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Group Edge Choosability of Planar Graphs without Adjacent Short Cycles

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Abstract In this paper, we prove that 2-degenerate graphs and some planar graphs without adjacent short cycles are group $(\Delta(G)+1)$-edge-choosable, and some planar graphs with large girth and maximum degree are group $\Delta(G)$-edge-choosable.

Keywords Group edge coloring, list coloring, planar graphs, short cycles, girth

MR(2010) Subject Classification 05C10, 05C15, 05C25

1 Introduction

All graphs considered in this paper are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $d_G(v)$, we denote the degree of v in G. For a plane graph G, $F(G)$ denotes its face set and $d_G(f)$ denotes the degree of a face f in G. The *girth* $g(G)$ of a graph G is the length of its smallest cycle, or $+\infty$ if G is a forest. Throughout this paper, a k-, k^+- and k^--*vertex* (resp. *face*) is a vertex (resp. face) of degree k, at least k and at most k, respectively. An i-*alternating cycle* is a cycle of even length in which alternate vertices have degree i. We say a graph G is k-*degenerate* if $\delta(H) \leq k$ for every subgraph $H \subseteq G$. Any undefined notation follows that of Bondy and Murty [1].

In 1992, Jaeger et al. [5] introduced a concept of group connectivity as a generalization of nowhere zero flows and its dual concept group coloring. They proposed the definition of group colorability of graphs as the equivalence of group connectivity of M, where M is a cographic matroid. Let G be a graph and let A be an Abelian group. Denote $F(G, A)$ to be the set of all functions $f : E(G) \mapsto A$ and D to be an arbitrary orientation of $E(G)$. We say that G is A-*colorable* under the orientation D if for any function $f \in F(G, A)$, G has an (A, f)-coloring, namely, a vertex coloring $c : V(G) \mapsto A$ such that $c(u) - c(v) \neq f(uv)$ for every directed edge

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uv from u to v. Lai and Zhang [8] pointed out that, for any Abelian group A, a graph G is A-colorable under the orientation D if and only if G is A-colorable under every orientation of $E(G)$. This means the group colorability of a graph is independent of the orientation of $E(G)$. The *group chromatic number* of a graph G, denoted by $\chi_g(G)$, is the minimum m such that G is A-colorable for any Abelian group A of order at least m. Clearly, $\chi(G) \leq \chi_g(G)$, where $\chi(G)$ is the chromatic number of G. Lai and Zhang [10] proved that $\chi_g(G) \leq 5$ for every planar graph G and Král' et al. [7] constructed planar graphs with the group chromatic number five. This implies that the well-known Four-Colors Theorem for ordinary colorings cannot be extended to group colorings. Nevertheless, some theorems for ordinary vertex colorings, such as Brooks' theorem, still can be extended. The following theorem is due to Lai et al. [9]. Note that for an even cycle C_{2n}, we have $\chi_g(C_{2n}) = 3$ by Theorem 1.1, but $\chi(C_{2n}) = 2$.

Theorem 1.1 *For any connected simple graph G,*

$$\chi_g(G) \leq \Delta(G) + 1,$$

where the equality holds if and only if G is either a cycle or a complete graph.

In 2004, Král' and Nejedlý [6] considered list group coloring as an extension of list coloring and group coloring. Let G be a graph, A be an Abelian group of order at least k and let $L : V(G) \mapsto 2^A$ be a k-uniform list assignment of $V(G)$. Denote $F(G, A)$ to be the set of all functions $f : E(G) \mapsto A$ and D to be an arbitrary orientation of $E(G)$. We say that G is *group k-choosable* under the orientation D if for any function $f \in F(G, A)$, G has an (A, L, f)-coloring, which is an (A, f)-coloring c so that $c(v) \in L(v)$ for every $v \in V(G)$. Note that the choice of an orientation of edges of G is not essential in this definition either. The *group choice number* of a graph G, denoted by $\chi_{gl}(G)$, is the minimum k such that G is group k-choosable. Král' and Nejedlý [6] showed that $\chi_{gl}(G) = 2$ if and only if G is a forest. Omidi [11] proved that the group choice number of a graph without K_5-minor or $K_{3,3}$-minor and with girth at least 4 (resp. 6) is at most 4 (resp. 3). Chuang et al. [2] established the group choosability version of Brooks' theorem, which extends Theorem 1.1.

Theorem 1.2 *For any connected simple graph G,*

$$\chi_{gl}(G) \leq \Delta(G) + 1,$$

where the equality holds if and only if G is either a cycle or a complete graph.

In this paper, we study the group version of edge coloring and list edge coloring. The *line graph* of a graph G, denoted by $\mathcal{L}(G)$, is a graph such that each vertex of $\mathcal{L}(G)$ represents an edge of G and two vertices of $\mathcal{L}(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. For an edge $uv \in E(G)$, we use e_{uv} to denote the vertex in $\mathcal{L}(G)$ that represents uv in G. Clearly, the edge chromatic number $\chi'(G)$ of a graph G is equal to the vertex chromatic number $\chi(\mathcal{L}(G))$ of its line graph $\mathcal{L}(G)$. In view of this, the group version of edge coloring and list edge coloring can be defined naturally. For an Abelian group A of order at least k, we say that G is *group A-edge-colorable* if $\mathcal{L}(G)$ is group A-colorable, and say that G is *group k-edge-choosable* if $\mathcal{L}(G)$ is group k-choosable. By $\chi'_g(G) = \chi_g(\mathcal{L}(G))$ and $\chi'_{gl}(G) = \chi_{gl}(\mathcal{L}(G))$, we denote the *group edge chromatic number* and the *group edge choice* *number* of a graph G. To begin with, we introduce a basic theorem.

Theorem 1.3 *For any connected simple graph G,*

$$\Delta(G) \le \chi'_g(G) \begin{cases} = \chi'_{gl}(G) = 1, & \text{if } G = K_2; \\ = \chi'_{gl}(G) = 2, & \text{if } G \text{ is a path of length at least } 2; \\ = \chi'_{gl}(G) = 3, & \text{if } G \text{ is a cycle}; \\ \le \chi'_{gl}(G) \le 2\Delta(G) - 2, & \text{if } \Delta(G) \ge 3. \end{cases}$$

Proof Since $\chi'_{gl}(G) \ge \chi'_g(G) \ge \chi'(G) \ge \Delta(G)$, the left inequality in above theorem holds. If G is a path (resp. cycle), then $\mathcal{L}(G)$ is also a path (resp. cycle). By Theorems 1.1 and 1.2, we have $\chi'_g(G) = \chi'_{gl}(G) = 1$ if $G = K_2$, and $\chi'_g(G) = \chi'_{gl}(G) = 2$ if G is a path of length at least 2, and $\chi'_g(G) = \chi'_{gl}(G) = 3$ if G is a cycle. If $\Delta(G) \ge 3$ and G is not a star, then $\mathcal{L}(G)$ is neither a cycle nor a complete graph, which implies, by Theorems 1.1 and 1.2, that $\chi'_g(G) \le \chi'_{gl}(G) = \chi_{gl}(\mathcal{L}(G)) \le \Delta(\mathcal{L}(G)) \le 2\Delta(G) - 2$. If $\Delta(G) \ge 3$ and G is star, then

$$\chi'_g(G) \le \chi'_{gl}(G) = \chi_{gl}(\mathcal{L}(G)) \le \Delta(\mathcal{L}(G)) + 1 = \Delta(G) < 2\Delta(G) - 2$$

by Theorem 1.2. □

From Theorem 1.3, we deduce that

$$\chi'_g(G) \le \chi'_{gl}(G) \le \Delta(G) + 1$$

for every graph with maximum degree 3 and $\chi'_g(G) = \chi'_{gl}(G)$ for every graph with maximum degree 2. These evidences motivate us to conjecture the analogue of Vizing's theorem on edge chromatic number and list edge coloring Conjecture on edge choice number.

Conjecture 1.4 *For any simple graph G, $\Delta(G) \le \chi'_g(G) \le \Delta(G) + 1$.*

Conjecture 1.5 *For any simple graph G, $\chi'_g(G) = \chi'_{gl}(G)$.*

In the next section, we would confirm Conjecture 1.4 for 2-degenerate graphs and some planar graphs without adjacent short cycles, and confirm Conjecture 1.5 for some planar graphs with large girth.

A graph G is *group $(\Delta(G) + i)$-edge-critical* if $\chi'_{gl}(G) > \Delta(G) + i$ and $\chi'_{gl}(H) \le \Delta(H) + i$ for every proper subgraph $H \subset G$, where i is a nonnegative integer. The $(\Delta(G) + i)$-edge-critical graph with respect to list edge coloring can be defined similarly. In most of the articles concerning list $(\Delta + 1)$-edge coloring of planar graphs in the literature (such as [3] and [4]), it was proved that a 3-alternating cycle C cannot appear in a critical graph G because if such a cycle C do exist, then $G - E(C)$ is $(\Delta + 1)$-edge choosable and every edge of C has at least two available colors, since it is incident with $\Delta(G) + 1$ edges, of which $\Delta(G) - 1$ edges are colored, which implies that one can extend the list $(\Delta + 1)$-edge coloring of $G - E(C)$ to G by the fact that even cycles are 2-edge-choosable. However, this essential technique is invalid for (list) group edge coloring since every cycle is not group 2-edge-choosable by Theorem 1.3.

2 Main Results and Their Proofs

We begin with a useful lemma, which will be frequently used in the next proofs. Meanwhile, it implies Conjecture 1.4 for 2-degenerate graphs.

Lemma 2.1 *If i is a nonnegative integer and G is a group $(\Delta(G)+i)$-edge-critical graph, then G is connected and $d_G(u)+d_G(v) \geq \Delta(G)+i+2$ for any edge $uv \in E(G)$.*

Proof The connectivity of G directly follows from its definition. Suppose that there is an edge $uv \in E(G)$ such that $d_G(u)+d_G(v) \leq \Delta(G)+i+1$. For an Abelian group A of order at least $\Delta(G)+i$, a $(\Delta(G)+i)$-uniform list assignment $L : V(\mathcal{L}(G)) \mapsto 2^A$ and a function $f \in F(\mathcal{L}(G), A)$, $\mathcal{L}(G)$ is not (A, L, f)-colorable but $\mathcal{L}(G-uv)$ is (A, L, f)-colorable by our assumption. Let c be an (A, L, f)-coloring of $\mathcal{L}(G-uv)$. In $\mathcal{L}(G)$, the only uncolored vertex under c is e_{uv}, which is adjacent to $m = d_G(u)+d_G(v)-2 \leq \Delta(G)+i-1$ colored vertices, say $e_1, e_2, \ldots, e_m$. Without loss of generality, we assume that e_{uv} is the head of each edge e_ie_{uv} in $\mathcal{L}(G)$ under a given orientation D of $E(\mathcal{L}(G))$, where $1 \leq i \leq m$. Now assign e_{uv} a color in $S = L(e_{uv}) - \bigcup_{i=1}^{m}\{c(e_i) - f(e_ie_{uv})\}$. Since $|S| \geq \Delta(G)+i-m \geq 1$, we can extended c to an (A, L, f)-coloring of $\mathcal{L}(G)$. This implies that G is group $(\Delta(G)+i)$-edge-choosable, a contradiction. □

Corollary 2.2 *If i is a nonnegative integer and G is a group $(\Delta(G)+i)$-edge-critical graph, then $\delta(G) \geq i+2$.*

Corollary 2.3 *Every 2-degenerate graph is group $(\Delta(G)+1)$-edge-choosable.*

Theorem 2.4 *Let G be a planar graph such that G does not contain an i-cycle adjacent to a j-cycle, where $3 \leq i \leq s$ and $3 \leq j \leq t$. If*

(1) $s=3$, $t=3$ *and* $\Delta(G) \geq 8$, *or*

(2) $s=3$, $t=4$ *and* $\Delta(G) \geq 6$, *or*

(3) $s=4$, $t=5$ *and* $\Delta(G) \geq 5$, *or*

(4) $s=4$, $t=7$,

then G is group $(\Delta(G)+1)$-edge-choosable.

Proof The proof is carried out by contradiction and discharging method. Suppose that G is a minimum counterexample to the theorem. By Lemma 2.1, G is a connected and group $(\Delta(G)+1)$-edge-critical planar graph with $\delta(G) \geq 3$.

By Euler's formula, for any $n > 2m > 0$, we have

$$\sum_{v \in V(G)} \left[\left(\frac{n}{2}-m\right)d_G(v)-n\right] + \sum_{f \in F(G)} (m \cdot d_G(f)-n) = -2n < 0. \tag{2.1}$$

Assign each vertex $v \in V(G)$ an initial charge $c(v) = (\frac{n}{2}-m)d_G(v)-n$ and each face $f \in F(G)$ an initial charge $c(f) = md_G(f)-n$. By (2.1), we have $\sum_{x \in V(G) \cup F(G)} c(x) < 0$. To prove the theorem, we are going to construct a new charge function c' on $V(G) \cup F(G)$ according to some defined discharging rules, which only move charge around but do not affect the total charges so that the final charge $c'(x)$ of each element $x \in V(G) \cup F(G)$ is nonnegative after discharging, and this contradiction completes the proof of the theorem. In the following, a face $f \in F(G)$ is *simple* if the boundary of f is a cycle. By $m_v(f)$, we denote the number of times through v by a face f in a clockwise order. Obviously, if v is a non-cut vertex or f is a simple face, then $m_v(f) = 1$.

(1) Let S be the set of 3-vertices, 4-vertices and 5-vertices in G. By Lemma 2.1, S is an independent set in G, since $\Delta(G) \geq 8$. Now we choose $m=2$ and $n=6$ in (2.1) and define the

discharging rules as follows:

R1.1 From each 4^+-face f to its incident vertex $v \in S$, transfer $m_v(f)$.

R1.2 From each 8^+-vertex u to its adjacent 3-vertex v, transfer $\frac{1}{2}$ if uv is incident with a 3-cycle.

Without loss of generality, we always assume that v is a non-cut vertex and f is simple in the following arguments (because during the calculational part of discharging, the case when v is a cut vertex that is incident with a non-simple face f is equivalent to the case when v is incident with $m_v(f)$ simple faces with the same degree of f, and the case when f is a non-simple face that is incident with a cut vertex v is equivalent to the case when f is incident with $m_v(f)$ non-cut vertices with the same degree of v). If $d_G(v) = 3$, then by Lemma 2.1, v is adjacent to three 8^+-vertices. If v is incident with a 3-face, then v is also incident with two 4^+-faces since no two 3-cycles are adjacent in G. This implies that $c'(v) \geq c(v) + 2 \times \frac{1}{2} + 2 \times 1 = 0$ by R1.1 and R1.2. If v is incident with no 3-faces, then $c'(v) \geq c(v) + 3 \times 1 = 0$ by R1.1. If $4 \leq d_G(v) \leq 5$, then v is incident with at least two 4^+-faces, which implies that $c'(v) \geq c(v) + 2 \times 1 = 0$. If $6 \leq d_G(v) \leq 7$, then it is easy to see that $w'(v) = w(v) \geq 0$. If $d_G(v) \geq 8$, then G is incident with at most $\lfloor \frac{d_G(v)}{2} \rfloor$ 3-faces since no two 3-cycles are adjacent in G. This implies that v may transfer charges to at most $\lfloor \frac{d_G(v)}{2} \rfloor$ 3-vertices by R1.2, since no two 3-vertices are adjacent in G. Thus, we have $c'(v) \geq d_G(v) - 6 - \frac{1}{2} \lfloor \frac{d_G(v)}{2} \rfloor \geq 0$ for $d_G(v) \geq 8$. If $d_G(f) = 3$, then it is trivial that $c'(f) = c(f) = 0$. If $d_G(f) \geq 4$, then f may transfer charges to at most $\lfloor \frac{d_G(f)}{2} \rfloor$ vertices by R1.1, since S is an independent set in G. This implies that $c'(f) \geq 2d_G(f) - 6 - \lfloor \frac{d_G(f)}{2} \rfloor \geq 0$ for $d_G(f) \geq 4$.

(2) We choose $m = 3$ and $n = 10$ in (2.1) and define the discharging rules as follows:

R2.1 From each 6^+-vertex to its adjacent 3-vertex, transfer $\frac{1}{3}$.

R2.2 From each 4-face f to its incident vertex v, transfer $m_v(f)$ if $d_G(v) = 3$, $\frac{1}{2}m_v(f)$ if $d_G(v) = 4$.

R2.3 From each 5^+-face f to its incident vertex v, transfer $\frac{3}{2}m_v(f)$ if $d_G(v) = 3$, $m_v(f)$ if $d_G(v) = 4$.

R2.4 From each 5^+-face to its adjacent 3-face, transfer $\frac{1}{3}$.

If $d_G(v) = 3$, then by Lemma 2.1, v is adjacent to three 6^+-vertices since $\Delta(G) \geq 6$. If v is incident with a 3-face, then v is incident with two 5^+-face by our assumption. This implies that $c'(v) \geq c(v) + 3 \times \frac{1}{3} + 2 \times \frac{3}{2} = 0$ by R2.1 and R2.3. If v is incident with no 3-faces, then v is incident with three 4^+-faces, which implies that $c'(v) \geq c(v) + 3 \times \frac{1}{3} + 3 \times 1 = 0$ by R2.1, R2.2 and R2.3. If $d_G(v) = 4$, then consider two subcases. If v is incident with a 3-face, then v is incident with at least two 5^+-faces, which implies that $c'(v) \geq c(v) + 2 \times 1 = 0$ by R2.3. If v is incident with no 3-faces, then v is incident with four 4^+-faces, which implies that $c'(v) \geq c(v) + 4 \times \frac{1}{2} = 0$ by R2.2 and R2.3. If $d_G(v) = 5$, then it is easy to see that $c'(v) = c(v) = 0$. If $d_G(v) \geq 6$, then by R2.1, we have $c'(v) \geq 2d_G(v) - 10 - \frac{1}{3}d_G(v) \geq 0$. If $d_G(f) = 3$, then f is adjacent to three 5^+-faces by our assumption, which implies that $c'(f) \geq c(f) + 3 \times \frac{1}{3} = 0$ by R2.4. If $d_G(f) \geq 4$, then f is incident with at most $\lfloor \frac{d_G(f)}{2} \rfloor$ 4^--vertices since no two 4^--vertices are adjacent in G by Lemma 2.1. This implies that $c'(f) \geq c(f) - 2 \times 1 = 0$ for $d_G(f) = 4$ by R2.2, and $c'(v) \geq 3d_G(f) - 10 - \frac{1}{3}d_G(f) - \frac{3}{2} \lfloor \frac{d_G(f)}{2} \rfloor > 0$ for $d_G(f) \geq 5$ by R2.3 and R2.4.

(3) We choose $m = 2$ and $n = 6$ in (2.1) and define the discharging rules as follows:

R3.1 From each 5-face f to its incident vertex v, transfer $m_v(f)$ if $d_G(v) = 3$, $\frac{1}{2}m_v(f)$ if $d_G(v) = 4$, $\frac{1}{5}m_v(f)$ if $d_G(v) = 5$.

R3.2 From each 6^+-face f to its incident vertex v, transfer $\frac{3}{2}m_v(f)$ if $d_G(v) = 3$, $m_v(f)$ if $d_G(v) = 4$, $\frac{1}{3}m_v(f)$ if $d_G(v) = 5$.

If $d_G(v) = 3$ and v is incident with a 4^--face, then v is incident with two 6^+-faces by our assumption, which implies that $c'(v) \geq c(v) + 2 \times \frac{3}{2} = 0$ by R3.2. If $d_G(v) = 3$ and v is incident with no 4^--faces, then by R3.1 and R3.2, we have $c'(v) \geq c(v) + 3 \times 1 = 0$. If $d_G(v) = 4$ and v is incident with a 4^--face, then v is incident with at least two 6^+-faces, which implies that $c'(v) \geq c(v) + 2 \times 1 = 0$ by R3.2. If $d_G(v) = 4$ and v is incident with no 4^--faces, then by R3.1 and R3.2, we have $c'(v) \geq c(v) + 4 \times \frac{1}{2} = 0$. If $d_G(v) = 5$ and v is incident with at least one 4^--face, then v is incident with either three 6^+-faces, implying that $c'(v) \geq c(v) + 3 \times \frac{1}{3} = 0$ by R3.2, or two 5^+-faces and two 6^+-faces, implying that $c'(v) \geq c(v) + 2 \times \frac{1}{5} + 2 \times \frac{1}{3} > 0$ by R3.1 and R3.2. If $d_G(v) = 5$ and v is incident with no 4^--faces, then by R3.1 and R3.2, we have $c'(v) \geq c(v) + 5 \times \frac{1}{5} = 0$. If $d_G(v) \geq 6$ or $3 \leq d_G(f) \leq 4$, then it is clear that $c'(v) = c(v) \geq 0$ and $c'(f) = c(f) \geq 0$. If $d_G(f) = 5$ and f is incident with no 3-vertices, then by R3.1, we have $c'(f) \geq c(f) - 5 \times \frac{1}{2} > 0$. If $d_G(f) = 5$ and f is incident with at least one 3-vertex, then f is incident with at least two 5^+-vertices since a 3-vertex cannot be adjacent to a 4^--vertex in G by Lemma 2.1. This implies that $c'(f) \geq c(f) - 2 \times \frac{1}{5} - 3 \times 1 > 0$ by R3.1. If $d_G(f) \geq 6$, then $d_G(f) - n_3 - n_4 \geq n_3$ by Lemma 2.1 since $\Delta(G) \geq 5$, where n_i is the number of i-vertices that are incident with f in G. This implies that $c'(f) \geq 2d_G(f) - 6 - \frac{3}{2}n_3 - n_4 - \frac{1}{3}(d_G(f) - n_3 - n_4) = d_G(f) - 6 - \frac{2}{3}(2n_3 + n_4 - d_G(f)) + \frac{1}{6}n_3 \geq 0$ by R3.2.

(4) We shall assume $\Delta(G) \geq 4$ in this part because the cases when $\Delta(G) \leq 3$ have been proved in Theorem 1.3. We now choose $m = 2$ and $n = 6$ in (2.1) and define the discharging rules as follows:

R4.1 From each face f of degree between 5 and 7 to its incident vertex v, transfer $m_v(f)$ if $d_G(v) = 3$, $\frac{1}{2}m_v(f)$ if $d_G(v) \geq 4$.

R4.2 From each 8^+-face f to its incident vertex v, transfer $\frac{3}{2}m_v(f)$ if $d_G(v) = 3$, $m_v(f)$ if $d_G(v) \geq 4$.

Since the above discharging rules are highly similar to the ones in part (3), we can check that $c'(v) \geq 0$ for all $v \in V(G)$ and $c'(f) \geq 0$ for $3 \leq d_G(f) \leq 4$ by the same analysis as in the previous part. Therefore, we shall only consider 5^+-faces. Since no two 3-vertices are adjacent in G by Lemma 2.1, $n_3 \leq \lfloor \frac{d_G(f)}{2} \rfloor$ for any $f \in F(G)$, where n_3 is the number of 3-vertices that are incident with f in G. If $5 \leq d_G(f) \leq 7$, then by R4.1, we have

$$c'(f) \geq 2d_G(f) - 6 - n_3 - \frac{1}{2}(d_G(f) - n_3) \geq \frac{3}{2}d_G(f) - 6 - \frac{1}{2}\left\lfloor \frac{d_G(f)}{2} \right\rfloor \geq 0.$$

If $d_G(f) \geq 8$, then

$$c'(f) \geq 2d_G(f) - 6 - \frac{3}{2}n_3 - 1 \times (d_G(f) - n_3) \geq d_G(f) - 6 - \frac{1}{2}\left\lfloor \frac{d_G(f)}{2} \right\rfloor \geq 0$$

by R4.2. This completes the proof of the theorem. □

As immediate corollaries of Theorem 2.4, we have the following two results.

Corollary 2.5 *Every planar graph with girth $g(G) \geq 5$ is group $(\Delta(G)+1)$-edge-choosable.*

Corollary 2.6 *Every planar graph with girth $g(G) \geq 4$ and maximum degree $\Delta(G) \geq 6$ is group $(\Delta(G)+1)$-edge-choosable.*

Another interesting topic concerting group edge colorings and list group edge colorings is to determine which class of graphs satisfies $\chi'_g(G) = \chi'_{gl}(G)$. We end this paper by proving the following theorem, which confirms Conjecture 1.5 for some planar graphs with large girth and maximum degree.

Theorem 2.7 *Let G be a planar graph with maximum degree $\Delta(G) \geq \Delta \geq 3$. If $g(G) \geq 4 + \lceil \frac{8}{\Delta-2} \rceil$, then $\chi'_g(G) = \chi'_{gl}(G) = \Delta(G)$.*

Proof In fact, we just need to prove that $\chi'_{gl}(G) = \Delta(G)$ here. Suppose, to the contrary, that G is a group $\Delta(G)$-edge-critical graph. Let $c(v) = 2d_G(v) - 6$ for $v \in V(G)$ and let $c(f) = d_G(f) - 6$ for $f \in V(G)$. By (2.1), we have $\sum_{x \in V(G) \cup F(G)} c(x) < 0$. Now we redistribute the charge of the vertices and faces of G according to the following discharging rules:

R1 From each vertex of maximum degree to its adjacent 2-vertex, transfer $2 - \frac{6}{\Delta}$.

R2 From each face f to its incident 2-vertex v, transfer $(\frac{6}{\Delta} - 1)m_v(f)$.

Let $c'(x)$ be the final charge of the element x after discharging. If $d_G(v) = 2$, then by Lemma 2.1, the two neighbors of v are both $\Delta(G)$-vertices, which implies that $c'(v) \geq c(v) + 2 \times (2 - \frac{6}{\Delta}) + 2 \times (\frac{6}{\Delta} - 1) = 0$ by R1 and R2. If $3 \leq d_G(v) \leq \Delta(G) - 1$ (if exists), then it is clear that $c'(v) = c(v) \geq 0$. If $d_G(v) = \Delta(G)$, then by R1, one can easily deduce that $c'(v) \geq 2\Delta(G) - 6 - \Delta(G)(2 - \frac{6}{\Delta}) \geq 0$ since $\Delta(G) \geq \Delta$. Suppose that f is a face in G. Similarly as in the proof of Theorem 2.4, we can assume, without loss of generality, that f is simple. By Lemma 2.1, f is incident with at most $\lfloor \frac{d_G(f)}{2} \rfloor$ 2-vertices. This implies that $c'(f) \geq d_G(f) - 6 - (\frac{6}{\Delta} - 1)\lfloor \frac{d_G(f)}{2} \rfloor \geq \frac{3\Delta-6}{2\Delta} g(G) - 6 \geq \frac{3\Delta-6}{2\Delta} \cdot \frac{4\Delta}{\Delta-2} - 6 = 0$ by R2. Therefore, $c'(x) \geq 0$ for every $x \in V(G) \cup F(G)$. This contradiction completes the proof. $\square$

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References

[1] Bondy, J. A., Murty, U. S. R.: Graph Theory with Applications, North-Holland, New York, 1976

[2] Chuang, H., Lai, H.-J., Omidi, G. R., et al.: On group choosability of graphs I. *Ars Combin.*, to appear

[3] Cohen, N., Havet, F.: Planar graphs with maximum degree $\Delta \geq 9$ are $(\Delta+1)$-edge-choosable: A short proof. *Discrete Math.*, **310**, 3049–3051 (2010)

[4] Hou, J., Liu, G., Cai, J.: Edge-choosability of planar graphs without adjacent triangles or without 7-cycles. *Discrete Math.*, **309**, 77–84 (2009)

[5] Jaeger, F., Linial, N., Payan, C., et al.: Group connectivity of graphs—A nonhomogeneous analogue of nowhere-zero flow properties. *J. Combin. Theory Ser. B*, **56**, 165–182 (1992)

[6] Král', D., Nejedlý, P.: Group coloring and list group coloring are Π_2^P-complete. *Lecture Notes in Comput. Sci.*, **3153**, 274–287 (2004)

[7] Král', D., Pangrác, O., Voss, H.-J.: A note on group colorings. *J. Graph Theory*, **50**, 123–129 (2005)

[8] Lai, H.-J., Zhang, X.: Group colorability of graphs. *Ars Combin.*, **62**, 299–317 (2002)

[9] Lai, H.-J., Li, X., Yu, G.: An inequality for the group chromatic number of a graph. *Discrete Math.*, **307**, 3076–3080 (2007)

[10] Lai, H.-J., Zhang, X.: Group chromatic number of graphs without K_5-minors. *Graphs Combin.*, **18**, 147–154 (2002)

[11] Omidi, G. R.: A note on group choosability of graphs with girth at least 4. *Graphs Combin.*, **27**, 269–273 (2011)