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LIGHT 3-CYCLES IN 1-PLANAR GRAPHS WITH DEGREE RESTRICTIONS

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Abstract. In this paper, we prove that the 3-cycle is light in the family of 1-planar graphs with minimum vertex degree at least 5 and minimum edge degree at least 12. This generates a known result of Fabrici and Madaras [8].

1. Introduction

All graphs considered in the paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$ to denote the set of vertices, the set of edges, the set of faces, the minimum degree and the maximum degree of a plane graph G, respectively. The *degree* of an edge uv in G is the value of $d_G(u)+d_G(v)$. A k-, k^+- and k^--*vertex* (resp. *face*) is a vertex (resp. face) of degree k, at least k and at most k, respectively. For other undefined concepts we refer the reader to [2].

A graph is *1-planar* if it can be drawn on a plane so that each edge is crossed by at most one other edge. The notion of 1-planarity was introduced by Ringel [14], who proved that each 1-planar graph is vertex 7-colorable. This is the first result on the colorings of 1-planar graphs, and from then on, many authors started to investigate the coloring problems (see [1, 3, 4, 6, 15, 16, 17, 21, 23, 24]) and the structural properties (see [5, 8, 9, 10, 11, 13, 18, 19, 20, 22]) of 1-planar graphs. One of possible approaches in the study of local graph structures can be formalized in the following way (see [12]):

Let H be a connected graph and $\mathcal{G}$ be a family of graphs. If for any graph $G\in\mathcal{G}$, G contains a subgraph $K\simeq H$ such that $\max_{x\in V(K)}\{d_G(x)\}$ is bounded above by a constant independent of G, then we say that H is *light* in $\mathcal{G}$; otherwise H is *heavy* in $\mathcal{G}$. By $\mathcal{L}(\mathcal{G})$, we denote the set of light graphs in the family $\mathcal{G}$, and by $\mathcal{P}_\delta^1(\varepsilon)$, we denote the family of 1-planar graphs with minimum vertex degree at least δ and minimum edge degree at least ε.

In 2007, Fabrici and Madaras [8] completely determined the set of light graphs in the family $\mathcal{P}_4^1(8)$; they are P_1, P_2 and P_3. Recently, the set of light

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graphs in the family $\mathcal{P}_5^1(10)$ is also completely determined (see [8, 7, 19]); they are P_1, P_2, P_3, P_4 and S_3, but $\mathcal{L}(\mathcal{P}_6^1(12))$ and $\mathcal{L}(\mathcal{P}_7^1(14))$ are still undetermined. In the reference [8], Fabrici and Madaras proved that the 3-cycle C_3 is light in $\mathcal{P}_6^1(12)$. As $\mathcal{P}_5^1(12)$ is a superfamily of $\mathcal{P}_6^1(12)$, we are to prove the following main theorem in this paper, which generates the result of Fabrici and Madaras.

Theorem 1. *Each 1-planar graph with minimum vertex degree at least 5 and minimum edge degree at least 12 contains a 3-cycle $C = [x_1x_2x_3]$ such that $\min\{d(x_1), d(x_2), d(x_3)\} \leq 7$ and $\max\{d(x_1), d(x_2), d(x_3)\} \leq 20$.*

Before giving a proof of the above theorem, we introduce some useful notations. Let G be a 1-planar graph. From now on, we always assume that G has been drawn on a plane so that the 1-planarity of G is satisfied. The *associated plane graph* of G, denoted by $G^\times$, is the graph obtained from G by turning all crossings of G into new 4-vertices, and those new 4-vertices are called *false vertices*. The face that is incident with no false vertex in $G^\times$ is called *true face*, and otherwise, we call it *false face*.

2. The proof of Theorem 1

Suppose that G is a counterexample to the Theorem 1 and $G^\times$ is the associated plane graph of G. Assign an initial charge c to each element $x \in V(G^\times) \bigcup F(G^\times)$ as follows:

$$c(x) = \begin{cases} d_{G^\times}(x) - 6, & \text{if } x \in V(G^\times); \\ 2d_{G^\times}(x) - 6, & \text{if } x \in F(G^\times), \end{cases}$$

By Euler's formula on $G^\times$, $\sum_{x\in V(G^\times)\bigcup F(G^\times)} c(x) < 0$. We now redistribute the charges among $V(G^\times) \bigcup F(G^\times)$ according to the rules defined below. Before stating them, we introduce some useful notations. In what follows, a *special* 4-face is a 4-face in $G^\times$ that is incident with two 4-vertices and two 5-vertices. By *big*, *mid* and *small* vertices, we denote the vertices v in $G^\times$ with $d(v) \geq 21$, $6 \leq d(v) \leq 20$ and $4 \leq d(v) \leq 5$, respectively. Let v_1, v_2 and v_3 be three consecutive neighbors of a big vertex v in $G^\times$. If v_1 and v_3 are both 4-vertices and v_2 is a 5-vertex with $v_1v_2, v_2v_3 \in E(G^\times)$, then v_2 is called a *special* 5-vertex that is adjacent to v. The discharging rules are as follows.

- Rule 1 Each mid vertex v sends $1 - \frac{6}{d_{G^\times}(v)}$ to each of its incident faces;
- Rule 2 Each big vertex v sends $\frac{2}{3}$ to each of its incident faces;
- Rule 3 Each big vertex v sends $\frac{8}{63}$ to each of its adjacent special 5-vertices;
- Rule 4 Each 3-face f with positive charge $c_2(f)$ after applications of Rules 1 and 2 sends $\frac{2}{3}c_2(f)$ to each of its incident 4-vertices, $\frac{1}{3}c_2(f)$ to each of its incident 5-vertices if f is incident with two small vertices, or $c_2(f)$ to the unique small vertex that is incident with f if f is incident with exactly one small vertex;
- Rule 5 Each special 4-face sends $\frac{2}{3}$ to each of its incident 4-vertices and $\frac{1}{3}$ to each of its incident 5-vertices;

Rule 6 Each non-special 4^+-face with positive charge after applications of Rules 1 and 2 redistributes this charge uniformly among its incident small vertices.

Let $c'(x)$ be the final charge of $x \in V(G^\times) \bigcup F(G^\times)$ after applications of the above rules. We now prove that $c'(x) \geq 0$ for each $x \in V(G^\times) \bigcup F(G^\times)$, therefore, $\sum_{x \in V(G^\times) \bigcup F(G^\times)} c(x) = \sum_{x \in V(G^\times) \bigcup F(G^\times)} c'(x) \geq 0$, which is a contradiction.

Since no two 4-vertices and no two 5-vertices are adjacent in $G^\times$, any 3-face f in $G^\times$ is incident with at most one 4-vertex and at most one 5-vertex. Hence, by Rule 4, f has a nonnegative final charge. By Rules 1, 5 and 6, it is also easy to check that the final charges of all mid vertices and all 4^+-faces are nonnegative. Thus in the following, we only need estimate the final charges of the small vertices and big vertices.

Claim 1. Each non-special 4^+-face in $G^\times$ sends at least $\frac{2}{3}$ to each of its incident small vertices.

Proof. Let f be a non-special 4^+-face in $G^\times$. Note that after applications of Rules 1 and 2, the saved charge $c_2(f)$ of f is at least $c(f) = 2d(f) - 6 > 0$. If $d_{G^\times}(f) = 4$, then the number of small vertices that are incident with f is at most 3, since no two 4-vertices and no two 5-vertices are adjacent in $G^\times$, thus by Rule 6, f sends at least $\frac{2\times 4-6}{3} = \frac{2}{3}$ to each of its incident small vertices. If $d_{G^\times}(f) \geq 5$, then it is easier to check that f sends at least $\frac{2\times 5-6}{5} = \frac{4}{5} > \frac{2}{3}$ to each of its incident small vertices by Rule 6. □

Claim 2. Each true 3-face in $G^\times$ sends at least $\frac{17}{21}$ to each of its incident 5-vertices.

Proof. Let $f = [uvw]$ be a true 3-face in $G^\times$ with $d_{G^\times}(v) = 5$. Since G is a counterexample to the theorem and the minimum edge degree of G is at least 12, $\max\{d_{G^\times}(v), d_{G^\times}(w)\} \geq 21$ and $\min\{d_{G^\times}(v), d_{G^\times}(w)\} \geq 7$. By Rules 1 and 2, f totally receives at least $1 - \frac{6}{7} + \frac{2}{3} = \frac{17}{21}$ from v and w. This implies that $c_2(f) \geq 2 \times 3 - 6 + \frac{17}{21} = \frac{17}{21}$ and this charge would be transferred to v by Rule 4. □

Claim 3. Each big vertex v is adjacent to at most $\lfloor \frac{1}{3} d_{G^\times}(v) \rfloor$ special 5-vertices in $G^\times$.

Proof. Suppose $d_{G^\times}(v) = 3r + s$, where r and s are two nonnegative integers with $0 \leq s < 3$. In what follows, we assume that $s > 0$. (The case when $s = 0$ can be dealt with similarly.) Let $v_1, v_2, \ldots, v_{3r+s}$ be the neighbors of v in a clockwise order. We now divide those $3r + s$ vertices into $r + 1$ parts $U_1, U_2, \ldots, U_r, U_{r+1}$ so that $U_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \leq i \leq r$ and $U_{r+1} = \{v_{3r+1}, \ldots, v_{3r+s}\}$. By the definitions of U_i's, one can check that U_i with $1 \leq i \leq r$ contains at most two special 5-vertices. If U_i contains exactly two special 5-vertices, then v_{3i-1} must be a 4-vertex, moreover, the two 5-vertices

v_{3i-2} and v_{3i} satisfy $v_{3i-2}v_{3i-1}, v_{3i-1}v_{3i} \in E(G^\times)$ and $v_{3i-2}v_{3i} \in E(G)$. This contradicts the fact that the minimum edge degree of G is at least 12. Hence, each U_i with $1 \le i \le r$ contains at most one special 5-vertex. Similarly, U_{r+1} also contains at most one special 5-vertex. If $s = 1$ and each U_i with $1 \le i \le r$ contains exactly one special 5-vertex, then $d_{G^\times}(v_{3r-2}) = d_{G^\times}(v_{3r}) = 4$, $d_{G^\times}(v_{3r-1}) = 5$ and $v_{3r-2}v_{3r-1}, v_{3r-1}v_{3r} \in E(G^\times)$. This implies that v_{3r+1} is not a special 5-vertex, because otherwise we have $v_{3r}v_{3r+1} \in E(G^\times)$, which implies $v_{3r-1}v_{3r+1} \in E(G)$, a contradiction. Similarly, we can prove that v_{3r+1} and v_{3r+2} are not special 5-vertices when $s = 2$ and each U_i with $1 \le i \le r$ contains exactly one special 5-vertex. Therefore, v is adjacent to at most r special 5-vertices in $G^\times$. □

In the following, we estimate the final charge of the big vertices $v \in V(G^\times)$. Since Rules 2 and 3 are the only rules that are involved with v, and Rules 2 and 3 can be applied to v at most $d_{G^\times}(v)$ times and $\lfloor \frac{1}{3} d_{G^\times}(v) \rfloor$ times by Claim 3, respectively, $c'(v) \ge d_{G^\times}(v) - 6 - \frac{2}{3} d_{G^\times}(v) - \frac{8}{63} \lfloor \frac{1}{3} d_{G^\times}(v) \rfloor \ge \frac{1}{189}(55 d_{G^\times}(v) - 1134) \ge \frac{1}{9} > 0$ for $d_{G^\times}(v) \ge 21$.

Until now, the remaining task is to check the nonnegativity of the final charge of small vertices. First, suppose that v is a 5-vertex. Let $v_1, \ldots, v_5$ be the neighbors of v in a clockwise sequence and let f_i be the face incident with the path $v_i v v_{i+1}$ in $G^\times$, where the addition on i are taken modulo 5. By $B(v), T(v)$ and $F(v)$, we denote the number of 4^+-faces, true 3-faces and false 3-faces that are incident with v, respectively. If $B(v) \ge 3$, then by Rule 5 and Claim 1, v receives at least $3 \times \frac{1}{3} = 1$ from its incident 4^+-faces, which implies that $c'(v) \ge 5 - 6 + 1 = 0$. If $B(v) = 2$ and f is incident with a non-special 4^+-face, then by Rule 5 and Claim 1, $c'(v) \ge 5 - 6 + \frac{1}{3} + \frac{2}{3} = 0$. If $B(v) = 2$ and f is incident with two special 4-faces, then those two 4-faces are adjacent in $G^\times$, since otherwise we would find two adjacent 4-vertices in $G^\times$, a contradiction. Without loss of generality, suppose that f_1 and f_2 are both special 4-faces. Since v_1, v_3 are false now and f_3, f_5 are 3-faces, v_4 and v_5 are true, which implies that f_4 is a true 3-face. Thus by Claim 2, f_4 sends at least $\frac{17}{21}$ to v, so $c'(v) \ge 5 - 6 + 2 \times \frac{1}{3} + \frac{17}{21} > 0$ by Rule 5. If $B(v) = 1$ and v is incident with a true 3-face, then by Rule 5, Claims 1 and 2, $c'(v) \ge 5 - 6 + \frac{1}{3} + \frac{17}{21} > 0$. If $B(v) = 1$ (here we assume that f_5 is a 4^+-face incident with v) and all 3-faces incident with v are false, then either v_2 and v_4 are both 4-vertices, or v_1, v_3 and v_5 are all 4-vertices. If the former case occurs, then f_5 is non-special and v_1, v_5 are not small. This implies that f_5 sends at least $\frac{2d_{G^\times}(f_5) - 6}{d_{G^\times}(f_5) - 2} \ge 1$ to v by Rule 6, and thus $c'(v) \ge 5 - 6 + 1 = 0$. If the latter case occurs, then v, v_2 and v_4 induce a 3-cycle in G with $d_{G^\times}(v) = 5$, which implies that v_2 and v_4 are 7^+-vertices and at least one of them is big. Without loss of generality, assume that v_4 is a big vertex, then by Rule 3, v_4 sends $\frac{8}{63}$ to v, since v is special. On the other hand, one can check that the saved charge of f_1, f_2, f_3 and f_4 after applying Rules 1 and 2 is at least $\frac{1}{7}$, $\frac{1}{7}$, $\frac{2}{3}$ and $\frac{2}{3}$, respectively, and f_5 sends at least $\frac{1}{3}$ to v

by Rules 5 and 6. Therefore, $c'(v) \geq 5-6+\frac{8}{63}+2\times\frac{1}{7}\times\frac{1}{3}+2\times\frac{2}{3}\times\frac{1}{3}+\frac{1}{3}=0$ by Rule 4. If $B(v)=0$ and $T(v)\geq 2$, then by Claim 2, $c'(v)\geq 5-6+2\times\frac{17}{21}>0$. If $B(v)=0$, $T(v)=1$ and $F(v)=4$ (assume that f_5 is true and v_1 is big), then by Rules 2 and 4, f_1 sends at least $\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$ to v, thus by Claim 2, we still have $c'(v)\geq 5-6+\frac{2}{9}+\frac{17}{21}>0$.

Now, suppose that v is a 4-vertex in $G^{\times}$. By Rule 5 and Claim 1, the final charge of v is nonnegative if v is incident with at least three 4^{+}-faces, so in the following, we assume that v is incident with at least two 3-faces. Let v_1, v_2, v_3, v_4 be the neighbors of v in a clockwise order. By f_i, we denote the face that is incident with vv_i and vv_{i+1}, where the subscripts are taken modulo 4. By $\tau(f \to v)$, we denote the charge transferred from f to its incident vertex v.

Case 1. v is incident with exactly two 3-faces.

First, suppose that $d_{G^{\times}}(f_1)=d_{G^{\times}}(f_3)=3$. Without loss of generality, assume that v is adjacent to at least two 5-vertices (the opposite case when v is adjacent to at most one 5-vertex can be dealt with similarly). Since the degree of any neighbor of a 5-vertex in G is at least 7, we can assume that $d_{G^{\times}}(v_2)=d_{G^{\times}}(v_3)=5$ and $\min\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_4)\}\geq 7$. Since $d_{G^{\times}}(v_1)\geq 7$, the saved charge of f_1 after applying Rules 1 and 2 is at least $\frac{1}{7}$, and two thirds of this charge would be transferred to v by Rule 4, that is to say, $\tau(f_1 \to v)\geq\frac{1}{7}\times\frac{2}{3}=\frac{2}{21}$. Similarly, we have $\tau(f_2\to v)\geq\frac{2}{3}$, $\tau(f_3\to v)\geq\frac{2}{21}$ and $\tau(f_4\to v)\geq\frac{8}{7}$. This implies that $c'(v)\geq -2+\frac{2}{21}+\frac{2}{3}+\frac{2}{21}+\frac{8}{7}=0$.

Second, suppose that $d_{G^{\times}}(f_1)=d_{G^{\times}}(f_2)=3$. Since v_1, v_2 and v_3 induce a 3-cycle in G, we either have $\max_{1\leq i\leq 3}\{d_{G^{\times}}(v_i)\}\geq 21$ or have $\min_{1\leq i\leq 3}\{d_{G^{\times}}(v_i)\}\geq 8$. If the former case occurs, then we can assume, without loss of generality, that $d_{G^{\times}}(v_1)\geq 21$. By Rules 2, 4, 5, 6 and Claim 1, we have $\tau(f_1\to v)\geq\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$, $\tau(f_3\to v)\geq\frac{2}{3}$ and $\tau(f_4\to v)\geq\frac{1}{3}\times(2+\frac{2}{3})=\frac{8}{9}$. This implies that $c'(v)\geq -2+\frac{4}{9}+\frac{2}{3}+\frac{8}{9}=0$. If the latter case occurs, then by Rules 1 and 4, $\tau(f_1\to v)\geq 2\times(1-\frac{6}{8})=\frac{1}{2}$ and $\tau(f_2\to v)\geq 2\times(1-\frac{6}{8})=\frac{1}{2}$, and by Rule 5 and Claim 1, $\tau(f_3\to v)\geq\frac{2}{3}$ and $\tau(f_4\to v)\geq\frac{2}{3}$. This implies that $c'(v)\geq -2+2\times\frac{1}{2}+2\times\frac{2}{3}>0$.

Case 2. v is incident with exactly three 3-faces.

Suppose that f_1, f_2 and f_3 are 3-faces and f_4 is a 4^{+}-face. If v is adjacent to no big vertices, then v is adjacent only to 8^{+}-vertices, since otherwise a light 3-cycle is found in G, a contradiction. By Rules 1 and 4, $\tau(f_i\to v)\geq 2\times(1-\frac{6}{8})=\frac{1}{2}$ for $i=1,2,3$. By Rule 5 and Claim 1, $\tau(f_4\to v)\geq\frac{2}{3}$. This implies that $c'(v)\geq -2+3\times\frac{1}{2}+\frac{2}{3}>0$. Therefore, we consider the case when v is adjacent to at least one big vertex. If v_2 or v_3, say v_2, is big, then either $d_{G^{\times}}(v_1)=d_{G^{\times}}(v_3)=d_{G^{\times}}(v_4)=6$ or $\max\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_3), d_{G^{\times}}(v_4)\}\geq 7$, since v_1, v_3 and v_4 induce a 3-path in G. If the former case occurs, by Rules 2 and 4, $\tau(f_1\to v)\geq\frac{2}{3}$ and $\tau(f_2\to v)\geq\frac{2}{3}$, and by Rule 5 and Claim 1, $\tau(f_4\to v)\geq\frac{2}{3}$. This implies that $c'(v)\geq -2+3\times\frac{2}{3}=0$. If the latter case occurs, then consider the worst case (one can easily check this fact) when $d_{G^\times}(v_1) = d_{G^\times}(v_4) = 5$ and $d_{G^\times}(v_3) = 7$. By Rules 1, 2, 3, 5 and Claim 1, we can still estimate that $\tau(f_1 \to v) \geq \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, $\tau(f_2 \to v) \geq \frac{2}{3} + \frac{1}{7} = \frac{17}{21}$, $\tau(f_3 \to v) \geq \frac{1}{7} \times \frac{2}{3} = \frac{2}{21}$ and $\tau(f_4 \to v) \geq \frac{2}{3}$. This implies that $c'(v) \geq -2 + \frac{4}{9} + \frac{17}{21} + \frac{2}{21} + \frac{2}{3} = \frac{1}{63} > 0$.

Case 3. v is incident with four 3-faces.

If v is adjacent only to 8^+-vertices, then by Rules 1 and 4, each face incident with v sends at least $2 \times (1 - \frac{6}{8}) = \frac{1}{2}$ to v, which implies that $c'(v) \geq -2 + 4 \times \frac{1}{2} = 0$. If $\min_{1 \leq i \leq 4}\{d_{G^\times}(v_i)\} \leq 7$, then there are at least two big vertices among the neighbors of v, because otherwise a light 3-cycle would appear in G. Without loss of generality, assume that both v_1 and v_3 are big vertices and that $d_{G^\times}(v_2) = 5$ and $d_{G^\times}(v_4) \geq 7$ (other cases can be handled similarly and much more easily). In this case, by Rules 1, 2 and 4, one can estimate that $\min_{i=1,2}\{\tau(f_i \to v)\} \geq \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ and $\min_{i=3,4}\{\tau(f_i \to v)\} \geq \frac{2}{3} + \frac{1}{7} = \frac{17}{21}$. This implies that $c'(v) \geq -2 + 2 \times \frac{4}{9} + 2 \times \frac{17}{21} > 0$. □

Since C_3 is heavy in the family of 1-planar graphs with minimum vertex degree at least 5, the bound 12 for the minimum edge degree in Theorem 1 cannot be improved to 10. In view of this, we end this paper by the following open problem.

Problem 1. Determine whether $C_3 \in \mathcal{L}(\mathcal{P}_5^1(11))$ or not.

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