Pseudo-outerplanar graphs and chromatic conjectures*

Jingjing Tian[a] , Xin Zhang[b†]

[a]Department of Applied Mathematics
Northwestern Polytechnical University, Xi'an 710072, P. R. China
[b]Department of Mathematics
Xidian University, Xi'an 710071, P. R. China

Abstract

In this paper, we verify the list edge coloring conjecture for pseudo-outerplanar graphs with maximum degree at least 5 and the equitable Δ-coloring conjecture for all pseudo-outerplanar graphs.

Keywords: *pseudo-outerplanar graph, list edge coloring, equitable coloring*

1 Introduction

All graphs considered in this paper are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $d_G(v)$, or $d(v)$ for brevity, we denote the degree of a vertex v in G. For other undefined notations, we refer the readers to [1].

For each edge $uv \in E(G)$, assign it a set $L(uv)$ of colors, called a *list* of uv. An edge coloring φ is an *edge L-coloring*, if $\varphi(xy) \in L(uv)$ for each edge $xy \in E(G)$. If $|L(xy)| = k$ for every $xy \in E(G)$, then an edge L-coloring is a *list edge k-coloring* and we say that G is *edge k-choosable*. The minimum integer k for which G has a list edge k-coloring, denoted by $\chi'_l(G)$, is the *list chromatic index* of G. It is trivial that $\chi'(G) \le \chi'_l(G)$. As far as the list edge coloring is concerned,

*Supported by the Fundamental Research Funds for the Central Universities (No. K5051370003), the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2013JQ1002), and the National Natural Science Foundation of China (No. 11301410,11071196).

†Corresponding author. Email address: xzhang@xidian.edu.cn.

Vizing, Gupta, Abertson and Collins, and Bollobás and Harris (see [14] for details) independently posed the following conjecture, which is well-known as List Edge Coloring Conjecture (LECC).

Conjecture 1.1. $\chi'_l(G) = \chi'(G)$ *for every graph G.*

As far as we know, Conjecture 1.1 has been proved for a few special cases, such as bipartite graphs [6], planar graphs with maximum degree at least 12 [3], series-parallel graphs [16] and outerplanar graphs [22]. On the other hand, Vizing's theorem implies that if G is a graph with maximum degree Δ, then $\Delta \le \chi'(G) \le \Delta + 1$. Thus, Vizing [21] posed a weaker conjecture than LECC, which is named Weak List Edge Coloring Conjecture (WLECC).

Conjecture 1.2. $\chi'_l(G) \le \Delta(G) + 1$ *for every graph G.*

Up to now, Conjecture 1.2 was confirmed for graphs with $\Delta(G) \le 4$ [12, 15] and some special graphs such as graphs with girth at least $8\Delta(G)(\log\Delta(G) + 1.1)$ [18], planar graphs with maximum degree at least 9 [2], and planar graphs with maximum degree $\Delta(G) \neq 5$ and without adjacent 3-cycles or with maximum degree $\Delta(G) \neq 5, 6$ and without 7-cycles [13]. However, the above two conjectures on list edge coloring remain very open.

A proper vertex coloring is *equitable* if the sizes of any two color classes differ by at most one, thus an equitable vertex coloring (or equitable coloring for short) is indeed a partition of vertices among the different colors so that they are as evenly as possible. The *equitable chromatic number* of a graph G, denoted by $\chi_{eq}(G)$, is the smallest number k such that G has an equitable coloring with k colors. Note that an equitably k-colorable graph may admit no equitably k'-colorings for some $k' > k$ (the balanced complete k-partite graph with n vertices is such an example), therefore, another chromatic parameter for equitable coloring of graphs is defined naturally. We call the smallest k such that G has equitable k'-colorings for every integer $k' \ge k$ the *equitable chromatic threshold* of G, denoted by $\chi^*_{eq}(G)$. In 1970, Hajnal and Szemerédi [7] answered a question of Erdős by proving every graph G with $\Delta(G) \le r$ has an equitable $(r + 1)$-coloring, which implies $\chi^*_{eq}(G) \le \Delta(G) + 1$ for every graph G. Three years later, Meyer [19] considered an equitable version of Brooks' Theorem and made the following Equitable Coloring Conjecture (ECC).

Conjecture 1.3. *For any connected graph G, except the complete graph and the odd cycle,* $\chi_{eq}(G) \le \Delta(G)$.

In 1994, Chen, Lih and Wu [4] posed the following Equitable Δ-coloring Conjecture (EΔCC), which is stronger than Conjecture 1.3, since $\chi^*_{eq}(G) \ge \chi_{eq}(G)$,

Conjecture 1.4. *If G is a connected graph with maximum degree Δ other than $K_{\Delta+1}$, $K_{\Delta,\Delta}$ and odd cycle, then $\chi^*_{eq}(G) \leq \Delta(G)$.*

Although Conjectures 1.3 and 1.4 were confirmed for many classes of graphs such as graphs with $\Delta \leq 3$ [4, 5] or $\Delta = 4$ [17], bipartite graphs [9], planar graphs with maximum degree at least 9 [11], series-parallel graphs [24] and outerplanar graphs [23], they are still much open. One can refer to the survey contributed by Lih [8] for more progresses concerning the research on equitable coloring of graphs.

A graph is *pseudo-outerplanar* if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. This notion was firstly introduced by Zhang, Liu and Wu [26], where is proved that the class of outerplanar graphs is the intersection of the classes of pseudo-outerplanar graphs and series-parallel graphs and the edge coloring and the linear arboricity of pseudo-outerplanar were considered. Recently, Zhang [25] also proved that every pseudo-outerplanar graphs with maximum degree $\Delta \geq 5$ is totally $(\Delta + 1)$-choosable.

In this paper, we aim to confirm Conjecture 1.1 for pseudo-outerplanar graphs with maximum degree at least 5 by Theorem 2.5, Conjectures 1.2 for all pseudo-outerplanar graphs by Theorem 2.6, and Conjectures 1.3 and 1.4 for all pseudo-outerplanar graphs by Theorem 2.10.

2 Main results and their proofs

Lemma 2.1. [26] *Every pseudo-outerplanar graph with minimum degree at least 2 contains one of the first seventeen configurations in Figure* 1.

Lemma 2.2. *Every pseudo-outerplanar graph contains one of the configurations in Figure* 1*, where the degree of a solid vertex is exactly shown, and the degree of a hollow vertex is at least the number of edges incident to the hollow vertex in the figure, and moreover, hollow vertices may be not distinct while solid vertices are distinct.*

Proof. Let G be a pseudo-outerplanar graph. If $\delta(G) \geq 2$, then G contains one of the configurations among G_1–G_{17} by Lemma 2.1. We now assume that $\delta(G) = 1$ and G contains none of the above configurations. Due to the absence of the configuration G_{18}, G has only one vertex of degree 1, say v. Let u be the neighbor of v in G. If $\delta(G - v) = 1$, then $d(u) = 2$, which implies the appearance of the configuration G_{19} in G. Therefore, $\delta(G - v) \geq 2$ and then $G - v$ contains one of the configurations among G_1–G_{17} by Theorem 4.2 of [26], say G_i. If v is not adjacent

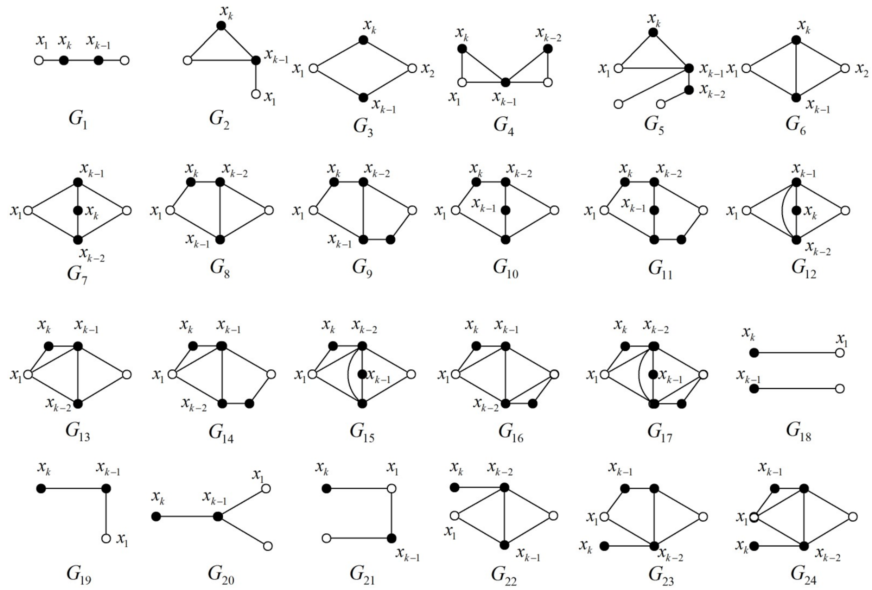

Figure 1: The unavoidable structures of pseudo-outerplanar graphs

to any solid vertex in G_i, then the configuration G_i occurs in G. If v is adjacent to some solid vertex in G_i, then one of the configurations among G_{19}–G_{23} appears in G. □

By Lemma 2.2, we immediately have the following corollary.

Corollary 2.3. *Every pseudo-outerplanar graph contains either a vertex of degree at most two or a configuration* G_6 *as shown in Figure* 1.

Lemma 2.4. [16] *Let* G *be the graph from Figure* 2 *and let* L *be an edge-list assignment for* G *such that* $|L(e)| \geq 2$ *if* e *is incident with* x *or* z*, and* $|L(e)| \geq 4$ *otherwise. Then* G *admits an edge* L*-coloring.*

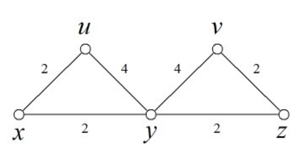

Figure 2: A special graph with the numbers of remaining colors

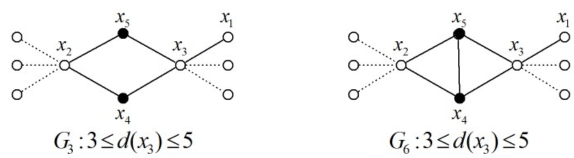

Figure 3: Label the vertices in S when G contains G_3 or G_6 and $k = \Delta = 5$

Theorem 2.5. *Let G be a pseudo-outerplanar graph with maximum degree Δ. If $\Delta \le k$ and $k \ge 5$, then $\chi'_l(G) \le k$.*

Proof. Suppose, to the contrary, that G is a minimum counterexample to this theorem. It is easy to see that $\delta(G) \ge 2$. If G contains G_3, then we delete the four edges in this configuration from G and denote the resulted graph by G'. Since G is the minimum counterexample to this theorem, G' has a list edge k-coloring ϕ. Since 4-cycles are edge 2-choosable, we can color the four deleted edges from their lists so that the extended coloring is still a list edge k-coloring. Hence G does not contain G_3. If G contains an edge uv with $d(u) + d(v) \le k + 1$, then $G - uv$ has a list edge k-coloring ϕ since G is the minimum counterexample to this theorem. Since the edge uv is incident with at most $k - 1$ colored edges under ϕ, one can easily extend ϕ to a list edge k-coloring of G by coloring uv with a color from its list. Therefore, $d(u) + d(v) \ge k + 2$ for every edge $uv \in E(G)$. The above two facts along with Lemma 2.2 imply that $k = 5$ and G contains the configuration G_{17}: a 7-path $x'uxvywy'$ such that $d(u) = d(v) = d(w) = 2$, $d(x) = d(y) = 5$ and $xx', yy', xy, x'y, xy' \in E(G)$. Here we shall also assume that $d(x') = d(y') = 5$ since x' and y' are adjacent to 2-vertices in G. Let L be the list assigned to the edges of G with $|L(e)| = k$ for every edge $e \in E(G)$ and let $G' = G - \{u, v, w, x, y\}$. By the minimality of G, G' has a list edge k-coloring ϕ under the list L. Let $A_\phi(e)$ be the set of the available colors in $L(e)$ to color an edge $e \in \{ux', ux, vx, vy, wy, wy', xx', xy, yy', x'y, xy'\}$ so that the color received by e is different with the colors incident with e under ϕ. It is easy to see that $|A_\phi(xx')|, |A_\phi(yy')|, |A_\phi(x'y)|, |A_\phi(xy')| \ge 3$. Without loss of generality, assume that $|A_\phi(xx')| = |A_\phi(yy')| = |A_\phi(x'y)| = |A_\phi(xy')| = 3$. We now claim that one can color $x'y$ and xy' from their available lists so that the extended coloring θ satisfies $|A_\theta(xx')|, |A_\theta(yy')| \ge 2$ by discussing the following two cases. First, if $A_\phi(x'y) \cap A_\phi(xy') \ne \emptyset$, then we construct the above coloring θ by coloring $x'y$ and xy' with a same coloring from $A_\phi(x'y) \cap A_\phi(xy')$. Second, if $A_\phi(x'y) \cap A_\phi(xy') = \emptyset$, then there are two colors $\alpha \in A_\phi(x'y)$ and $\beta \in A_\phi(xy')$ so that $\{\alpha, \beta\} \nsubseteq A_\phi(xx')$ and $\{\alpha, \beta\} \nsubseteq A_\phi(yy')$, thus we can construct the above coloring θ by coloring $x'y$ and

xy' with α and β. One can also check that the extended partial coloring θ satisfies $|A_\theta(ux')|, |A_\theta(wy')| \geq 2$, $|A_\theta(xy)| \geq 3$ and $|A_\theta(ux)|, |A_\theta(vx)|, |A_\theta(vy)|, |A_\theta(wy)| \geq 4$. Without loss of generality, we assume that all of the above equalities hold (otherwise we meet easier cases and can deal with them much more easily). We claim that θ can be extended by coloring wy, wy' and yy' properly to another partial coloring λ of G which satisfies $|A_\lambda(ux')|, |A_\lambda(xx')|, |A_\lambda(xy)|, |A_\lambda(vy)| \geq 2$ and $|A_\lambda(ux)|, |A_\lambda(vx)| \geq 4$. First, if $A_\theta(yy') \nsubseteq A_\theta(xy)$, then we color yy' with $\lambda(yy') \in A_\theta(yy') \setminus A_\theta(xy) \neq \emptyset$, wy' with $\lambda(wy') \in A_\theta(wy') \setminus \{\lambda(yy')\}$ and wy with $\lambda(wy) \in A_\theta(wy) \setminus \{\lambda(wy'), \lambda(yy')\}$. Second, if $A_\theta(yy') \subseteq A_\theta(xy)$, then we color wy with $\lambda(wy) \in A_\theta(wy) \setminus A_\theta(xy) \neq \emptyset$, wy' with $\lambda(wy') \in A_\theta(wy') \setminus \{\lambda(wy)\}$ and yy' with $\lambda(yy') \in A_\theta(yy') \setminus \{\lambda(wy')\}$ (note that $\lambda(wy) \notin A_\theta(yy')$). In either case, one can confirm that the partial coloring λ of G satisfies the above required conditions. Therefore, λ can be extended to a final list edge k-coloring φ of G by Lemma 2.4. □

Theorem 2.6. *Every pseudo-outerplanar graph with maximum degree Δ is edge $(\Delta + 1)$-choosable.*

Proof. This is an immediate corollary from Theorem 2.5 and the fact that every graph with maximum degree $\Delta = 3$ [12] or $\Delta = 4$ [15] is edge $(\Delta + 1)$-choosable. □

Corollary 2.7. *LECC holds for pseudo-outerplanar graph with maximum degree at least 5.*

Corollary 2.8. *WLECC holds for all pseudo-outerplanar graph.*

Lemma 2.9. [27] *Let $S = \{v_1, v_2, \ldots, v_k\}$ where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in graph G. If $G - S$ has an equitable k-coloring, and $|N_G(v_i) - S| \leq k - i$ with $1 \leq i \leq k$, then G has an equitable k-coloring.*

Theorem 2.10. *Every connected pseudo-outerplanar graph with maximum degree Δ has an equitable coloring with k colors for every $k \geq \max\{\Delta, 5\}$.*

Proof. We prove the theorem by induction on the order of G. If $\Delta \leq 4$, then the result holds by the Hajnal-zemerédi Theorem on equitable coloring, so we assume that $k \geq \Delta \geq 5$ in the following arguments. Since G is pseudo-outerplanar, G contains one of the 24 configurations by Lemma 2.2.

If G contains a configuration among G_4, G_5, G_7–G_{17} and G_{22}–G_{24}, then label the vertices v_1, v_{k-2}, v_{k-1} and v_k as they are in the Figure 2 and fill the remaining unspecified positions in S as described in Lemma 2.9 from highest to lowest indices by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices thus far chosen for S. This can be done by using Corollary 2.3. Since $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq k$ and $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9.

If G contains a configuration among G_1, G_2 and G_{18}–G_{21}, then we first label the vertices v_1, v_{k-1} and v_k as they are in the Figure 1. Let H_i be the graph derived from G by deleting the labeled vertices in the configuration G_i if G contains G_i. By Corollary 2.3, we consider two cases for $i \in \{1, 2, 18, 19, 20, 21\}$. If H_i contains a 2-vertex, then label this vertex by v_{k-2}. If H_i contains a pair of adjacent 3-vertices, then label these two vertices by v_2 and v_{k-2}. In either case, we fill the remaining unspecified positions in S as described in Lemma 2.9 from highest to lowest indices by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices thus far chosen for S. Since $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq k$ and $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9.

If G contains a configuration G_3 or G_6 and $k \geq 6$, then we first label the vertices v_1, v_2, v_{k-1} and v_k as they are in the Figure 1. If H_3 or H_6 contains a 2-vertex, then label this vertex by v_{k-2}. If H_3 or H_6 contains a pair of adjacent 3-vertices, then label these two vertices by v_3 and v_{k-2}. In either case, we fill the remaining unspecified positions in S as described in Lemma 2.9 from highest to lowest indices by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices thus far chosen for S. Since $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq k$ and $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9.

If G contains a configuration G_3 or G_6 and $k = \Delta = 5$, then one neighbor of the 2-vertices in this configuration is of degree at least 3, otherwise G is isomorphic to C_4, contradicting the fact that $\Delta = 5$. Under this condition, we label the vertices v_1, v_2, v_3, v_4 and v_5 as they are in the Figure 3. It is easy to see that $|N_G(v_i)-S| \leq k-i$ for all $1 \leq i \leq 5$. Since $G-S$ has an equitable k-coloring by the induction hypothesis, G has an equitable k-coloring by Lemma 2.9. □

Since ECC and EΔCC holds for graphs with maximum degree 3 [4, 5] and 4 [17], we immediately have the following corollaries.

Corollary 2.11. *ECC and EΔCC holds for all pseudo-outerplanar graphs.*

Corollary 2.12. *Every connected pseudo-outerplanar graph except K_4 with maximum degree Δ has an equitable coloring with k colors for every $k \geq \max\{\Delta, 3\}$.*

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, GTM 244, 2008.

[2] O. V. Borodin, An extension of Kotzig theorem and the list edge coloring of planar graphs (in Russian), *Matem. Zametki*, 48, (1990), 22–48.

[3] O. V. Borodin, A. V. Kostochka, D. R. Woodall, List edge and list total coloring of multigraphs, *J. Combin. Theory Ser. B*, 71, (1997), 184-204.

[4] B.-L. Chen, K.-W. Lih, P. L. Wu, Equitable coloring and the maximum degree, *European J. Combin.*, 15, (1994) 443–447.

[5] B.-L. Chen, C.-H. Yen, Equitable Δ-coloring of graphs, *Discrete Mathematics*, 312, (2012) 1512–1517.

[6] F. Galvain, The list chromatic index of a bipartite multigraph, *J. Combin. Theory Ser. B.*, 63, (1995) 153–158.

[7] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős, In: Combinatorial Theory and its Applications (P. Erdős, A. Rényi and V. T. Sós, eds), North-Holand, London, 1970, pp. 601–623.

[8] K.-W. Lih, Equitable coloring of graphs, In: Handbook of Combinatorial Optimization (P. M. Pardalos, D.-Z. Du, R. Graham, eds), 2nd ed., Springer, 2013, doi: 10.1007/978-1-4419-7997-1_25.

[9] K.-W. Lih, P.-L. Wu, On equitable coloring of bipartite graphs, *Discrete Mathematics*, 151, (1996), 155–160.

[10] W. Meyer, Equitable coloring, *Amer. Math. Monthly*, 80, (1973), 920–922.

[11] K. Nakprasit, Equitable colorings of planar graphs with maximum degree at least nine, *Discrete Mathematics*, 312, (2012) 1019–1024.

[12] A.J. Harris, Problems and conjectures in extremal graph theory, Ph.D. Dissertation, Cambridge University, UK, 1984.

[13] J. Hou, G. Liu, J. Cai, Edge-choosability of planar graphs without adjacent triangles or without 7-cycles, *Discrete Mathematics*, 309, (2009), 77–84.

[14] T. R. Jensen, B. Toft, Graph Coloring Problems, John Wiley & Sons, 1995.

[15] M. Juvan, B. Mohar, R. Škrelkovski, Graphs of degree 4 are 5-edge-choosable, *J. Graph Theory*, 32, (1999), 250–262.

[16] M. Juvan, B. Mohar, R. Thomas, List edge-coloring of series-parallel graphs, *The Electronic Journal of Combinatorics*, 6, (1999), #R42.

[17] H. A. Kierstead, A. V. Kostochka. Every 4-colorable graph with maximum degree 4 has an equitable 4-coloring. *J. Graph Theory*, 71, (2012), 31–48.

[18] A. V. Kostochka, List edge chromatic number of graphs with large grith, *Discrete Math.*, 101, (1992), 189–201.

[19] W. Meyer, Equitable coloring, *Amer. Math. Monthly*, 80 (1973) 920–922.

[20] M. J. Pelsmajer, Equitable list coloring for graphs of maximum degree 3, *J. Graph Theory*, 47, (2004), 1–8.

[21] V. G. Vizing, Vertex cloouring with given coloours (in Russian), *Discrete Analysis*, 29, (1976), 3–10.

[22] W. Wang, K. W. Lih, Choosability, edge-choosability and total choosability of outerplanar graphs, *European J. Combin.*, 22, (2001), 71–78.

[23] H. P. Yap, Y. Zhang, The equitable Δ-coloring coniecture holds for outerplanar graphs, *Bull. Inst. Math. Acad. Sin.*, 25, (1997), 143–149.

[24] X. Zhang, J.-L. Wu, On equitable and equitable list colorings of series-parallel graphs, *Discrete Mathematics*, 311, (2011), 800–803.

[25] X. Zhang, List total coloring of pseudo-outerplanar graphs, *Discrete Mathematics*, 313, (2013), 2297-2306.

[26] X. Zhang, G. Liu, J.-L. Wu, Edge covering pseudo-outerplanar graphs with forests. *Discrete Mathematics*, 312, (2012), 2788–2799.

[27] J. Zhu, Y. Bu, Equitable list coloring of planar graphs without short cycles, *Theoretical Computer Science*, 407, (2008), 21-28.