A note on the weight of triangle in 1-planar graphs with minimum degree 6*

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Abstract

It is proved that every 1-planar graph with minimum degree at least 6 contains a triangle uvw with $d(u)+d(v)+d(w) \leq 22$, or with $d(u) = 6, d(v) = 7$ and $d(w) = 10$, or with $d(u) = 7, d(v) = 8$ and $d(w) = 8$. Moreover, it is also proved that every plane graph with independent crossings with minimum degree 6 contains a triangle uvw with $d(u) = d(v) = d(w) = 6$.

Keywords: *1-planar graph; independent crossings; light subgraph; triangle*

1 Introduction

All graphs considered in this paper are simple and undirected. A *plane graph* is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. By $V(G)$ and $F(G)$, we denote the set of vertices of a graph G and the set of faces of a plane graph G, respectively. The *degree* of a vertex $v \in V(G)$ or a face $f \in F(G)$, denoted by $d_G(v)$ or $d_G(f)$, is the number of vertices that are adjacent to v or the number of edges that are incident with f in G , respectively. A k- or k^+-*vertex* (resp. *face*) is a vertex (resp. face) of degree k or at least k, respectively. A 3-cycle of type $(\leq d_1, \leq d_2, \leq d_3)$ is a triangle uvw with $d(u) \leq d_1, d(v) \leq d_2, d(w) \leq d_3$. For other undefined concepts we refer the reader to [1].

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Let H be a connected graph and let $\mathcal{G}$ be a family of graphs. If for any graph $G \in \mathcal{G}$, G contains a subgraph $K \simeq H$ such that

$$\max_{x \in V(K)} \{d_G(x)\} \leq t_h < +\infty \text{ and } \sum_{x \in V(K)} d_G(x) \leq t_w < +\infty, \quad (1)$$

then we say H is *light* in $\mathcal{G}$, and otherwise say H is *heavy* in $\mathcal{G}$. The smallest integers t_h and t_w satisfying (1), denoted by $h(H, \mathcal{G})$ and $w(H, \mathcal{G})$, are the *height* and the *weight* of H in $\mathcal{G}$, respectively.

A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planarity was first introduced by Ringel [6] in 1965 while trying to simultaneously color the vertices and faces of a plane graph so that any pair of adjacent/incident elements receive different colors. As a superclass of planar graphs, 1-planar graphs show similar behavior as planar graphs. For example, the size of any 1-planar graph G is at most $4|V(G)| - 8$ and this bound is tight [2], which implies that every 1-planar graph contains a vertex of degree at most 7. Note that every planar graph contains a vertex of degree at most 5. Thus, 1-planar graphs with minimum degree at least 6 are non-planar. The local structures of 1-planar graphs with high minimum degree were investigated by many authors including [2, 3, 4, 5, 7, 8, 10, 11]. In particular, Fabrici and Madaras [2] proved that the triangle is light in the class of 1-planar graphs with minimum degree δ if and only if $\delta \geq 6$, and moreover, every 1-planar graph with minimum degree at least 6 contains a triangle with vertices of degree at most 10[1].

In this paper, we mainly consider the weight of triangle in the class of 1-planar graph with minimum degree at least 6 and prove that every 1-planar graph with minimum degree at least 6 contains a triangle of degree sum at most 23, which generalizes a result in [2].

2 Main result and its proof

In this section, we always assume that every 1-planar graph is a *1-plane graph* — a graph embedded in the plane with its 1-planarity satisfied. The *associated plane graph* of a 1-plane graph G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices.

Theorem 1. *Every 1-planar graph G with minimum degree at least 6 contains a triangle uvw with $d(u) + d(v) + d(w) \leq 22$, or with $d(u) = 6, d(v) = 7$ and $d(w) = 10$, or with $d(u) = 7, d(v) = 8$ and $d(w) = 8$.*

[1] There exits a correctable error in the proof of this result in [2]. Specifically, the discharging rules in that proof shall be altered by the rules in our proof of Theorem 1.

Proof. Let G be a hypothetical counterexample to the result and let $G^{\times}$ be its associated plane graph. We first assign an initial charge c to each element $x \in V(G^{\times}) \cup F(G^{\times})$ as follows:

$$c(x) = \begin{cases} d_{G^{\times}}(x) - 6, & \text{if } x \in V(G^{\times}); \\ 2d_{G^{\times}}(x) - 6, & \text{if } x \in F(G^{\times}), \end{cases} \tag{2}$$

Next, we are to redistribute the charge of the vertices and the faces of $G^{\times}$ according to the discharging rules defined below, which only move charge around but do not affect the total charges, so that after discharging the final charge c' of each element in $V(G^{\times}) \cup F(G^{\times})$ is nonnegative. This leads to a contradiction: $\sum_{x \in V(G^{\times}) \cup F(G^{\times})} c(x) = \sum_{x \in V(G^{\times}) \cup F(G^{\times})} c'(x) \geq 0$.

Rule 1 Each 4^{+}-face f sends 1 to each of its incident 4-vertices;

Rule 2 For a 7^{+}-vertex v, if v is incident two adjacent 3-faces $f_1 = xvz$ and $f_2 = yvz$ and z is a 4-vertex in $G^{\times}$, then v sends z a charge of $2 - \frac{12}{d_{G^{\times}}(v)}$ if $d_{G^{\times}}(v)$ is even, and $2 - \frac{10}{d_{G^{\times}}(v)-1}$ if $d_{G^{\times}}(v)$ is odd;

Rule 3 For a 7^{+}-vertex v, if v is incident with a 3-face $f_1 = xvz$ that shares a common edge vz with a 4^{+}-face f_2 and z is a 4-vertex in $G^{\times}$, then v sends z a charge of $\frac{3}{2} - \frac{9}{d_{G^{\times}}(v)}$ if $d_{G^{\times}}(v)$ is even, and $1 - \frac{5}{d_{G^{\times}}(v)-1}$ if $d_{G^{\times}}(v)$ is odd.

Since 3-faces and 6-vertices are not involved in the rules, they have nonnegative final charges. For a 4^{+}-face f, $c'(f) \geq 2d_{G^{\times}}(f) - 6 - \left\lfloor \frac{d_{G^{\times}}(f)}{2} \right\rfloor \geq \frac{3}{2}d_{G^{\times}}(f) - 6 \geq 0$, since f, which is incident with at most $\left\lfloor \frac{d_{G^{\times}}(f)}{2} \right\rfloor$ 4-vertices, only sends out charges by Rule 1. For a 7^{+}-vertex v, v only sends out charges to its adjacent 4-vertices by Rule 2 or Rule 3. Let $\alpha(v)$ and $\beta(v)$ be the times the Rule 2 and Rule 3 are applied to v, respectively. It is easy to see that the subgraph induced by the faces that are incident with v can be decomposed into many parts, each of which is one of the five clusters in Figure 1, and any two parts of which are adjacent only if they have a common edge vw so that w is a 6^{+}-vertex. In Figure 1, the hollow vertices are all 4-vertices and the solid ones are 6^{+}-vertices, and all marked faces are 4^{+}-faces. By $n_i(v)$, we denote the number of i-clusters that are incident with v. Since $\alpha(v) = n_1(v), \beta(v) = n_2(v) + 2n_4(v)$, $2n_1(v) + 2n_2(v) + n_3(v) + 3n_4(v) + n_5(v) \leq d_{G^{\times}}(v)$, and $n_2(v) + n_3(v) + n_4(v) + n_5(v) \geq 1$ if $d_{G^{\times}}(v)$ is odd, we have $2\alpha(v) + \frac{3}{2}\beta(v) \leq d_{G^{\times}}(v)$, and $2\alpha(v) + \beta(v) \leq d_{G^{\times}}(v) - 1$ if $d_{G^{\times}}(v)$ is odd. Therefore, by Rule 2 and Rule 3, if $d_{G^{\times}}(v)$ is even, then $c(v) \geq d_{G^{\times}}(v) - 6 - \left[2 - \frac{12}{d_{G^{\times}}(v)}\right]\alpha(v) - \left[\frac{3}{2} - \frac{9}{d_{G^{\times}}(v)}\right]\beta(v) = (1 - \frac{6}{d_{G^{\times}}(v)})\left[d_{G^{\times}}(v) - (2\alpha(v) + \frac{3}{2}\beta(v))\right] \geq 0$, and if $d_{G^{\times}}(v)$ is odd, then $c(v) \geq d_{G^{\times}}(v) - 6 - \left[2 - \frac{10}{d_{G^{\times}}(v)-1}\right]\alpha(v) - \left[1 - \frac{5}{d_{G^{\times}}(v)-1}\right]\beta(v) = (d_{G^{\times}}(v) - 6)\left[1 - \frac{1}{d_{G^{\times}}(v)-1}(2\alpha(v) + \beta(v))\right] \geq 0$. Hence in the following, we only need calculate the final charges of 4-vertices.

Let v be a 4-vertex and let v_1, v_2, v_3 and v_4 be its neighbors in clockwise orientation. By f_i, we denote the face that is incident with the edges vv_i and vv_{i+1}

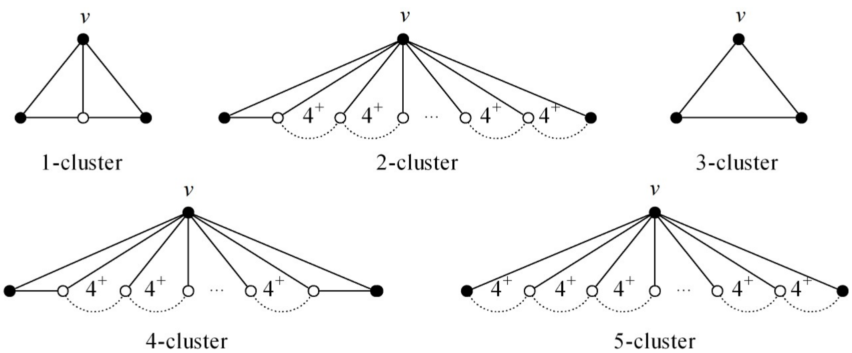

Figure 1: Five types of clusters

(indices are taken modulo 4) in $G^{\times}$. If v is incident with at least two 4^+-faces, then $c'(v) \geq -2 + 2 \times 1 = 0$ by Rule 1. If v is incident with exactly three 3-faces, then we assume, without loss of generality, that f_1, f_2, f_3 are 3-faces and f_4 is a 4^+-face. If $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} = 6$, then we assume, without loss of generality, that $d_{G^\times}(v_2) = d_{G^\times}(v_3) = 6$, because other cases can be processed similarly. In this case, we have $d_{G^\times}(v_1) \geq 11$ and $d_{G^\times}(v_4) \geq 11$, otherwise a 3-cycle of degree sum at most 22 occurs in G. Therefore, $c'(v) \geq -2 + 1 + 2 \times (1 - \frac{5}{11-1}) = 0$ by Rules 1 and 3. If $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} = 7$ and $\min\{d_{G^\times}(v_1), d_{G^\times}(v_4)\} \geq 7$, then $c'(v) \geq -2 + 1 + 2 \times (2 - \frac{10}{7-1}) + 2 \times (1 - \frac{5}{7-1}) = 0$ by Rules 1, 2 and 3. If $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} = 7$ and $\min\{d_{G^\times}(v_1), d_{G^\times}(v_4)\} = 6$, then to avoid a 3-cycle of degree sum at most 22, we have $\max\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} \geq 9$. Therefore, $c'(v) \geq -2+1+2\times(2-\frac{10}{7-1})+(2-\frac{10}{9-1}) > 0$ by Rule 2. If $\min\{d_{G^\times}(v_2), d_{G^\times}(v_3)\} \geq 8$. then by Rules 1 and 2, $c'(v) \geq -2 + 1 + 2 \times (2 - \frac{12}{8}) = 0$.

We now suppose that v is incident with four 3-faces. If v is adjacent to at least two 11^+-vertices, then by Rule 2, $c'(v) \geq -2 + 2 \times (2 - \frac{10}{11-1}) = 0$. Thus, if v is adjacent to a 6-vertex (also assume that v is adjacent to at most one 11^+-vertex), then v is adjacent to one 8-vertex and two 9^+-vertices, or adjacent to three 9^+-vertices, otherwise a 3-cycle of type $(6, 8, 8)$ or $(6, 7, \leq 10)$ occurs in G. Therefore, by Rule 2, we have $c'(v) \geq -2 + 2 \times (2 - \frac{12}{8}) + 2 \times (2 - \frac{10}{9-1}) > 0$. Suppose that v is adjacent only to 7^+-vertices. If at least two of v_1, v_2, v_3 and v_4 are 7-vertices, then at least two of v_1, v_2, v_3 and v_4 are 9^+-vertices, because otherwise a 3-cycle of degree sum 22 would appear in G, a contradiction. Therefore, $c'(v) \geq -2+2\times(2-\frac{10}{7-1}) + 2 \times (2 - \frac{10}{9-1}) > 0$. If exactly one of v_1, v_2, v_3 and v_4 is a 7-vertex, then v is adjacent to at least one 9^+-vertex to avoid the occurrence of a 3-cycle of the type $(7, \leq 8, \leq 8)$ in G. Therefore, $c'(v) \geq -2 + (2 - \frac{10}{7-1}) + 2 \times (2 - \frac{12}{8}) + (2 - \frac{10}{9-1}) > 0$

by Rule 2. If v is adjacent only to 8^+-vertices, then by Rule 2, we have $c'(v) \geq -2 + 4 \times (2 - \frac{12}{8}) = 0$ in final. □

3 Remarks

In fact, Theorem 1 implies that the weight of triangle in 1-planar graph with minimum degree at least 6 is at most 23. Since every triangle in the adjacency/incidence vertex-face graph of the soccer ball graph (which is a 1-planar graph with minimum degree 6) contains a vertex of degree at least 10, we deduce the following:

Corollary 2. $22 \leq w(C_3, \mathcal{P}_6^1) \leq 23$ *and* $h(C_3, \mathcal{P}_6^1) = 10$*, where* $\mathcal{P}_6^1$ *is the class of 1-planar graphs with minimum degree at least* 6*.*

If $w(C_3, \mathcal{P}_6^1) = 23$, then the extremal graph (the graph G in $\mathcal{P}_6^1$ with the degree sum of every triangle in G being at least 23) must contain a 3-cycle of type $(6, 7, 10)$ or $(7, 8, 8)$ by Theorem 1. This gives us a possible direction to construct such an extremal graph (if exists).

If we change the class of graphs in Theorem 1 to 1-planar graphs with minimum degree 7, the method of the proof (even if we change the initial charge function c to: $c(x) = 2d_{G^\times}(x) - 14$ if $x \in V(G^\times)$ and $c(x) = 5d_{G^\times}(x) - 7$ if $x \in F(G^\times)$) is still not sufficient to exclude the 3-cycles of type $(7, 8, 8)$ in the result. Therefore, if $w(C_3, \mathcal{P}_7^1) \leq 22$ or $w(C_3, \mathcal{P}_6^1) = 22$, we may need a different treatment to prove them.

However, to determine the height and the weight of triangle in the class of plane graphs with independent crossings, which is the class of graphs that can be embedded in the plane so that the end-vertices of any two pairs of crossing edges are disjoint and is also the subclass of 1-planar graphs, is easier. Specially, we can prove

Theorem 3. *Every plane graph with independent crossings with minimum degree 6 contains a triangle uvw with* $d(u) = d(v) = d(w) = 6$*.*

Theorem 3 implies that the height and the weight of triangle in the class of plane graphs with independent crossings with minimum degree 6 is 6 and 18, respectively, since there exist 6-regular plane graphs with independent crossings, see [9].

The proof of Theorem 3 also proceeds by discharging on the associated plane graph $G^\times$ of the counterexample G via assigning initial charge c as in the proof of Theorem 1 to each element $x \in V(G^\times) \cup F(G^\times)$ and defining discharging rule as follows: each 4^+-face sends 2 to each of its incident 4-vertices and each 7^+-vertex sends 1 to each of its adjacent 4-vertices.

Since every face f is incident with at most $\left\lfloor \frac{1}{4} d_{G^\times}(f) \right\rfloor$ 4-vertices and every vertex in $G^\times$ is adjacent to at most one 4-vertex, $c'(f) \geq 2d_{G^\times}(f) - 6 - 2\left\lfloor \frac{1}{4} d_{G^\times}(f) \right\rfloor \geq 0$ for $d_{G^\times}(f) \geq 4$ and $c'(v) \geq d_{G^\times}(v) - 6 - 1 \geq 0$ for $d_{G^\times}(v) \geq 7$. For a 4-vertex v, if v is incident with a 4^+-face, then $c'(v) \geq -2 + 2 = 0$, and if v is incident only with 3-faces, then v is adjacent to at least two 7^+-vertices, which implies $c'(v) \geq -2 + 2 \times 1 = 0$. Therefore, we have $\sum_{x \in V(G^\times) \cup F(G^\times)} c(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) \geq 0$, a contradiction.

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