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Note

On r-equitable chromatic threshold of Kronecker products of complete graphs[☆]

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ABSTRACT

Let r and k be positive integers. A graph G is r-equitably k-colorable if its vertex set can be partitioned into k independent sets, any two of which differ in size by at most r. The r-equitable chromatic threshold of a graph G, denoted by $\chi^*_{r=}(G)$, is the minimum k such that G is r-equitably k'-colorable for all $k' \geq k$. Let $G \times H$ denote the Kronecker product of graphs G and H. In this paper, we completely determine the exact value of $\chi^*_{r=}(K_m \times K_n)$ for general m, n and r. As a consequence, we show that for $r \geq 2$, if $n \geq \frac{1}{r-1}(m+r)(m+2r-1)$ then $K_m \times K_n$ and its spanning supergraph $K_{m(n)}$ have the same r-equitable colorability, and in particular $\chi^*_{r=}(K_m \times K_n) = \chi^*_{r=}(K_{m(n)})$, where $K_{m(n)}$ is the complete m-partite graph with n vertices in each part.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a positive integer k, let $[k] = \{1, 2, \dots, k\}$. A (proper) *k-coloring* of G is a mapping $f : V(G) \to [k]$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a k-coloring. We call the set $f^{-1}(i) = \{x \in V(G): f(x) = i\}$ a *color class* for each $i \in [k]$. Notice that each color class in a proper coloring is an independent set, i.e., a subset of $V(G)$ of pairwise non-adjacent vertices, and hence a k-coloring is a partition of $V(G)$ into k independent sets. For a fixed positive integer r, an *r-equitable k-coloring* of G is a k-coloring for which any two color classes differ in size by at most r. A graph is *r-equitably k-colorable* if it has an r-equitable k-coloring. The *r-equitable chromatic number* of G, denoted by $\chi_{r=}(G)$, is the smallest integer k such that G is r-equitably k-colorable. For a graph G, the *r-equitable chromatic threshold* of G, denoted by $\chi^*_{r=}(G)$, is the smallest integer k such that G is r-equitably k'-colorable for all $k' \geq k$. Although the concept of r-equitable colorability seems a natural generalization of usual equitable colorability (corresponding to $r = 1$) introduced by Meyer [9] in 1973, it was first proposed recently by Hertz and Ries [6,7], where the authors generalized the characterizations of usual equitable colorability of trees [2] and forests [1] to r-equitable colorability. Quite recently, Yen [12] proposed a necessary and sufficient condition for a complete multipartite graph G to have an r-equitable k-coloring and also gave exact values of $\chi_{r=}(G)$ and $\chi^*_{r=}(G)$. In particular, they obtained the following results for $K_{m(n)}$, where $K_{m(n)}$ denotes the complete m-partite graph with n vertices in each part.

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Lemma 1 (*[12]*). *For integers $n, r \geq 1$ and $k \geq m \geq 2$, $K_{m(n)}$ is r-equitably k-colorable if and only if $\left\lceil \frac{n}{\lfloor k/m \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil k/m \rceil} \right\rfloor \leq r$.*

Lemma 2 (*[12]*). *For integers $n, r \geq 1$ and $m \geq 2$, we have $\chi^*_{r=}(K_{m(n)}) = m\lceil \frac{n}{\theta+r} \rceil$, where θ is the minimum positive integer such that $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$.*

The special case of Lemmas 1 and 2 for $r = 1$ was obtained by Lin and Chang [8].

For two graphs G and H, the *Kronecker product* $G \times H$ of G and H is the graph with vertex set $\{(x, y): x \in V(G), y \in V(H)\}$ and edge set $\{(x, y)(x', y'): xx' \in E(G) \text{ and } yy' \in E(H)\}$. In this paper, we analyze the r-equitable colorability of Kronecker product of two complete graphs. We refer to [3,5,8,11] for more studies on the usual equitable colorability of Kronecker products of graphs.

In [4], Duffus et al. showed that if $m \leq n$ then $\chi(K_m \times K_n) = m$. From this result, Chen [3] got that $\chi_=(K_m \times K_n) = m$ for $m \leq n$. Indeed, let $V(K_m \times K_n) = \{(x_i, y_j): i \in [m], j \in [n]\}$. Then we can partition $V(K_m \times K_n)$ into m sets $\{(x_i, y_j): j \in [n]\}$ with $i = 1, 2, \ldots, m$, all of which have equal size and are clearly independent. Similarly, for any $r \geq 1$, $\chi_{r=}(K_m \times K_n) = m$ for $m \leq n$. However, it is much more difficult to determine the exact value of $\chi^*_{r=}(K_m \times K_n)$, even for $r = 1$.

Lemma 3 (*[8]*). *For positive integers $m \leq n$, we have $\chi^*_=(K_m \times K_n) \leq \lceil \frac{mn}{m+1} \rceil$.*

In the same paper, Lin and Chang determined the exact values of $\chi^*_=(K_2 \times K_n)$ and $\chi^*_=(K_3 \times K_n)$. Note that the case when $m = 1$ is trivial since $K_1 \times K_n$ is the empty graph I_n and hence $\chi^*_=(K_1 \times K_n) = 1$. Recently, those results have been improved to the following.

Theorem 4 (*[10]*). *For integers $n \geq m \geq 2$,*

$$\chi^*_=(K_m \times K_n) = \begin{cases} \left\lceil \dfrac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m-1 \pmod{m+1}; \\ m\left\lceil \dfrac{n}{s^*} \right\rceil, & \text{if } n \equiv 0, 1, m \pmod{m+1}, \end{cases}$$

where s^* is the minimum positive integer such that $s^* \nmid n$ and $m\lceil \frac{n}{s^*} \rceil \leq \lceil \frac{mn}{m+1} \rceil$.

From the definition of s^*, we see that $s^* \neq 1$ and hence $s^* \geq 2$. Let $\theta = s^* - 1$. Then we can restate Theorem 4 as follows.

Theorem 5. *For integers $n \geq m \geq 2$,*

$$\chi^*_=(K_m \times K_n) = \begin{cases} \left\lceil \dfrac{mn}{m+1} \right\rceil, & \text{if } n \equiv 2, \ldots, m-1 \pmod{m+1}; \\ m\left\lceil \dfrac{n}{\theta+1} \right\rceil, & \text{if } n \equiv 0, 1, m \pmod{m+1}, \end{cases}$$

where θ is the minimum positive integer such that $\theta + 1 \nmid n$ and $m\lceil \frac{n}{\theta+1} \rceil \leq \lceil \frac{mn}{m+1} \rceil$.

A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is a *spanning subgraph* of G if it has the same vertex set as G.

Corollary 6. *If $n \geq m$ and $n \equiv 2, \ldots, m-1 \pmod{m+1}$ then $\chi^*_=(K_m \times K_n) < \chi^*_=(K_{m(n)})$.*

Proof. Since $K_m \times K_n$ is a spanning subgraph of $K_{m(n)}$, $\chi^*_=(K_m \times K_n) \leq \chi^*_=(K_{m(n)})$. Therefore, the corollary follows if we can show $\chi^*_=(K_m \times K_n) \neq \chi^*_=(K_{m(n)})$. Let $n = (m+1)s + t$ with $s = \lfloor \frac{n}{m+1} \rfloor$ and $2 \leq t \leq m-1$. We have $\lceil \frac{mn}{m+1} \rceil = \lceil \frac{m(m+1)s+mt}{m+1} \rceil = \lceil \frac{m(m+1)s+(m+1)t-t}{m+1} \rceil = ms + t + \lceil \frac{-t}{m+1} \rceil = ms + t$. By Theorem 5, $\chi^*_=(K_m \times K_n) = \lceil \frac{mn}{m+1} \rceil = ms + t$ and hence m is not a factor of $\chi^*_=(K_m \times K_n)$. On the other hand, by Lemma 2, m is a factor of $\chi^*_=(K_{m(n)})$. Therefore, $\chi^*_=(K_m \times K_n) \neq \chi^*_=(K_{m(n)})$ and hence the proof is complete. □

The main purpose of this paper is to obtain the exact value of $\chi^*_{r=}(K_m \times K_n)$ for any $r \geq 1$, which we state as the following theorem.

Theorem 7. *For any integers $n \geq m \geq 2$ and $r \geq 1$,*

$$\chi^*_{r=}(K_m \times K_n) = \begin{cases} n - r\left\lfloor \dfrac{n}{m+r} \right\rfloor, & \text{if } n \equiv 2, \ldots, m-1 \pmod{m+r} \text{ and} \\ & \left\lceil \dfrac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \dfrac{n}{\lceil n/(m+r) \rceil} \right\rfloor > r; \\ m\left\lceil \dfrac{n}{\theta+r} \right\rceil, & \text{otherwise}, \end{cases}$$

where θ is the minimum positive integer such that $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$ and $m\lceil \frac{n}{\theta+r} \rceil \leq \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$.

Theorem 7 agrees with Theorem 5 when $r = 1$. First, $n - \lfloor \frac{n}{m+1} \rfloor = n + \lceil \frac{-n}{m+1} \rceil = \lceil \frac{(m+1)n-n}{m+1} \rceil = \lceil \frac{mn}{m+1} \rceil$. Second, we claim that $n \equiv 2, \ldots, m-1 (\mathrm{mod}\ m+1)$ implies $\left\lceil \frac{n}{\lfloor n/(m+1) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+1) \rceil} \right\rfloor > 1$. Let $n = (m+1)s + t$ with $s = \lfloor \frac{n}{m+1} \rfloor$ and $2 \le t \le m-1$. Then $(m+1)s < n < (m+1)(s+1)$ and hence

$$\left\lceil \frac{n}{\lfloor n/(m+1) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+1) \rceil} \right\rfloor = \left\lceil \frac{n}{s} \right\rceil - \left\lfloor \frac{n}{s+1} \right\rfloor \ge (m+2) - m \ge 2.$$

Finally, we need to check that two definitions of θ in Theorems 5 and 7 are equivalent. Clearly, $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+1} \rceil$ if and only if $\theta + 1 \nmid n$. Since $m\lceil \frac{n}{m+1} \rceil$ is an integer and $m\lceil \frac{n}{m+1} \rceil \ge \frac{mn}{m+1}$, we have $m\lceil \frac{n}{m+1} \rceil \ge \lceil \frac{mn}{m+1} \rceil$. As we have already shown $n - \lfloor \frac{n}{m+1} \rfloor = \lceil \frac{mn}{m+1} \rceil$, we see that $\min\{n - \lfloor \frac{n}{m+1} \rfloor, m\lceil \frac{n}{m+1} \rceil\} = \lceil \frac{mn}{m+1} \rceil$. This shows that the two definitions of θ are equivalent.

For fixed integers m and $r \ge 2$, Theorem 7 can be simplified when n is sufficiently large. Compared to Corollary 6, the following theorem indicates that the behaviors of $\chi^*_{r=}(K_{m(n)})$ and $\chi^*_{r=}(K_m \times K_n)$ with $r \ge 2$ are quite different from the case when $r = 1$.

Theorem 8. *For any integers $n \ge m \ge 2$ and $r \ge 2$, if $n \ge \frac{1}{r-1}(m+r)(m+2r-1)$ then $\chi^*_{r=}(K_m \times K_n) = \chi^*_{r=}(K_{m(n)})$, and moreover, $K_m \times K_n$ and $K_{m(n)}$ have the same r-equitable colorability, that is, $K_m \times K_n$ is r-equitably k-colorable if and only if $K_{m(n)}$ is r-equitably k-colorable.*

2. Proofs of Theorems 7 and 8

Let us begin with the following

Lemma 9. *Let m, n and r be positive integers and let $n = (m+r)s + t$, where $s = \lfloor \frac{n}{m+r} \rfloor$. Then*

$$\min\left\{n - r\left\lfloor \frac{n}{m+r} \right\rfloor, m\left\lceil \frac{n}{m+r} \right\rceil\right\} = \begin{cases} ms + t, & 0 \le t \le m-1, \\ m(s+1), & m \le t \le m+r-1. \end{cases}$$

Proof. Clearly, $n - r\lfloor \frac{n}{m+r} \rfloor = (m+r)s + t - rs = ms + t$ and

$$m\left\lceil \frac{n}{m+r} \right\rceil = \begin{cases} ms, & t = 0, \\ ms + m, & t = 1, \ldots, m+r-1. \end{cases}$$

The lemma follows. □

Now we give an upper bound for $\chi^*_{r=}(K_m \times K_n)$, a generalization of Lemma 3.

Lemma 10. *For positive integers $m \le n$ and r, we have $\chi^*_{r=}(K_m \times K_n) \le \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$.*

Proof. Let $\Gamma = \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$ and let k be any integer with $k \ge \Gamma$. We need to show that $K_m \times K_n$ is r-equitably k-colorable. Since $\chi^*_{r=}(K_m \times K_n) \le \chi^*_{=}(K_m \times K_n)$ and $\lceil \frac{mn}{m+1} \rceil \le n$, Lemma 3 implies $\chi^*_{r=}(K_m \times K_n) \le n$. Therefore, we further may assume $k \le n$ and hence $\Gamma \le k \le n$. Let $V(K_m \times K_n) = \{(x_i, y_j) \colon i \in [m], j \in [n]\}$ and $n = (m+r)s + t$, where $s = \lfloor \frac{n}{m+r} \rfloor$.

Case 1: $0 \le t \le m-1$.

By Lemma 9, $\Gamma = ms + t$. Let $V_j = \{(x_i, y_j) \colon i \in [m]\}$ for $1 \le j \le k - ms$. By the definition of Kronecker products, each V_j is an independent set. Let $n' = n - (k - ms)$. Since $ms + t = \Gamma \le k \le n$, we have $ms \le n' \le n - t = (m+r)s$ and hence

$$m \le \left\lfloor \frac{n'}{s} \right\rfloor \le \left\lceil \frac{n'}{s} \right\rceil \le m + r. \quad (1)$$

Let $U_i = \{(x_i, y_j) \colon k - ms + 1 \le j \le n\}$ for $i = 1, 2, \ldots, m$. Clearly each U_i is an independent set of size n'. Therefore, we can partition each U_i with $i = 1, 2, \ldots, m$ into s independent sets, each of which has size $\lfloor \frac{n'}{s} \rfloor$ or $\lceil \frac{n'}{s} \rceil$. In this way, we partition $\cup_{i=1}^{m} U_i$ into ms independent sets and all of these sets have sizes between m and $m + r$ because of (1). Since each V_j with $1 \le j \le k - ms$ is of size m, combining $V_1, \ldots, V_{k-ms}$ with these ms independent sets gives an r-equitable k-coloring of $K_m \times K_n$.

Case 2: $m \le t \le m + r - 1$.

By Lemma 9, $\Gamma = m(s+1)$ and hence $m(s+1) \le k \le n$. Let $V_j = \{(x_i, y_j) \colon i \in [m]\}$ for $1 \le j \le k - m(s+1)$. Clearly, each V_j is an independent set of size m. Let $n' = n - (k - m(s+1))$. Since $m(s+1) \le k \le n$, we have $m(s+1) \le n' \le n = (m+r)s + t \le (m+r)(s+1)$ and hence

$$m \le \left\lfloor \frac{n'}{s+1} \right\rfloor \le \left\lceil \frac{n'}{s+1} \right\rceil \le m + r. \quad (2)$$

Let $U_i = \{(x_i, y_j): k - m(s+1) + 1 \le j \le n\}$ for $i = 1, 2, \dots, m$. Clearly each U_i is an independent set of size n'. Similar to that of Case 1, from (2), we can partition $\cup_{i=1}^{m} U_i$ into $m(s+1)$ independent sets of sizes between m and $m+r$. Combining $V_1, \dots, V_{k-m(s+1)}$ with these $m(s+1)$ independent sets gives an r-equitable k-coloring of $K_m \times K_n$. □

Since $K_m \times K_n$ is a spanning subgraph of $K_{m(n)}$, any r-equitable k-coloring of $K_{m(n)}$ yields an r-equitable k-coloring of $K_m \times K_n$. The following lemma indicates that the converse is also true under the assumption that k is less than the upper bound given in Lemma 10.

Lemma 11. *For positive integers $m \ge 2, s, \theta, n$ and r, if $K_m \times K_n$ is r-equitably k-colorable for some $k < \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$, then $K_{m(n)}$ is also r-equitably k-colorable.*

Proof. Let $V(K_m \times K_n) = V(K_{m(n)}) = \{(x_i, y_j): i \in [m], j \in [n]\}$. Let c be any r-equitable k-coloring of $K_m \times K_n$ with $k < \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$. It suffices to show that each color class of c is a subset of $\{(x_i, y_j): j \in [n]\}$ for some $i \in [m]$. Let ℓ denote the number of color classes, each of which is a subset of $\{(x_i, y_j): i \in [m]\}$ for some $j \in [n]$. Note that, each independent set of $V(K_m \times K_n)$ is either a subset of $\{(x_i, y_j): j \in [n]\}$ for some $i \in [m]$ or a subset of $\{(x_i, y_j): i \in [m]\}$ for some $j \in [n]$. Therefore, we only need to prove $\ell = 0$. Suppose to the contrary that $\ell > 0$ and let $U_1, \dots, U_\ell$ be such color classes defined above. Since any two color classes of c differ in size by at most r and some color class, say U_1, contains at most m vertices, each color class is of the size at most $m + r$. For each $i \in [m]$, let k_i be the number of color classes contained in $W_i = \{(x_i, y_j): j \in [n]\} \setminus \bigcup_{p=1}^{\ell} U_p$. Since $|W_i| \ge n - \ell$, we have $k_i \ge \lceil \frac{n-\ell}{m+r} \rceil$. Therefore, $k = k_1 + \cdots + k_m + \ell \ge m\lceil \frac{n-\ell}{m+r} \rceil + \ell$.

Define $a_q = m\lceil \frac{n-q}{m+r} \rceil + q$ for $q \ge 0$. Since $a_{q+m+r} = m\lceil \frac{n-q-m-r}{m+r} \rceil + q + m + r = m\lceil \frac{n-q}{m+r} \rceil + q + r = a_q + r > a_q$, the minimum of $\{a_q: q \ge 0\}$ exists and is achieved by a_q for some $q \in \{0, 1, \dots, m + r - 1\}$. Therefore, $k \ge a_\ell \ge \min\{a_0, a_1, \dots, a_{m+r-1}\}$. Let $n = (m + r)s + t$ with $s = \lfloor \frac{n}{m+r} \rfloor$. Now, $a_q = m\lceil \frac{n-q}{m+r} \rceil + q = ms + m\lceil \frac{t-q}{m+r} \rceil + q$. We will distinguish two cases.

Case 1: $0 \le t \le m - 1$.

We claim in this case that $\min\{a_0, a_1, \dots, a_{m+r-1}\} = ms + t$ and hence $k \ge ms + t$. Clearly, $a_t = ms + t$. If $0 \le q \le t - 1$ then $a_q = ms + m\lceil \frac{t-q}{m+r} \rceil + q \ge ms + m > ms + t$. If $t + 1 \le q \le m + r - 1$ then $t - q \ge 0 - (m + r - 1) > -(m + r)$ and hence $a_q = ms + m\lceil \frac{t-q}{m+r} \rceil + q \ge ms + q > ms + t$. On the other hand, by Lemma 9, we have $\min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\} = ms + t$. This is a contradiction to our assumption that $k < \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$.

Case 2: $m \le t \le m + r - 1$.

We claim in this case that $\min\{a_0, a_1, \dots, a_{m+r-1}\} = m(s + 1)$ and hence $k \ge m(s + 1)$. Clearly, $a_0 = ms + m\lceil \frac{t}{m+r} \rceil = m(s + 1)$. If $1 \le q \le t - 1$ then $a_q = ms + m\lceil \frac{t-q}{m+r} \rceil + q \ge ms + m + 1 > m(s + 1)$. If $t \le q \le m + r - 1$ then $a_q = ms + m\lceil \frac{t-q}{m+r} \rceil + q = ms + q \ge ms + t \ge m(s + 1)$. Similarly, by Lemma 9, we have $\min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\} = m(s+1)$, a contradiction. □

Lemmas 10 and 11 reduce the r-equitable colorability of $K_m \times K_n$ to that of $K_{m(n)}$. We need the following two results on r-equitable colorability of $K_{m(n)}$.

Lemma 12. *If m, n, r and θ are positive integers with $m \ge 2$ and $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$, then $K_{m(n)}$ is not r-equitably $\left(m\lceil \frac{n}{\theta+r} \rceil - i\right)$-colorable for $1 \le i < m$.*

Proof. Let $q = \lceil \frac{n}{\theta+r} \rceil$. If $\theta + r \mid n$, then $\lceil \frac{n}{\theta+r} \rceil = \frac{n}{\theta+r} \le \frac{n}{\theta+1}$, yielding $\lceil \frac{n}{\theta+r} \rceil \le \lfloor \frac{n}{\theta+1} \rfloor$, a contradiction to the assumption of this lemma. Hence $\theta + r \nmid n$. Now we have $q = \lceil \frac{n}{\theta+r} \rceil > \frac{n}{\theta+1} \ge \frac{n}{\theta+r} > \lfloor \frac{n}{\theta+r} \rfloor = q - 1$. Consequently, $\frac{n}{q} < \theta + 1$ and $\frac{n}{q-1} > \theta + r$. Note that we may assume $q - 1 \ne 0$ since the lemma trivially follows when $q = 1$. Therefore, $\left\lceil \frac{n}{\lfloor (mq-i)/m \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil (mq-i)/m \rceil} \right\rfloor = \lceil \frac{n}{q-1} \rceil - \lfloor \frac{n}{q} \rfloor \ge (\theta + r + 1) - \theta = r + 1$ for $1 \le i < m$. By Lemma 1, $K_{m(n)}$ is not r-equitably $\left(m\lceil \frac{n}{\theta+r} \rceil - i\right)$-colorable. □

Lemma 13. *For positive integers $m \ge 2, s, \theta, n$ and r, if $K_{m(n)}$ is not r-equitably k-colorable for some $k \ge m\lceil \frac{n}{\theta+r} \rceil$, then there is a positive integer θ' such that $\lfloor \frac{n}{\theta'+1} \rfloor < \lceil \frac{n}{\theta'+r} \rceil$, $\lceil \frac{n}{\theta'+r} \rceil = \lceil \frac{k}{m} \rceil$ and $\theta' < \theta$.*

Proof. By Lemma 1, $\left\lceil \frac{n}{\lfloor k/m \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil k/m \rceil} \right\rfloor > r$. Hence, $\frac{n}{\lfloor k/m \rfloor} > \theta' + r > \theta' + r - 1 > \cdots > \theta' + 1 > \frac{n}{\lceil k/m \rceil}$ for some nonnegative integer θ' and so

$$\left\lceil \frac{k}{m} \right\rceil > \frac{n}{\theta' + 1} > \cdots > \frac{n}{\theta' + r} > \left\lfloor \frac{k}{m} \right\rfloor. \tag{3}$$

If $\theta' = 0$ then the first inequality of (3) implies $k > mn$ and hence $K_{m(n)}$ is clearly r-equitably k-colorable, a contradiction. Thus, $\theta' > 0$. By (3), we see $\lceil \frac{k}{m} \rceil > \lfloor \frac{k}{m} \rfloor$ and hence $\lceil \frac{k}{m} \rceil = \lfloor \frac{k}{m} \rfloor + 1$. Also from (3), we have $\lceil \frac{n}{\theta'+r} \rceil = \lceil \frac{k}{m} \rceil$ and $\lfloor \frac{n}{\theta'+1} \rfloor = \lfloor \frac{k}{m} \rfloor < \lceil \frac{n}{\theta'+r} \rceil$. Finally, $\frac{n}{\theta'+r} > \lfloor \frac{k}{m} \rfloor \geq \lfloor \frac{m}{m} \lceil \frac{n}{\theta+r} \rceil \rfloor = \lceil \frac{n}{\theta+r} \rceil \geq \frac{n}{\theta+r}$ implying $\theta' < \theta$. □

Proof of Theorem 7. Let $\Gamma = \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$ and $n = (m+r)s + t$, where $s = \lfloor \frac{n}{m+r} \rfloor$. We divide the proof into two cases.

Case 1: $n \equiv 2, \ldots, m-1 \pmod{m+r}$ and $\left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor > r$.

Note that $2 \leq t \leq m-1$ from the first condition of this case. By Lemmas 10 and 9, $\chi^*_{r=}(K_m \times K_n) \leq \Gamma = ms + t$. Let $k = ms + t - 1$. We need to show that $K_m \times K_n$ is not r-equitably k-colorable. Noting $k < \Gamma$, it suffices to show that $K_{m(n)}$ is not r-equitably k-colorable by Lemma 11.

Since $2 \leq t \leq m-1$ and $k = ms + t - 1$, we have $ms < k < m(s+1)$. Consequently, $\lfloor \frac{k}{m} \rfloor = s$ and $\lceil \frac{k}{m} \rceil = s+1$. Since $\lfloor \frac{n}{m+r} \rfloor = s$ and $\lceil \frac{n}{m+r} \rceil = s+1$, we have $\left\lceil \frac{n}{\lfloor k/m \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil k/m \rceil} \right\rfloor = \lceil \frac{n}{s} \rceil - \lfloor \frac{n}{s+1} \rfloor = \left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor > r$ from the last condition of this case. Therefore, by Lemma 1, $K_{m(n)}$ is not r-equitably k-colorable. This completes the proof of this case.

Case 2: $n \equiv 0, 1, m, m+1, \ldots, m+r-1 \pmod{m+r}$, or $n \equiv 2, \ldots, m-1 \pmod{m+r}$ and $\left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor \leq r$.

Since $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$, by Lemma 12, $K_{m(n)}$ is not r-equitably $\left(m\lceil \frac{n}{\theta+r} \rceil - 1\right)$-colorable. Since $m\lceil \frac{n}{\theta+r} \rceil - 1 < \Gamma$ from the definition of θ in Theorem 7, by Lemma 11, $K_m \times K_n$ is not r-equitably $\left(m\lceil \frac{n}{\theta+r} \rceil - 1\right)$-colorable. In the following, we prove that $K_m \times K_n$ is r-equitably k-colorable for all $k \geq m\lceil \frac{n}{\theta+r} \rceil$, which implies that $\chi^*_{r=}(K_m \times K_n) = m\lceil \frac{n}{\theta+r} \rceil$.

Suppose to the contrary that $K_m \times K_n$ (and hence $K_{m(n)}$) is not r-equitably k-colorable for some $k \geq m\lceil \frac{n}{\theta+r} \rceil$. By Lemma 10, $k < \Gamma$. By Lemma 13, there is a positive integer θ' such that $\lfloor \frac{n}{\theta'+1} \rfloor < \lceil \frac{n}{\theta'+r} \rceil$, $\lceil \frac{n}{\theta'+r} \rceil = \lceil \frac{k}{m} \rceil$ and $\theta' < \theta$. By the minimality of θ, $m\lceil \frac{n}{\theta'+r} \rceil > \Gamma$. We show that each of the following three subcases yields a contradiction.

Subcase 2.1: $n \equiv 0, 1 \pmod{m+r}$, i.e., $t = 0, 1$.

By Lemma 9, $\Gamma = ms + t$. Since $k < \Gamma$ we see $k < ms + t \leq ms + 1$, and hence $k \leq ms$. Therefore, $m\lceil \frac{n}{\theta'+r} \rceil = m\lceil \frac{k}{m} \rceil \leq ms \leq \Gamma$. This is a contradiction.

Subcase 2.2: $n \equiv m, \ldots, m+r-1 \pmod{m+r}$, i.e., $t = m, \ldots, m+r-1$.

By Lemma 9, $\Gamma = m(s+1)$. Hence $k < m(s+1)$ and $m\lceil \frac{n}{\theta'+r} \rceil = m\lceil \frac{k}{m} \rceil \leq m(s+1) \leq \Gamma$. This is a contradiction.

Subcase 2.3: $n \equiv 2, \ldots, m-1 \pmod{m+r}$ and $\left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor \leq r$.

By Lemma 9, $\Gamma = ms + t$. If $k \leq ms$ then $m\lceil \frac{n}{\theta'+r} \rceil = m\lceil \frac{k}{m} \rceil \leq ms \leq \Gamma$, a contradiction. Now assume that $k > ms$. Since $k < \Gamma = ms + t$, we have $ms < k < ms + t < m(s+1)$, yielding $\lfloor \frac{k}{m} \rfloor = s$ and $\lceil \frac{k}{m} \rceil = s+1$. Consequently, by the second condition of this subcase, $\left\lceil \frac{n}{\lfloor k/m \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil k/m \rceil} \right\rfloor = \lceil \frac{n}{s} \rceil - \lfloor \frac{n}{s+1} \rfloor = \left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor \leq r$. Therefore, $K_{m(n)}$ is r-equitably k-colorable. This is a contradiction. □

Proof of Theorem 8. Comparing Theorem 7 with Lemma 2, it suffices to show, for the first part, that under the assumption of this theorem, the following two statements hold:

(i) $\left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor \leq r$;

(ii) if $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$ then $m\lceil \frac{n}{\theta+r} \rceil \leq \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$.

By the assumption that $n \geq \frac{1}{r-1}(m+r)(m+2r-1)$, we have $(r-1)\frac{n}{m+r} \geq m + 2r - 1$, yielding

$$(r-1)\left\lfloor \frac{n}{m+r} \right\rfloor > (r-1)\frac{n}{m+r} - (r-1) \geq (m+2r-1) - (r-1) = m+r.$$

Multiplying the first and last term of the inequality by $\lceil \frac{n}{m+r} \rceil$ gives

$$(r-1)\left\lfloor \frac{n}{m+r} \right\rfloor \left\lceil \frac{n}{m+r} \right\rceil > (m+r)\left\lceil \frac{n}{m+r} \right\rceil \geq n \geq \left(\left\lceil \frac{n}{m+r} \right\rceil - \left\lfloor \frac{n}{m+r} \right\rfloor\right)n.$$

Dividing by $\lfloor \frac{n}{m+r} \rfloor \lceil \frac{n}{m+r} \rceil$ leads to $\frac{n}{\lfloor n/(m+r) \rfloor} - \frac{n}{\lceil n/(m+r) \rceil} < r - 1$. Hence, $\left\lceil \frac{n}{\lfloor n/(m+r) \rfloor} \right\rceil - \left\lfloor \frac{n}{\lceil n/(m+r) \rceil} \right\rfloor < r + 1$, which implies (i).

Now we assume further $\lfloor \frac{n}{\theta+1} \rfloor < \lceil \frac{n}{\theta+r} \rceil$ and show $m\lceil \frac{n}{\theta+r} \rceil \leq \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$. If $\frac{n}{\theta+1} - \frac{n}{\theta+r} \geq 1$ then $\lfloor \frac{n}{\theta+1} \rfloor \geq \lfloor \frac{n}{\theta+r} + 1 \rfloor \geq \lceil \frac{n}{\theta+r} \rceil$, a contradiction. Hence $\frac{n}{\theta+1} - \frac{n}{\theta+r} < 1$. Multiplying by $(\theta+1)(\theta+r)$ gives $(\theta+1)(\theta+r) > (r-1)n \geq (m+r)(m+2r-1)$, implying $\theta > m+r-1$. Hence $m\lceil \frac{n}{\theta+r} \rceil \leq m\lceil \frac{n}{m+r} \rceil$. It remains to show $m\lceil \frac{n}{\theta+r} \rceil \le n - r\lfloor \frac{n}{m+r} \rfloor$. Since $\theta > m + r - 1$ and $n \ge \frac{1}{r-1}(m+r)(m+2r-1)$, we have

$$\begin{aligned} m\left\lceil \frac{n}{\theta+r} \right\rceil + r\left\lfloor \frac{n}{m+r} \right\rfloor - n &\le m\left\lceil \frac{n}{m+2r-1} \right\rceil + \left(r\frac{n}{m+r} - n \right) \\ &\le m\left(1 + \frac{n}{m+2r-1}\right) - m\frac{n}{m+r} \\ &= m\left(1 - \frac{(r-1)n}{(m+r)(m+2r-1)}\right) \\ &\le 0, \end{aligned}$$

as desired.

Since $K_m \times K_n$ is a spanning subgraph of $K_{m(n)}$, $K_{m(n)}$ has an r-equitable k-coloring only if $K_m \times K_n$ has an r-equitable k-coloring. Suppose that $K_m \times K_n$ is r-equitably k-colorable for some integer k. If $k \ge \chi^*_{r=}(K_m \times K_n)$ then $k \ge \chi^*_{r=}(K_{m(n)})$, since $\chi^*_{r=}(K_m \times K_n) = \chi^*_{r=}(K_{m(n)})$, and hence $K_{m(n)}$ is r-equitably k-colorable. If $k < \chi^*_{r=}(K_m \times K_n)$, then $k < \min\{n - r\lfloor \frac{n}{m+r} \rfloor, m\lceil \frac{n}{m+r} \rceil\}$ by Lemma 10. Therefore, Lemma 11 implies that $K_{m(n)}$ is r-equitably k-colorable. This completes the proof of Theorem 8. □

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