d(y)\} \leq 8$ and y, u, u' are all intermediate vertices with $yu' \in E(G^\times)$;
- $d(x) \geq 13$, $\min\{d(u), d(y)\} = 6$, $\max\{d(u), d(y)\} \leq 8$ and y, u, y' are all intermediate vertices with $uy' \in E(G^\times)$;
- $d(x) \geq 11$, $\min\{d(u), d(y)\} = 7$, $\max\{d(u), d(y)\} \leq 8$ and y, u, u' are all semi-intermediate vertices with $yu' \in E(G^\times)$;
- $d(x) \geq 11$, $\min\{d(u), d(y)\} = 7$, $\max\{d(u), d(y)\} \leq 8$ and y, u, y' are all semi-intermediate vertices with $uy' \in E(G^\times)$,

then x sends $\frac{5}{24}$ to w through yz and yu;

Rule 8 Let $f = [xyz]$ and $g = [uyz]$ be two adjacent false 3-faces. If z is a 4-vertex, then y sends to z a charge of

$$\begin{cases} \frac{5}{12}, & \text{if } d(y) = 7; \\ \frac{5}{6}, & \text{if } d(y) = 8; \\ \frac{5}{4}, & \text{if } 9 \leq d(y) \leq 12; \\ \frac{5}{3}, & \text{if } d(y) \geq 13; \end{cases}$$

Rule 9 Let $f = [xyz]$ be a false 3-face having a common edge yz with a big face β. If z is a 4-vertex, then y sends to z a charge of

$$\begin{cases} \frac{5}{12}, & \text{if } d(y) = 7; \\ \frac{5}{8}, & \text{if } 8 \leq d(y) \leq 12; \\ \frac{5}{6}, & \text{if } d(y) \geq 13. \end{cases}$$

In the following, we estimate the final charge c' of vertices and faces after the charge redistribution and prove $c'(x) \geq 0$ for each $x \in V(G^\times) \bigcup F(G^\times)$. By Rules 1 and 2, any 3-face f in $G^\times$ receive $\frac{1}{3}$ from each of its incident vertices, which implies the final charge of f is exactly zero. For a big face f in $G^\times$ (recall that $\zeta(f)$ denotes the number of 4-vertices incident with f), it would send $\frac{11}{12}\zeta(f)$ to its incident 4-vertices by Rule 3. Besides, if f is incident with a 4-vertex v, then f send out $2 \times \frac{5}{24} = \frac{5}{12}$ through uv and vw by Rule 5, where u and w denote the neighbors of v on the boundary of f. Since f is incident with $d(f) - 2\zeta(f)$ true edges (namely, an edge of $G^\times$ containing no 4-vertex), by Rule 4, a total charge of $\frac{5}{12}(d(f) - 2\zeta(f))$ would be sent out from f through the true edges incident with f. On the other hand, f receive $\frac{1}{3}$ from each of $d(f) - \zeta(f)$ true vertices incident with it. Since $\zeta(f) \leq \frac{d(f)}{2}$, $c'(f) \geq 3d(f) - 10 - \frac{11}{12}\zeta(f) - \frac{5}{12}\zeta(f) - \frac{5}{12}(d(f) - 2\zeta(f)) + \frac{1}{3}(d(f) - \zeta(f)) = \frac{35}{12}d(f) - \frac{5}{6}\zeta(f) - 10 \geq \frac{5}{2}d(f) - 10 \geq 0$ for $d(f) \geq 4$.

By Lemma 2.1 along with Rules 1,8 and 9, one can check that $c'(v) \geq 0$ for all vertices of degree between 6 and 10. For a big vertex v in $G^\times$, denote $F(v)$ to be the subgraph induced by the faces that are incident with v. As we state at the beginning of Section 2.2, $F(v)$ can be decomposed into a combination of the five clusters in Figure 1. By n_i and m_i, we denote the number of i-clusters contained in $F(v)$ and the charges sent out from v through an i-cluster. If there is a 2-cluster in $F(v)$, then v send $\frac{5}{6}$ to y (see Figure 1) by Rule 9 and at most $\frac{5}{24}$ through xy by Rule 7, so $m_2 \leq \frac{5}{6} + \frac{5}{24} = \frac{25}{24}$. Similarly, we can prove that $m_3 \le \frac{5}{12}$ by Rule 6, $m_4 \le 2 \times \frac{5}{6} + 2 \times \frac{5}{24} = \frac{25}{12}$ by Rules 7, 9 and $m_5 = 0$. We now estimate the value of m_1 much more carefully. First, we show the following observation.

Observation. *If v is incident with a 1-cluster as in Figure 1 and has sent out some charge through xy by Rule 7, then it would not send out any charge through the edge yz.*

Proof. Denote u to be the neighbor of v in G such that uv crosses xz in G at the point y. Suppose, on the contrary, that v send out some charge through yz. By the definitions of the rules, uyz is a 3-face in $G^\times$ and z is an intermediate vertex in $G^\times$. Since v has sent out charge through xy by Rule 7, by the definition of Rule 7, we have $xu \in E(G^\times)$, $\min\{d(x), d(u)\} = 6$ and $\max\{d(x), d(u)\} \le 8$. Furthermore, xu is also incident with a 3-cycle, say xuw, in G such that w is an intermediate vertex in $G^\times$ different from z. Now, the four distinct vertices x, z, u, w form a 4-cycle $[uwxz]$ in G with a chord ux, and therefore, the configuration (b) occurs in G. This contradiction verifies this observation. $\|$

By Rules 7, 8 and the above observation, we immediately have $m_1 \le \frac{5}{3} + \frac{5}{24} = \frac{15}{8}$. Therefore, by Rule 1 and the inequality (2.7) in Section 2.1, we have $c'(v) \ge 2d(v) - 10 - \frac{1}{3}d(v) - \frac{15}{8}n_1 - \frac{25}{24}n_2 - \frac{5}{12}n_3 - \frac{25}{12}n_4 \ge \frac{5}{3}d(v) - 10 - \frac{15}{16}(2n_1 + 2n_2 + n_3 + 3n_4 + n_5) + \frac{25}{48}(n_2 + n_3 + n_4 + n_5) \ge \frac{35}{48}d(v) - 10 + \frac{25}{48}(n_2 + n_3 + n_4 + n_5) > 0$ for $d(v) \ge 14$. If $d(v) = 13$, then by the inequality (2.8) in Section 2.1, we also have $c'(v) \ge \frac{35}{48}d(v) - 10 + \frac{25}{48} = 0$ in final. For vertices of degree 11 or 12 (they are semi-big but not big), we can also check the nonnegativity of their final charges. Proof of them are left to the readers, since they use the same argument as in the previous analysis on the big vertices.

Now, the only missed case is when v is a 4-vertex in $G^\times$ (namely, v is a false vertex). As we know, the initial charge of a 4-vertex v is -2, so if v is incident with at least three big faces, then by Rules 2 and 3, the final charge $c'(v)$ of v is at least $-2-\frac{1}{3}+3\times\frac{11}{12} = \frac{5}{12} > 0$. In the following, we discuss three other cases.

Case 1. v is incident with exactly two 3-faces.

First, suppose that f_1 and f_2 are 3-faces. Since no two 6-vertices are adjacent in G, at least two of v_1, v_2 and v_3 are 7^+-vertices. Thus by Rules 8 and 9, each of the two 7^+-vertices among v_1, v_2 and v_3 would send at least $\frac{5}{12}$ to v. Therefore, $c'(v) \ge -2 - 2 \times \frac{1}{3} + 2 \times \frac{11}{12} + 2 \times \frac{5}{12} = 0$ by Rules 2, 3, 8 and 9.

Second, suppose that f_1 and f_3 are 3-faces. In this case, one can also show that there are at least two 7^+-vertices among v_1, v_2, v_3 and v_4. Thus by Rules 2, 3, 8 and 9, we still have $c'(v) \ge -2 - 2 \times \frac{1}{3} + 2 \times \frac{11}{12} + 2 \times \frac{5}{12} = 0$.

Case 2. v is incident with exactly three 3-faces.

Without loss of generality, we assume that f_1, f_2, f_3 are 3-faces and f_4 is a 4^+-face (recall the definitions of f_i in Section 2.1). By Rule 3, f_4 shall send $\frac{11}{12}$ to v.

First, suppose that at least two of v_1, v_2, v_3 and v_4, say v_1 and v_4 (other cases can be dealt with similarly), are big vertices. Thus at least one of v_2 and v_3 must be 7^+-vertex since they are adjacent in G. Therefore, by Rules 2, 3 and 9, $c'(v) \ge -2-3\times\frac{1}{3}+2\times\frac{5}{6}+\frac{5}{12}+\frac{11}{12} = 0$.

Next, suppose that only one of v_1, v_2, v_3 and v_4 is big vertex. If v_2 or v_3, say v_2, is big, then at least one of v_1, v_3 and v_4 should be a 7^+-vertex since they three form a 3-path in G. By Rules 2, 3, 8 and 9, $c'(v) \ge -2 - 3 \times \frac{1}{3} + \frac{5}{3} + \frac{5}{12} + \frac{11}{12} = 0$. If v_1 or v_4, say v_1, is big, then all of v_2, v_3 and v_4 are intermediate. If they are all 7^+-vertices, then $c'(v) \ge -2 - 3 \times \frac{1}{3} + \frac{5}{6} + 3 \times \frac{5}{12} + \frac{11}{12} = 0$ by Rules 2, 8 and 9. Thus we assume that at least one of v_2, v_3 and v_4 is a 6-vertex. Here, we only consider the case when $d(v_3) = 6$ and leave the discussions on the rest two cases to the readers, since they are quite similar. First, suppose $d(v_2) \geq 9$. By Rules 3 and 8, v_1, v_2 and f_4 shall send $\frac{5}{6}$, $\frac{5}{4}$ and $\frac{11}{12}$ to v, respectively, thus, $c'(v) \geq -2 - 3 \times \frac{1}{3} + \frac{5}{6} + \frac{5}{4} + \frac{11}{12} = 0$. Second, suppose $d(v_2) \leq 8$. We now consider the face f_2' (recall its definition in Section 2.1). If f_2' is a big face in $G^\times$, then by Rule 4, f_2' sends $\frac{5}{12}$ to v through the edge v_2v_3. If f_2' is a true 3-face, then v_2' (recall the corresponding definition in Section 2.1) must be a big vertex, because otherwise the configuration (b) would appear in G, meanwhile, v receives $\frac{5}{12}$ from v_2' through the edge v_2v_3 by Rule 6. If f_2' is false 3-face, then we consider the faces f_2^L and f_2^R (recall their definitions in Section 2.1). If f_2^L is a big face, then by Rule 5, f_2^L sends $\frac{5}{24}$ to v through the edge v_2v_3. If f_2^L is a 3-face, then it must be false since it is incident with a false vertex v_2'. Since v_2, v_3, v_4 and v_3'' (recall the definitions of v_i'' in Section 2.1) form a chordal 4-cycle with a chord v_2v_3 in G, v_3'' must be a big vertex, and then v shall receive $\frac{5}{24}$ from v_3'' through the edge v_2v_3 by Rule 7. Similarly, v would receive another $\frac{5}{24}$ through the edge v_2v_3 from either the face f_2^R or the vertex v_2''. Hence, through the edge v_2v_3, v shall totally receive a charge of $2 \times \frac{5}{24} = \frac{5}{12}$. Since neither v_2 nor v_4 can be a 6-vertex (because each of them is adjacent to the 6-vertex v_3 in G), each of v_2 and v_4 shall send $\frac{5}{12}$ to v by Rules 8 and 9. Thus, by Rules 2, 3 and 9, we still have $c'(v) \geq -2 - 3 \times \frac{1}{3} + \frac{5}{12} + 2 \times \frac{5}{12} + \frac{5}{6} + \frac{11}{12} = 0$.

At last, suppose none of v_1, v_2, v_3 and v_4 is big. If v_2 or v_3, say v_2, is a 6-vertex, then v_1, v_3 and v_4 are 7^+-vertices since each of them is adjacent to v_2. Furthermore, $d(v_3) \geq 9$, because otherwise the configuration (b) would appear in G, so by Rules 2, 3, 8 and 9, we have $c'(v) \geq -2 - 3 \times \frac{1}{3} + 2 \times \frac{5}{12} + \frac{5}{4} + \frac{11}{12} = 0$. We now assume $\min\{d(v_2), d(v_3)\} \geq 7$. If $\max\{d(v_2), d(v_3)\} \geq 9$ (without loss of generality, assume $d(v_3) \geq 9$), then by Rules 2, 3, 8 and 9, $c'(v) \geq -2 - 3 \times \frac{1}{3} + 2 \times \frac{5}{4} + \frac{11}{12} > 0$ when $d(v_2) \geq 9$, $c'(v) \geq -2 - 3 \times \frac{1}{3} + 2 \times \frac{5}{12} + \frac{5}{4} + \frac{11}{12} = 0$ when $d(v_2) \leq 8$ and $d(v_1) \geq 7$, and $c'(v) \geq -2 - 3 \times \frac{1}{3} + \frac{5}{12} + \frac{5}{12} + \frac{5}{4} + \frac{11}{12} = 0$ when $d(v_2) \leq 8$ and $d(v_1) = 6$ (note that in this case, a charge of at least $\frac{5}{12}$ shall be transferred to v through the edge v_1v_2). Therefore, we assume $\max\{d(v_2), d(v_3)\} \leq 8$ in the following. First, suppose at least one of v_2 and v_3 is a 8-vertex. Without loss of generality, suppose $d(v_2) = 7$ and $d(v_3) = 8$. By Rule 8, v_2 and v_3 shall send $\frac{5}{12}$ and $\frac{5}{6}$ to v, respectively. By a similar argument as above, v shall also receive $\frac{5}{12}$ either from the vertex v_1 when $d(v_1) \geq 7$ or through the edge v_1v_2 when $d(v_{(1}) = 6$, and another $\frac{5}{12}$ either from the vertex v_4 when $d(v_4) \geq 7$ or through the edge v_3v_4 when $d(v_{(4}) = 6$. Thus, by Rules 2 and 3, we have $c'(v) \geq -2 - 3 \times \frac{1}{3} + \frac{5}{12} + \frac{5}{6} + 2 \times \frac{5}{12} + \frac{11}{12} = 0$.

Second, suppose $d(v_2) = d(v_3) = 7$. Under this hypothesis, at least one of v_1 and v_4 should be semi-big, because otherwise a configuration (c) would appear in G. If v_1 and v_4 are 8^+-vertices, then by Rules 2, 3, 8 and 9, $c'(v) \geq -2 - 3 \times \frac{1}{3} + 2 \times \frac{5}{12} + 2 \times \frac{5}{8} + \frac{11}{12} = 0$. If one of v_1 and v_4, say v_4, is a 7^--vertex, then by a similar argument as before, we can show that v receives $\frac{5}{12}$ through the edges v_1v_2 and another $\frac{5}{12}$ through the edges v_2v_3. Therefore, by Rules 2, 3, 8 and 9, we have $c'(v) \geq -2 - 3 \times \frac{1}{3} + 2 \times \frac{5}{12} + \frac{5}{8} + 2 \times \frac{5}{12} + \frac{11}{12} > 0$.

Case 3. v is incident with four 3-faces.

If at least two of v_1, v_2, v_3 and v_4 are big vertices, then by Rules 2 and 8, $c'(v) \geq -2 - 4 \times \frac{1}{3} + 2 \times \frac{5}{3} = 0$.

If only one of v_1, v_2, v_3, v_4, say v_1, is a big vertex, then we can assume that v_2, v_3 and v_4 are 7^+-vertices. Otherwise, without loss of generality, suppose $d(v_2) = 6$. Since no two 6-vertices are adjacent in G, $d(v_3) \geq 7$ and $d(v_4) \geq 7$. If v_3 and v_4 are 8^+-vertices, then $c'(v) \geq -2 - 4 \times \frac{1}{3} + \frac{5}{3} + 2 \times \frac{5}{6} = 0$ by Rules 2 and 8. We now assume that one of v_3 and v_4, say v_3, is a 7-vertex. If now $d(v_4) \geq 9$, then $c'(v) \geq -2 - 4 \times \frac{1}{3} + \frac{5}{3} + \frac{5}{12} + \frac{5}{4} = 0$

Rules 2 and 8. If $d(v_4) \leq 8$, then by a similar argument as in Case 1, we can show that v receives $\frac{5}{12}$ through the edges v_2v_3 and another $\frac{5}{12}$ through the edges v_3v_4. Therefore, by Rules 2 and 8, $c'(v) \geq -2 - 4 \times \frac{1}{3} + \frac{5}{3} + 2 \times \frac{5}{12} + 2 \times \frac{5}{12} = 0$. Hence, we can assume $\min\{d(v_2), d(v_3), d(v_4)\} \geq 7$. If at least one of v_2, v_3 and v_4 is a 8^+-vertex, then by Rules 2 and 8, $c'(v) \geq -2 - 4 \times \frac{1}{3} + 2 \times \frac{5}{12} + \frac{5}{6} = 0$. If $d(v_2) = d(v_3) = d(v_4) = 7$, then by a similar argument as in Case 1, one can show that a charge of $\frac{5}{12}$ would be transferred to v through each of the edges v_2v_3 and v_3v_4. Hence, by Rules 2 and 8, we have $c'(v) \geq -2 - 4 \times \frac{1}{3} + 3 \times \frac{5}{12} + 2 \times \frac{5}{12} > 0$.

We now consider the last case when v_1, v_2, v_3 and v_4 are intermediate vertices. If they all are 8^+-vertices, then by Rules 2 and 8, $c'(v) \geq -2 - 4 \times \frac{1}{3} + 4 \times \frac{5}{6} = 0$. If one of them, say v_1, is a 6-vertex, then v_2, v_3 and v_4 are 9^+-vertex, because otherwise the configuration (b) would appear in G. This implies that $c'(v) \geq -2 - 4 \times \frac{1}{3} + 3 \times \frac{5}{4} > 0$ by Rules 2 and 8. We now assume that $d(v_1) = 7$ and $\min\{d(v_2), d(v_3), d(v_4)\} \geq 7$. If at least two of v_2, v_3 and v_4 are 9^+-vertices, then by Rules 2 and 8, $c'(v) \geq -2 - 4 \times \frac{1}{3} + 2 \times \frac{5}{12} + 2 \times \frac{5}{4} = 0$. Thus, we assume that at least two of v_2, v_3, v_4, say v_2 and v_3, are 8^--vertices. In this case, v_4 should be a semi-big vertex because otherwise the configuration (c) would occur in G. If $d(v_2) = d(v_3) = 8$, then by Rules 2 and 8, $c'(v) \geq -2 - 4 \times \frac{1}{3} + \frac{5}{12} + 2 \times \frac{5}{6} + \frac{5}{4} = 0$. If $\min\{d(v_2), d(v_3)\} = 7$, then by a similar argument as in Case 1, one can prove that a charge of $\frac{5}{12}$ would be transferred to v through each of the edges v_1v_2 and v_2v_3. This implies that $c'(v) \geq -2 - 4 \times \frac{1}{3} + 3 \times \frac{5}{12} + 2 \times \frac{5}{12} + \frac{5}{4} = 0$.

Hence, we deduce that $\sum_{x \in V(G^\times) \bigcup F(G^\times)} c'(x) \geq 0$. This contradiction completes the proof. □

Corollary 2.3. *$K_4^- \in \mathcal{L}(\mathcal{P}_6^1(13))$ and $h(K_4^-, \mathcal{P}_6^1(13)) \leq 12$.*

Corollary 2.4. *$K_4^- \in \mathcal{L}(\mathcal{P}_7^1)$ and $h(K_4^-, \mathcal{P}_7^1) \leq 10$.*

Corollary 2.5. *$h(P_2, P_6^1) \leq 8$ and $w(P_2, P_6^1) \leq 15$.*

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