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On Edge Colorings of 1-Toroidal Graphs

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Abstract A graph is 1-toroidal, if it can be embedded in the torus so that each edge is crossed by at most one other edge. In this paper, it is proved that every 1-toroidal graph with maximum degree $\Delta \geq 10$ is of class one in terms of edge coloring. Meanwhile, we show that there exist class two 1-toroidal graphs with maximum degree Δ for each $\Delta \leq 8$.

Keywords 1-Toroidal graph, 1-planar graph, edge coloring

MR(2010) Subject Classification 05C10, 05C15

1 Introduction

All graphs considered in this paper are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $N_G(v)$ and $d_G(v)$, we denote the set of neighbors of v and the degree of v in G, respectively. For a vertex set $S \subseteq V(G)$, let $N_G(S) = \bigcup_{v\in S} N_G(v)$. For a plane graph G, $F(G)$ denotes its face set and $d_G(f)$ denotes the degree of a face f in G. Throughout this paper, a k-, k^+- and k^--vertex (resp. face) is a vertex (resp. face) of degree k, at least k and at most k, respectively. We say that u is a *k-neighbor* of v in G if $uv \in E(G)$ and $d_G(u) = k$. Any undefined notation follows that of Bondy and Murty [1].

A *proper edge coloring* of a graph is an assignment of colors to the edges of the graph so that no two adjacent edges receive the same color. The smallest number of colors needed in a proper edge coloring of a graph G is the *edge chromatic number*, denoted by $\chi'(G)$. A well-known theorem of Vizing states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every simple graph G. By this way, we can divide all simple graphs into two classes: a graph G is of *class one* if $\chi'(G) = \Delta(G)$, and is of *class two* if $\chi'(G) = \Delta(G) + 1$. Consequently, a major question in the area of edge

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colorings is to determine which of these two classes a given graph belongs. For a planar graph G, it is proved that G is of class one provided that $\Delta(G) \geq 7$ (see [13, 16]) and G can be of class two if $\Delta(G) \leq 5$ (see [14]). In this paper, we focus on non-planar graphs.

A graph G is 1-*embedded* in a surface if it can be drawn in the surface so that each edge is crossed by at most one other edge. In particular, a graph G is 1-*planar* if it can be 1-embedded in the plane and is 1-*toroidal* if it can be 1-embedded in the torus. Usually, it is assumed that G is non-planar here and elsewhere. The notion of 1-planar graphs was introduced by Ringel [12] while trying to simultaneously color the vertices and faces of a plane graph G such that any pair of adjacent/incident elements receive different colors. The structure and coloring of 1-planar graphs have been extensively studied by many authors including [2–10, 17–26]. In particular, Zhang and Wu [21] proved the following theorem.

Theorem 1.1 *Every* 1*-planar graph with maximum degree at least* 10 *is of class one.*

In this paper, we aim to extend Theorem 1.1 to the following theorem. Note that every 1-planar graph is 1-toroidal.

Theorem 1.2 *Every* 1*-toroidal graph with maximum degree at least* 10 *is of class one.*

Throughout this paper, for any 1-toroidal graph G, we always assume that G has already been embedded in a torus such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. We call such an embedding of G 1-*torus graph*. The *associated torus graph* $G^\times$ of a 1-torus graph G is the graph obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is *false* if it is a new added vertex and is *true* otherwise. We call a face in $G^\times$ *false* or *true* according to whether it is incident with a false vertex or not.

2 Proof of Theorem 1.2

For our purpose, the following lemma of Zhang and Wu is a useful starting point.

Lemma 2.1 ([21]) *If G is a* 1*-torus graph and $G^\times$ is the associated torus graph of G, then the following hold*:

(1) *For any two false vertices u and v in $G^\times$, $uv \notin E(G^\times)$.*

(2) *If there is a* 3*-face $uvwu$ in $G^\times$ such that $d_G(v) = 2$, then u and w are both true vertices.*

(3) *If $d_G(u) = 3$ and v is a false vertex in $G^\times$, then either $uv \notin E(G^\times)$ or uv is not incident with two* 3*-faces.*

(4) *If a* 3*-vertex v in G is incident with two* 3*-faces and adjacent to two false vertices in $G^\times$, then v is incident with a* 5^+*-face.*

(5) *For any* 4*-vertex u in G, u is incident with at most three false* 3*-faces.*

(6) *For any k-vertex u in G, where $k \geq 5$, u is incident with at most $2\lfloor \frac{k}{2} \rfloor$ false* 3*-faces.*

Here it should be remarked that the corresponding lemma in [21] is proved for 1-plane graphs, but the proofs are also available for 1-torus graphs.

A graph G is *critical* if it is connected, of class two and $\chi'(G - e) < \chi'(G)$ for every edge $e \in E(G)$. A critical graph with maximum degree Δ is a Δ-*critical graph*. The following four lemmas on the structures of Δ-critical graphs play important roles in our proof of Theorem 1.2.

In particular, Lemma 2.2 is well known as Vizing's adjacency lemma, which will be cited as VAL throughout this section.

Lemma 2.2 ([15]) *Let G be a Δ-critical graph. If v and w are adjacent vertices of G and $d_G(v) = k$, then*

(1) *if $k < \Delta$, then w is adjacent to at least $(\Delta - k + 1)$ Δ-vertices;*

(2) *if $k = \Delta$, then w is adjacent to at least two Δ-vertices.*

Lemma 2.3 ([16]) *Let G be a Δ-critical graph. If $xy \in E(G)$ and $d_G(x) + d_G(y) = \Delta + 2$, then*

(1) *every vertex of $N_G(N_G(x, y))\backslash\{x, y\}$ is of degree at least $\Delta - 1$;*

(2) *if $d_G(x), d_G(y) < \Delta$, then every vertex of $N_G(N_G(x, y))\backslash\{x, y\}$ is a Δ-vertex.*

Lemma 2.4 ([11]) *Let G be a Δ-critical graph with $\Delta \geq 5$. If x is a 3-vertex in G, then there are at least two Δ-vertices in $N_G(x)$ that are not adjacent to any $(\Delta - 2)^-$-vertices except x.*

Lemma 2.5 ([11]) *Let G be a Δ-critical graph with $\Delta \geq 6$ and let x be a 4-vertex. If x is not adjacent to any $(\Delta - 2)$-vertex and one of the neighbors y of x is adjacent to $d_G(y) - (\Delta - 3)$ $(\Delta - 2)^-$-vertices, then each of the other three neighbors of x is adjacent to only one $(\Delta - 2)^-$-vertex, which is x.*

Let G be a counterexample to Theorem 1.2 with the smallest number of edges. One can observe that G is Δ-critical and $\delta(G) \geq 2$. For a vertex v in G, denote the degree of the neighbors of v in G as $\delta_1(v) \leq \delta_2(v) \leq \cdots \leq \delta_{d_G(v)}(v)$. In what follows, we process by the discharging method.

First of all, we assign an initial charge $c(v) = d_G(v) - 4$ to every vertex $v \in V(G)$ and $c(f) = d_{G^\times}(f) - 4$ to every face $f \in F(G^\times)$. By Euler's formula $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 0$ on the torus graph $G^\times$ and by the fact that $c(v) = 0$ for every $v \in V(G^\times)\backslash V(G)$, one can easily deduce that

$$\begin{aligned}\sum_{x \in V(G) \cup F(G^\times)} c(x) &= \sum_{v \in V(G)} (d_G(v) - 4) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 4) \\ &= \sum_{v \in V(G^\times)} (d_{G^\times}(v) - 4) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 4) \\ &= 0.\end{aligned}$$

Whereafter, we define discharging rules that only move charge around but do not affect the total charges. Let c' be the final charge function on the vertices and faces after discharging. We first prove that $c'(x) \geq 0$ for every $x \in V(G) \cup F(G^\times)$, and then show that there exists a vertex $v \in V(G)$ with $c'(v) > 0$. Therefore, we would get $0 = \sum_{x \in V(G) \cup F(G^\times)} c(x) = \sum_{x \in V(G) \cup F(G^\times)} c'(x) > 0$, a contradiction completing the proof.

Based on the discharging procedure described as above, the proof of Theorem 1.1 (see [21, pp. 126–128]) actually implies that every 1-toroidal graph with maximum degree at least 11 is of class one*. Therefore, we only need consider a single case $\Delta = 10$ in this section.

The discharging rules are defined as follows.

*In that paper, the authors proved the nonnegativity of every vertex and face, and in particular, they proved that the final charge of every Δ-vertex is at least $(\Delta - 10)/2$.

R1 If v is a 2-vertex in G and $uv \in E(G)$, then v receives $\frac{1}{2}$ from u and receives $\frac{1}{18}$ from each of the vertices in $N_G(u)\backslash\{v\}$ through u.

R2 If v is a 3-vertex and u is a 10-neighbor of v in G so that u is not adjacent to any 8^--vertices in G except v, then v receives $\frac{3}{4}$ from u.

R3 Let v be a 4-vertex in G and let v_1, v_2, v_3, v_4 be neighbors of v in G so that $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3) \leq d_G(v_4)$.

R3.1 If $d_G(v_1) = 8$, then v receives $\frac{1}{2}$ from each of v_2, v_3 and v_4.

R3.2 If $d_G(v_1) \geq 9$ and v_4 is adjacent to exactly three 8^--vertices in G, then v receives $\frac{3}{8}$ from each of v_1, v_2 and receives $\frac{3}{4}$ from v_3.

R3.3 If $d_G(v_1) \geq 9$ and any 10-neighbor of v is adjacent to at most two 8^--vertices in G, then v receives $\frac{3}{8}$ from each of the neighbors of v in G.

R4 If v is a 5-vertex in G, then v receives $\frac{1}{5}$ from each of the neighbors of v in G.

R5 Let v be a 6-vertex in G and let $v_1, v_2, v_3, v_4, v_5, v_6$ be neighbors of v in G so that $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3) \leq d_G(v_4) \leq d_G(v_5) \leq d_G(v_6)$.

R5.1 If $d_G(v_1) = 6$, then v receives $\frac{1}{5}$ from each of v_2, v_3, v_4, v_5 and v_6.

R5.2 If $d_G(v_1) = 7$, then v receives $\frac{3}{28}$ from each of v_1, v_2 and receives $\frac{11}{56}$ from each of v_3, v_4, v_5 and v_6.

R5.3 If $d_G(v_1) \geq 8$, then v receives $\frac{1}{6}$ from each of the neighbors of v in G.

R6 If v is a 7-vertex in G, then v receives $\frac{1}{56}$ from each of its 8-neighbors, $\frac{3}{70}$ from each of its 9-neighbors and $\frac{1}{7}$ from each of its 10-neighbors.

R7 If v is a 8-vertex or 9-vertex in G, then v receives $\frac{1}{10}$ from each of its 10-neighbors.

R8 If f is a false 3-face in $G^\times$, then f receives $\frac{1}{2}$ from each of its incident true vertices.

R9 Let f be a true 3-face in $G^\times$ and let v_1, v_2, v_3 be vertices that are incident with f so that $d_G(v_1) \leq d_G(v_2) \leq d_G(v_3)$.

R9.1 If $d_G(v_1) \leq 5$, then f receives $\frac{1}{2}$ from each of v_2 and v_3.

R9.2 If $d_G(v_1) \geq 6$, then f receives $\frac{1}{3}$ from each of v_1, v_2 and v_3.

R10 If f is a 5^+-face and v is a 3-vertex incident with f, then v receives $\frac{1}{2}$ from f.

We now prove that $c'(x) \geq 0$ for every $x \in F(G^\times) \cup V(G)$.

First of all, for every false 3-face f we have $c'(f) = -1 + 2 \times \frac{1}{2} = 0$ by R8, since f is incident with two true vertices in $G^\times$, and for every true 3-face f, we have $c'(f) \geq -1 + \min\{2 \times \frac{1}{2}, 3 \times \frac{1}{3}\} = 0$ by R9. The final charge of every 4-face in $G^\times$ is exactly 0, since 4-faces are not involved in the rules. For a 5^+-face f, the number of 3-vertices that are incident with f cannot exceed the half of the degree of f, since no two 3-vertices are adjacent in G (and thus in $G^\times$) by VAL. Therefore, $c'(f) \geq d_{G^\times}(f) - 4 - \frac{1}{2}\lfloor\frac{d_{G^\times}(f)}{2}\rfloor \geq 0$ by R10.

Let v be a vertex in G. If $d_G(v) = 2$ (suppose $N_G(v) = \{u, w\}$), then by VAL, every vertex in $N_G(u, w) \cup \{u, w\}$ is of degree 10 except v. Moreover, v is incident with no false 3-faces by (2) of Lemma 2.1. Thus by R1, R8 and R9, $c'(v) \geq -2 + 2 \times \frac{1}{2} + 2 \times 9 \times \frac{1}{18} = 0$.

If $d_G(v) = 3$, then by Lemma 2.4, there are at least two 10-vertices in $N_G(v)$ that are not adjacent to any 8^--vertices except v, so v receives at least $2 \times \frac{3}{4} = \frac{3}{2}$ from those 10-neighbors of v by R2. If v is incident with at most one false 3-face, then by R8 and R9, v sends at most $\frac{1}{2}$ to its incident faces and therefore, $c'(v) \geq -1 + \frac{3}{2} - \frac{1}{2} = 0$. If v is incident with at least two false 3-faces, then by (3) of Lemma 2.1, v is adjacent to exactly two false vertices and incident with exactly two false 3-faces in $G^{\times}$, to which v sends at most $2 \times \frac{1}{2} = 1$ by R8 and R9. By (4) of Lemma 2.1, v is also incident with a 5^+-face, from which v receives $\frac{1}{2}$ by R10. Therefore, $c'(v) \geq -1 - 1 + \frac{1}{2} + \frac{3}{2} = 0$.

If $d_G(v) = 4$, then by (5) of Lemma 2.1, v is incident with at most three false 3-faces, so by R8 and R9, v sends at most $3 \times \frac{1}{2} = \frac{3}{2}$ to its incident faces. On the other hand, we have $\delta_1(v) \geq 8$ by VAL (otherwise, v has a 7^--neighbor, thus by VAL, v is adjacent to at least $10-7+1 = 4$, 10-vertices in G, a contradiction). If $\delta_1(v) = 8$, then by R3.1, v receives $3\times\frac{1}{2} = \frac{3}{2}$ from its neighbors in G, so $c'(v) = 0 - \frac{3}{2} + \frac{3}{2} = 0$. If $\delta_1(v) \geq 9$, then either any 10-neighbor of v is adjacent to at most two 8^--vertices in G, in which case v receives $4 \times \frac{3}{8} = \frac{3}{2}$ from its neighbors in G by R3.3, or one 10-neighbor of v is adjacent to exactly three 8^--vertices in G by VAL, in which case v receives $2 \times \frac{3}{8} + \frac{4}{3} = \frac{25}{12} > \frac{3}{2}$ from its neighbors in G by R3.2. Therefore, we have $c'(v) \geq 0 - \frac{3}{2} + \frac{3}{2} = 0$.

If $d_G(v) = 5$, then by (6) of Lemma 2.1, v is incident with at most four false 3-faces, so by R8 and R9, v sends at most $4 \times \frac{1}{2} = 2$ to its incident faces. On the other hand, v receives $5 \times \frac{1}{5} = 1$ from its neighbors in G by R4. Thus, $c'(v) \geq 1 - 2 + 1 = 0$.

If $d_G(v) = 6$, then by VAL, $\delta_1(v) \geq 6$. Let v_1 be the neighbor of v in G with $d_G(v_1) = \delta_1(v)$. If $d_G(v_1) = 6$, then by R5.1, v receives $5 \times \frac{1}{5} = 1$ from its neighbors in G. If $d_G(v_1) = 7$, then by R5.2, v receives $2 \times \frac{3}{28} + 4 \times \frac{11}{56} = 1$ from its neighbors in G. If $d_G(v_1) \geq 8$, then by R5.3, v receives $6 \times \frac{1}{6} = 1$ from its neighbors in G. Note that v sends at most $6 \times \frac{1}{2} = 3$ to its incident faces by R8 and R9. Thus, $c'(v) \geq 2 + 1 - 3 = 0$.

If $d_G(v) = 7$, then by VAL, $\delta_1(v) \geq 5$. If $\delta_1(v) = 5$, then v shall be adjacent to at least $10 - 5 + 1 = 6$, 10-vertices in G by VAL, which implies that $\delta_2(v) = 10$. In this case, v sends at most $7 \times \frac{1}{2} = \frac{7}{2}$ to its incident faces by R8 and R9, and $\frac{1}{5}$ to its adjacent 5-vertex in G by R4. Meanwhile, v receives $6 \times \frac{1}{7} = \frac{6}{7}$ from its adjacent 10-vertices in G by R6. Thus, $c'(v) \geq 3 - \frac{7}{2} - \frac{1}{5} + \frac{6}{7} > 0$. If $\delta_1(v) = 6$, then $\delta_3(v) = 10$ by VAL. In this case, v sends at most $7 \times \frac{1}{2} = \frac{7}{2}$ to its incident faces by R8 and R9 and receives $5 \times \frac{1}{7} = \frac{5}{7}$ from its adjacent 10-vertices in G by R6. Let v_1 and v_2 be the neighbors of v in G with $d_G(v_1) = \delta_1(v)$ and $d_G(v_2) = \delta_2(v)$. First of all, v sends to v_1 at most $\frac{3}{28}$ by R5 and VAL. If $d_G(v_2) > 6$, then v sends none to v_2; otherwise, v_2 is a 6-vertex and thus v sends to v_2 at most $\frac{3}{28}$ by R5 and VAL. Therefore, we have $c'(v) \geq 3 - \frac{7}{2} + \frac{5}{7} - 2 \times \frac{3}{28} = 0$. If $\delta_1(v) = 7$, then $\delta_4(v) = 10$ by VAL. Note that v sends none to its neighbors in G and v is adjacent to four 10-vertices in G, from which v receives $4 \times \frac{1}{7} = \frac{4}{7}$ by R6, so by R8 and R9, we have $c'(v) \geq 3 + 4 \times \frac{1}{7} - 7 \times \frac{1}{2} > 0$. If $\delta_1(v) = 8$, then $\delta_5(v) = 10$ by VAL, which implies that v is adjacent to three 10-vertices in G, from which v receives $3 \times \frac{1}{7} = \frac{3}{7}$ by R6. Since v receives at least $\frac{1}{56}$ from each of its 8^+-neighbors by R6, we have $c'(v) \geq 3 + \frac{3}{7} + 4 \times \frac{1}{56} - 7 \times \frac{1}{2} = 0$ by R8 and R9. If $\delta_1(v) = 9$, then $\delta_6(v) = 10$ by VAL. Similarly, we can prove that $c'(v) \geq 3 + 2 \times \frac{1}{7} + 5 \times \frac{3}{70} - 7 \times \frac{1}{2} = 0$ by R6, R8 and R9. If $\delta_1(v) = 10$, then it is easy to check that $c'(v) \geq 3 - 7 \times \frac{1}{2} + 7 \times \frac{1}{7} > 0$ by R6, R8 and R9.

If $d_G(v) = 8$, then by VAL, $\delta_1(v) \geq 4$. If $\delta_1(v) = 4$, then $\delta_2(v) = 10$ by VAL. Since v sends at most $\frac{1}{2} \times 8 = 4$ to its incident faces by R8 and R9, and sends none to any of its 4-neighbors by R3.1, we have $c'(v) \geq 4 - 4 = 0$. If $\delta_1(v) = 5$, then $\delta_3(v) = 10$ by VAL, which implies that v has six 10-neighbors in G, from which v receives $6 \times \frac{1}{10} = \frac{3}{5}$ by R7. Since v sends at most $\frac{1}{5}$ to each of its k-neighbors by R4, R5.2, R5.3 and R6, where $k \leq 9$, we have

$c'(v) \geq 4+\frac{3}{5}-2\times\frac{1}{5}-\frac{1}{2}\times 8 > 0$ by R8 and R9. If $\delta_1(v)=6$, then $\delta_4(v)=10$ by VAL. Similarly, we have $c'(v) \geq 4+5\times\frac{1}{10}-3\times\frac{1}{6}-\frac{1}{2}\times 8 = 0$ by R5.2, R5.3 and R6–R9. If $\delta_1(v)=7$, then $\delta_5(v)=10$ by VAL. Similarly, we also have $c'(v) \geq 4+4\times\frac{1}{10}-4\times\frac{1}{56}-\frac{1}{2}\times 8 > 0$ by R6–R9. If $\delta_1(v)\geq 8$, then v sends at most $\frac{1}{2}\times 8=4$ to its incident faces by R8 and R9 and none to its neighbors in G, so we have $c'(v)\geq 4-4=0$.

If $d_G(v)=9$, then by VAL, $\delta_1(v)\geq 3$. If $\delta_1(v)=3$, then $\delta_2(v)=10$ by VAL. Since v sends at most $9\times\frac{1}{2}=\frac{9}{2}$ to its incident faces by R8 and R9 and none to its adjacent 3-vertex in G by R2, we have $c'(v)\geq 5-\frac{1}{2}\times 9=\frac{1}{2}>0$. If $\delta_1(v)=4$, then $\delta_3(v)=10$ by VAL, which implies that v has seven 10-neighbors in G, from which v receives $7\times\frac{1}{10}=\frac{7}{10}$ by R7. Since v sends at most $\frac{3}{8}$ to each of its k-neighbors by R3.2, R3.3, R4, R5.3 and R6, where $k\leq 9$, we have $c'(v)\geq 5+\frac{7}{10}-2\times\frac{3}{8}-\frac{1}{2}\times 9>0$. If $\delta_1(v)=5$, then $\delta_4(v)=10$ by VAL and we similarly have $c'(v)\geq 5+6\times\frac{1}{10}-3\times\frac{1}{5}-\frac{1}{2}\times 9>0$ by R4–R9. If $\delta_1(v)=6$, then $\delta_5(v)=10$ by VAL. In this case, one can similarly prove that $c'(v)\geq 5+5\times\frac{1}{10}-4\times\frac{1}{6}-\frac{1}{2}\times 9>0$ by R5–R9. If $\delta_1(v)\geq 7$, then $\delta_6(v)=10$ by VAL and we similarly have $c'(v)\geq 5+4\times\frac{1}{10}-5\times\frac{3}{70}-\frac{1}{2}\times 9>0$ by R6–R9.

If $d_G(v)=10$, then by VAL, $\delta_1(v)\geq 2$. If $\delta_1(v)=2$, then $\delta_2(v)=10$ by VAL, which implies that v has nine 10-neighbors in G, through which v sends out at most $9\times\frac{1}{18}=\frac{1}{2}$ by R1. Since v sends at most $10\times\frac{1}{2}=5$ to its incident faces by R8 and R9 and $\frac{1}{2}$ to its 2-neighbor by R1, we have $c'(v)\geq 6-\frac{1}{2}-5-\frac{1}{2}=0$. If $\delta_1(v)=3$, then $\delta_3(v)=10$ by VAL. Let u be a 3-neighbor of v. If v sends $\frac{3}{4}$ to u by R2, then $\delta_2(v)\geq 9$ by the definition of this rule. This implies that $c'(v)\geq 6-\frac{1}{2}\times 10-\frac{3}{4}-\frac{1}{10}>0$ by R2 and R7–R9. On the other hand, if v sends none to u, then $c'(v)\geq 6-\frac{1}{2}\times 10-\frac{3}{4}>0$ by R2–R9. If $\delta_1(v)=4$, then $\delta_4(v)=10$ by VAL. Let u be a 4-neighbor of v. If v sends $\frac{1}{2}$ to u by R3.1, then u has a 8-neighbor w in G. Since $d_G(u)+d_G(w)=12=\Delta(G)+2$, by (2) of Lemma 2.3, every neighbor of v in G except u and w is of degree 10 (note that we may have $vw\in E(G)$), which implies that $\delta_2(v)\geq 8$ and $\delta_3(v)=10$. Therefore, we have $c'(v)\geq 6-\frac{1}{2}\times 10-\frac{1}{2}-\frac{1}{10}>0$ by R3.1 and R7–R9. If v sends $\frac{3}{8}$ or $\frac{3}{4}$ to u by R3.2, then by the definition of this rule, u is not adjacent to any 8^--vertices in G and has a 10-neighbor z in G which is adjacent to exactly three 8^--vertices in G. Thus, by Lemma 2.5, v is adjacent to only one 8^--vertex in G, which is u. This implies that $\delta_2(v)\geq 9$. Therefore, we have $c'(v)\geq 6-\frac{1}{2}\times 10-\frac{3}{4}-2\times\frac{1}{10}>0$ by R3.2 and R7–R9. If v sends $\frac{3}{8}$ to u by R3.3, then by the definition of this rule, one can easily confirm that v is adjacent to at most two 8^--vertices in G, which implies that $\delta_3(v)\geq 9$. Thus, we have $c'(v)\geq 6-\frac{1}{2}\times 10-\frac{3}{8}-\frac{1}{2}-\frac{1}{10}>0$ by R3–R9. We now assume that v sends none to any of its 4-neighbors in G. In this case, we can easily obtain that $c'(v)\geq 6-\frac{1}{2}\times 10-2\times\frac{1}{5}>0$ by R4–R9. If $\delta_1(v)=5$, then $\delta_5(v)=10$ by VAL, which implies that $c'(v)\geq 6-\frac{1}{2}\times 10-4\times\frac{1}{5}>0$ by R4–R9. If $\delta_1(v)=6$, then $\delta_6(v)=10$ by VAL. Let u be a 6-neighbor of v. If v sends $\frac{1}{5}$ to u by R5.1, then by the definition of this rule, u has a 6-neighbor w in G. Since $d_G(u)+d_G(w)=12=\Delta(G)+2$, by (2) of Lemma 2.3, every neighbor of v in G except u and w is of degree 10 (note that we may have $vw\in E(G)$). This implies that $\delta_2(v)\geq 6$ and $\delta_3(v)=10$. Thus, we have $c'(v)\geq 6-\frac{1}{2}\times 10-2\times\frac{1}{5}>0$ by R5–R9. If v sends charges (at most $\frac{11}{56}$) to u by R5.2 or R5.3, then one can easily obtain that $c'(v)\geq 6-\frac{1}{2}\times 10-\frac{11}{56}-4\times\frac{1}{5}>0$ by R5–R9. We now assume that v sends none to any of its 6-neighbors in G. In this case, we immediately have $c'(v)\geq 6-\frac{1}{2}\times 10-5\times\frac{1}{7}>0$ by R6–R9, since v sends at most $\frac{1}{7}$ to each of its 7^+-neighbors in G. If $\delta_1(v) \geq 7$, then by R6–R9 and VAL, we similarly have $c'(v) \geq 6 - \frac{1}{2} \times 10 - \max\{6 \times \frac{1}{7}, 7 \times \frac{1}{10}, 8 \times \frac{1}{10}\} > 0$.

At this stage, we have proved that $c'(x) \geq 0$ for every $x \in V(G) \cup F(G^\times)$ and $c'(v) > 0$ for every 10-vertex v with $\delta_1(v) \geq 3$. If there exists a 10-vertex v with $\delta_1(v) \geq 3$, then we would get $\sum_{x \in V(G) \cup F(G^\times)} c'(x) > 0$, a contradiction.

Therefore, we shall assume that $\delta_1(v) = 2$ for every 10-vertex $v \in V(G)$. Let v be a 10-vertex that has one 2-neighbor u in G. By VAL, the vertices of $N_G(v) \backslash \{u\}$ are all of degree 10. Choose one vertex $w \in N_G(v) \backslash \{u\}$ such that $uw \notin E(G)$. By (1) of Lemma 2.3, every neighbor of w in G is of degree at least 9, which implies that $\delta_1(w) \geq 9$, contradicting our assumption that $\delta_1(w) = 2$ (note that w is a 10-vertex in G).

3 Class Two 1-toroidal Graphs

In this section, we focus on 1-toroidal graphs with small maximum degree. To begin with, we prove the following basic result.

Theorem 3.1 *Every* 1*-toroidal graph contains a vertex of degree at most* 8; *the bound* 8 *is best possible.*

Proof We prove this theorem by contradiction. Let G be a 1-toroidal graph with $\delta(G) \geq 9$. It is easy to see that $|E(G)| \geq \frac{9}{2}|V(G)|$. However, it is proved in [20] that $|E(G)| \leq 4|V(G)|$ for every 1-toroidal graph G. This contradiction implies that $\delta(G) \leq 8$. Furthermore, since the complete graph K_9 is 1-toroidal (see Figure 1), the bound 8 in the theorem is sharp. □

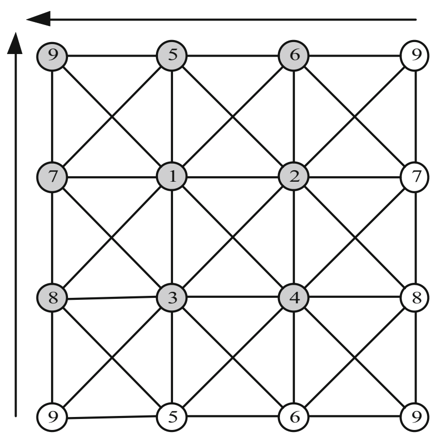

Figure 1 K_9 is a 1-toroidal graph

In [18], Zhang and Liu presented examples of 1-planar graphs (and thus 1-toroidal graph) of class two with maximum degree no more than 7. In this paper, we display that the complete graph K_9 is a class two (this can be easily checked) 1-toroidal graph with maximum degree 8. We conclude the following theorem.

Theorem 3.2 *There exist* 1*-toroidal graphs of class two with maximum degree* Δ *for each* $\Delta \leq 8$.

We end this paper with an interesting conjecture, the proof of which may need much more detailed discussions.

Conjecture 3.3 *Every* 1*-toroidal graph with maximum degree* 9 *is of class one.*

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