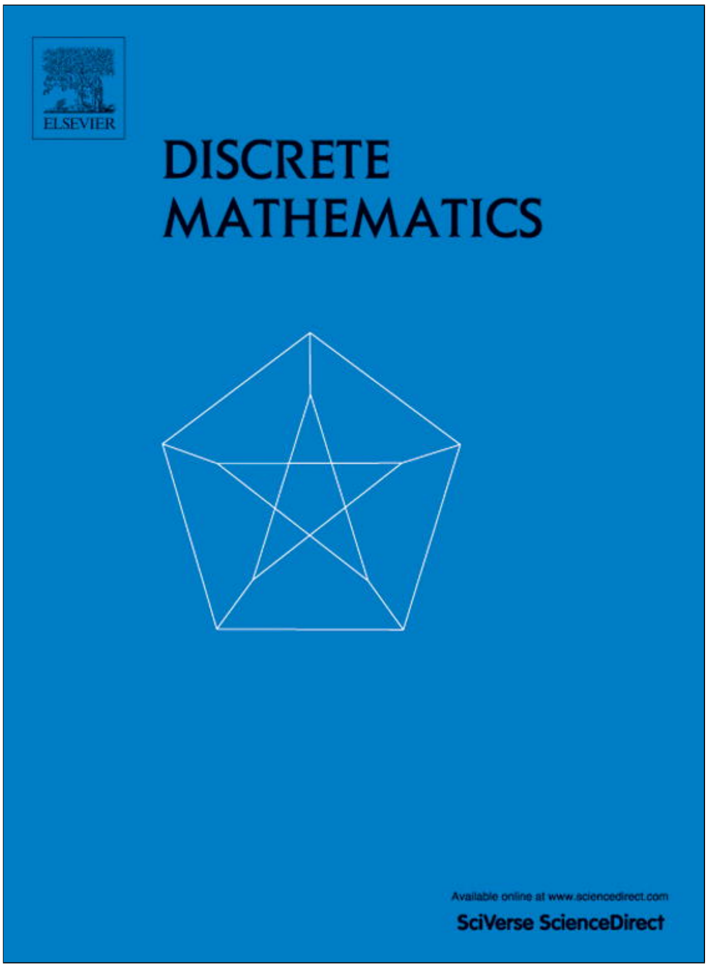
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Discrete Mathematics 312 (2012) 2788–2799

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Edge covering pseudo-outerplanar graphs with forests[☆]

Xin Zhang[a,b,*], Guizhen Liu[a], Jian-Liang Wu[a]

[a] School of Mathematics, Shandong University, Jinan, 250100, PR China

[b] Department of Mathematics, Xidian University, Xi'an, 710071, PR China

ARTICLE INFO

Article history:
Received 16 December 2009
Received in revised form 22 May 2012
Accepted 25 May 2012

Keywords:
Pseudo-outerplanar graphs
Edge decomposition
Edge chromatic number
Linear arboricity

ABSTRACT

A graph is *pseudo-outerplanar* if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this paper, we prove that each pseudo-outerplanar graph admits edge decompositions into a linear forest and an outerplanar graph, or a star forest and an outerplanar graph, or two forests and a matching, or $\max\{\Delta(G), 4\}$ matchings, or $\max\{\lceil \Delta(G)/2\rceil, 3\}$ linear forests. These results generalize known results on outerplanar graphs and $K_{2,3}$-minor-free graphs, since the class of pseudo-outerplanar graphs is larger than the class of $K_{2,3}$-minor-free graphs.

1. Introduction

In this paper, all graphs considered are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. Let $d_G(v)$ (or $d(v)$ for simplicity) denote the degree of a vertex $v \in V(G)$. A *block* of a graph G is a maximal 2-connected subgraph of G. A graph H is a *minor* of a graph G if a copy of H can be obtained from G via repeated edge deletion and/or edge contraction. For a subset $S \subseteq V(G) \cup E(G)$, $G[S]$ denotes the subgraph of G induced by S. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose deletion from G disconnects it. For other undefined concepts we refer the readers to [3].

An *outerplanar graph* is a graph that can be embedded on the plane in such a way that it has no crossings and that all its vertices lie on the outer face. In this paper, we introduce an extension of this concept. A graph is *pseudo-outerplanar* if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this embedding, the edges bounding the disk(s) are *boundary edges* and a disk is *closed* or *open* according to whether or not it contains the circle that constitutes its boundary. For example, Fig. 1 exhibits a pseudo-outerplanar embedding of a graph with two blocks: one is K_4 and the other is $K_{2,3}$. The drawing of K_4 in this embedding lies inside a closed disk but the one of $K_{2,3}$ in this embedding lies inside an open disk. In Fig. 1, the edges in bold are the boundary edges. A pseudo-outerplanar graph is *maximal* if it is not possible to add an edge such that the resulting graph is still pseudo-outerplanar. Thus, $K_{2,3}$ is not a maximal pseudo-outerplanar graph, since we can add two edges to $K_{2,3}$ and remain its pseudo-outerplanarity. One can easily check that each pseudo-outerplanar graph has a planar embedding, so the class of pseudo-outerplanar graphs forms a subclass of planar graphs. Actually, the definition of pseudo-outerplanar graphs is similar to that of 1-planar graphs (i.e. graphs that can be drawn on the plane so that each edge is crossed by at most one other edge), which was introduced by Ringel [10].

☆ Research supported by NSFC (10971121, 11101243, 61070230), RFDP (20100131120017) and GIIFSDU (yzc10040).

* Corresponding author at: School of Mathematics, Shandong University, Jinan, 250100, PR China.
E-mail addresses: sdu.zhang@yahoo.com.cn, xinzhang@mail.sdu.edu.cn (X. Zhang), gzliu@sdu.edu.cn (G. Liu), jlwu@sdu.edu.cn (J.-L. Wu).

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doi:10.1016/j.disc.2012.05.017

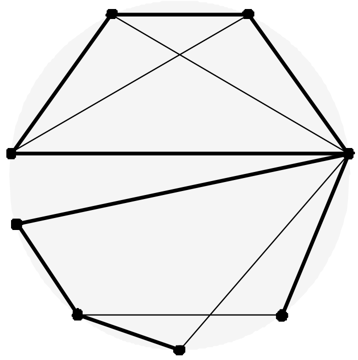


Fig. 1. An example of a pseudo-outerplanar graph.

Many classic problems in graph theory are considered for the class of planar graphs and its subclasses, such as the class of series–parallel graphs and the class of outerplanar graphs. Consider the problem of covering graphs with forests and a graph of bounded maximum degree, for example. We say that a graph is (t, d)-*coverable* if its edges can be covered by at most t forests and a graph of maximum degree d. In [2], et al. conjectured that every simple planar graph is (2, 4)-coverable and gave a example to show that there are infinitely many planar graphs that are not (2, 3)-coverable. This conjecture was recently confirmed by Gonçalves in [5]. In [2], it is also proved that every series–parallel graph is (2, 0)-coverable and that every $K_{2,3}$-minor-free graph is both (1, 3)-coverable and (2, 0)-coverable. Since a graph is outerplanar if and only if it is $\{K_4, K_{2,3}\}$-minor-free [6], every outerplanar graph is both (1, 3)-coverable and (2, 0)-coverable. It is interesting to know what can be said about pseudo-outerplanar graphs, a larger class than outerplanar graphs.

Edge-coloring is another classic problem in graph theory. In fact, we can regard edge-coloring problems as an edge decomposition problem. When we color the edges of a graph G, our actual task is to decompose the edge set $E(G)$ into many parts such that the graph induced by each part satisfies a property $\mathcal{P}$. Different properties $\mathcal{P}$ correspond to different types of edge-coloring. For example, a *proper edge k-coloring* of G is a decomposition of $E(G)$ into k subsets such that the graph induced by each subset is a matching in G. The minimum integer k such that G has a proper edge k-coloring, denoted by $\chi'(G)$, is the *edge chromatic number* of G. Vizing's Theorem states that for any graph G, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$. A graph G is of *class* 1 if $\chi'(G) = \Delta(G)$ and of *class* 2 if $\chi'(G) = \Delta(G) + 1$. Sanders and Zhao [11] showed that each planar graph with maximum degree at least 7 is of class 1. Juvan et al. [9] proved that each series–parallel graph (and each outerplanar graphs) with maximum degree at least 3 is of class 1. We ask whether each pseudo-outerplanar graph with large maximum degree is of class 1.

On the other hand, one can consider improper edge-colorings. Concerning this topic, Harary [7] introduced the concept of linear arboricity. A *linear forest* is a forest in which every connected component is a path. A *tree k-coloring* of G is a decomposition of $E(G)$ into k subsets such that the graph induced by each subset is a linear forest. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum integer k such that G has a tree k-coloring. Akiyama et al. [1] conjectured that $\mathrm{la}(G) = \lceil(\Delta(G)+1)/2\rceil$ for any regular graph G. It is obvious that $\mathrm{la}(G) \ge \lceil\Delta(G)/2\rceil$ for any graph G and that $\mathrm{la}(G) \ge \lceil(\Delta(G)+1)/2\rceil$ for any regular graph G. Hence the conjecture is equivalent to the following one.

Conjecture 1.1 (*Linear Arboricity Conjecture*). *For any graph G,* $\lceil\frac{\Delta(G)}{2}\rceil \le \mathrm{la}(G) \le \lceil\frac{\Delta(G)+1}{2}\rceil$.

Conjecture 1.1 has been proved true for all planar graphs [13,15]. However, it is still interesting to determine which planar graphs satisfy $\mathrm{la}(G) = \lceil\Delta(G)/2\rceil$. Wu [13] proved that it holds for planar graphs with maximum degree at least 13, and this bound 13 was later improved to 9 by Cygan et al. [4]. For subclasses of planar graphs, Wu [14] proved that $\mathrm{la}(G) = \lceil\Delta(G)/2\rceil$ for all series–parallel graphs (hence also for all outerplanar graphs) with maximum degree at least 3. Can the same conclusion extend to the class of pseudo-outerplanar graphs?

In Section 2, we give some relationships among three classes containing the outerplanar graphs; they are the $K_{2,3}$-minor-free graphs, the series–parallel graphs, and the pseudo-outerplanar graphs. In Section 3, we investigate the problem of covering pseudo-outerplanar graphs with forests and a graph of bounded maximum degree. In Section 4, some unavoidable structures of pseudo-outerplanar graphs are obtained. These structures will be applied to determine the edge chromatic number and the linear arboricity of pseudo-outerplanar graphs in Section 5.

2. Basic properties

Let G be a pseudo-outerplanar graph. In the remainder of this paper, we always assume that G has been drawn on the plane such that (1) for each block B of G, the vertices of B lie on a fixed circle and the edges of B lie inside the disk of this circle with each of them crossing at most one another; (2) the number of crossings in G is as small as possible. We call such a drawing *pseudo-outerplanar diagram* of G. Let G be a pseudo-outerplanar diagram and let B be a block of G. Denote by $v_1, v_2, \dots, v_{|B|}$ the vertices of B, which are lying in a clockwise sequence. Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \dots, v_j\}$ and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j] \setminus \{v_i, v_j\}$, where the subscripts and the additions are taken modular $|B|$.

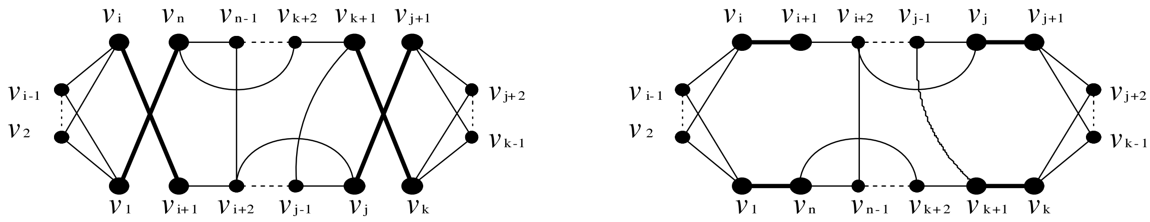


Fig. 2. Each Hamiltonian pseudo-outerplanar graph has a Hamiltonian diagram.

Lemma 2.1 ([6]). *If G is an outerplanar graph, then*

(a) $\delta(G) \le 2$,
(b) $\kappa(G) \le 2$.

Theorem 2.2. *If G is a pseudo-outerplanar graph, then*

(a) $\delta(G) \le 3$,
(b) $\kappa(G) \le 2$ *unless* $G \simeq K_4$.

Proof. The proof of (a) is left to Corollary 4.3; we only prove (b) here. If $|G| \le 4$, then this claim is trivial. Hence we may assume that G is a pseudo-outerplanar diagram with $|G| \ge 5$ and $\kappa(G) \ge 3$. If G has no crossings, then G is an outerplanar graph; thus Lemma 2.1 yields $\kappa(G) \le 2$, a contradiction. Therefore, we assume that there are two chords v_iv_j and v_kv_l in G that cross each other, and that v_i, v_k, v_j, v_l are lying in a clockwise sequence. Since $|G| \ge 5$, at least one of $\mathcal{V}(v_i, v_k)$, $\mathcal{V}(v_k, v_j)$, $\mathcal{V}(v_j, v_l)$ and $\mathcal{V}(v_l, v_i)$ is nonempty. Without loss of generality, assume that $\mathcal{V}(v_i, v_k) \neq \emptyset$. Since v_iv_j crosses v_kv_l, there is no edge between the two vertex sets $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$. Thus, $\{v_i, v_k\}$ separates $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$, contradicting $\kappa(G) \ge 3$. $\square$

It is well-known that every 2-connected outerplanar graph is Hamiltonian, but this does not hold for 2-connected pseudo-outerplanar graphs. The complete bipartite graph $K_{2,3}$ is just a counterexample. If the disk of a circle C is closed, then we call C a *closed circuit*. A 2-connected pseudo-outerplanar diagram is a *Hamiltonian diagram* if it is drawn so that all its vertices lie on a closed circuit C; and this closed circuit C is the *Hamiltonian boundary* of the diagram. By this definition, one can easily see that a non-Hamiltonian 2-connected pseudo-outerplanar graph cannot have a Hamiltonian diagram. We ask whether each Hamiltonian pseudo-outerplanar graph has a Hamiltonian diagram.

Theorem 2.3. *Let G be a pseudo-outerplanar diagram and let C be a Hamiltonian cycle of G. If C is not the boundary of G, then G has a Hamiltonian diagram such that C is the Hamiltonian boundary of this diagram.*

Proof. We proceed by induction on the order of G. Since G has a Hamiltonian cycle C with vertices $v_1, \ldots, v_n$ that is not the boundary of the pseudo-outerplanar diagram of G, there exists at least one crossing in the drawing of C, which is a sub-diagram of G. Suppose that v_jv_{j+1} and v_kv_{k+1} for $j < k$ cross each other and that $v_k, v_j, v_{k+1}, v_{j+1}$ lie in a clockwise order. Denote respectively by U and W the set of vertices from v_j to v_{k+1} and from v_{j+1} to v_k in the cyclic clockwise sequence of vertices on the outer boundary of G. Take the first graph in Fig. 2 for example, we have $C = v_1v_2, \ldots, v_nv_1, U = \{v_j, v_{j-1}, \ldots, v_{i+1}, v_1, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{k+1}\}$ and $W = \{v_{j+1}, v_{j+2}, \ldots, v_{k-1}, v_k\}$. Note that besides v_jv_{j+1} and v_kv_{k+1}, there is no other edge uw such that $u \in U$ and $w \in W$, by the definition of G. One can see that G_1 is a pseudo-outerplanar diagram with a Hamiltonian cycle C_1 having vertices $v_{k+1}, \ldots, v_n, v_1, \ldots, v_j$, while G_2 is a pseudo-outerplanar diagram with a Hamiltonian cycle C_2 having vertices $v_{j+1}, \ldots, v_k$. By the induction hypothesis, G_1 and G_2 have Hamiltonian diagrams with Hamiltonian boundaries C_1 and C_2, respectively. We now combine these two Hamiltonian diagrams and add two edges v_jv_{j+1} and v_kv_{k+1} (see the second graph in Fig. 2) to obtain a Hamiltonian diagram of G with Hamiltonian boundary $v_{k+1}v_{k+2}, \ldots, v_nv_1, \ldots, v_jv_{j+1}v_{j+2}, \ldots, v_{k-1}v_kv_{k+1}$, which is the cycle C. $\square$

Corollary 2.4. *Each Hamiltonian pseudo-outerplanar graph has a Hamiltonian diagram.*

A graph G is *quasi-Hamiltonian* if each block of G is Hamiltonian. Denote the class of pseudo-outerplanar graphs, quasi-Hamiltonian pseudo-outerplanar graphs, series–parallel graphs, $K_{2,3}$-minor-free graphs, and outerplanar graphs by $\mathcal{P}$, $\mathcal{P}_H$, $\mathcal{S}$, $\mathcal{M}_{2,3}$, and $\mathcal{O}$, respectively. The following basic relationship is obvious.

Remark 2.5. $\mathcal{P} \supset \mathcal{P}_H \supset \mathcal{O}$, $\mathcal{M}_{2,3} \bigcap \mathcal{S} = \mathcal{O}$.

In the following, we prove other relationships among these five classes of graphs.

Theorem 2.6. $\mathcal{P}_H \bigcap \mathcal{S} = \mathcal{O}$.

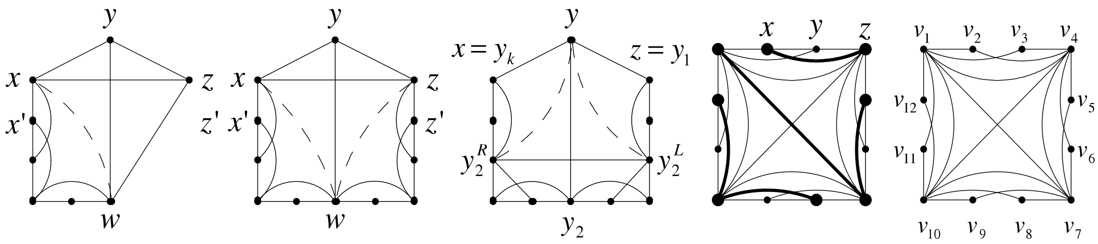


Fig. 3. Decomposability of pseudo-outerplanar graphs.

Proof. Let $G \in \mathcal{P}_H \bigcap \mathcal{S}$ and let B be a block of G. By Corollary 2.4, B has a Hamiltonian diagram, and actually this diagram is outerplanar. If there is a crossing, there would be four vertices u, v, x, y with uv and xy crossing in B. Since the diagram is Hamiltonian, there are four pairwise disjoint paths P_{ux}, P_{xv}, P_{vy} and P_{yu} that connect u to x, x to v, v to y and y to u. Thus, the two edges uv and vy and the four paths $P_{ux}, P_{xv}, P_{vy}, P_{yu}$ form a K_4-minor, which is impossible in a series–parallel graph. Hence B is an outerplanar graph. $\square$

Lemma 2.7 ([8]). *Let H be a graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2. If G is a H-minor-free graph, then each block of G is either K_4-minor-free or isomorphic to K_4.*

Corollary 2.8. *For any 2-connected graph $G \in \mathcal{M}_{2,3}$, either $G \in \mathcal{O}$ or $G \simeq K_4$.*

Proof. Since $G \in \mathcal{M}_{2,3}$, G is H-minor-free, where H is the graph in Lemma 2.7. Thus, by Remark 2.5 and Lemma 2.7, either $G \in \mathcal{O}$ or $G \simeq K_4$. $\square$

Theorem 2.9. *$\mathcal{M}_{2,3} \subset \mathcal{P}_H$.*

Proof. The inclusion of $\mathcal{M}_{2,3}$ in $\mathcal{P}_H$ directly follows from Corollary 2.8. The inequality comes from the graph $(K_1 \bigcup K_2) \vee \overline{K_2}$ that belongs to $\mathcal{P}_H$ but not to $\mathcal{M}_{2,3}$. $\square$

3. Decomposability

Let G be a pseudo-outerplanar diagram and let B be a block of G. Denote by $v_1, v_2, \ldots, v_{|B|}$ the vertices of B, which are lying in a clockwise sequence. The edges joining consecutive vertices in this list are *boundary edges* and other edges are *chords* of G. Since G is a pseudo-outerplanar diagram, all of the crossings are generated by one chord crossing another chord. Let $\mathcal{C}[v_i, v_j]$ be the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$ and let $\mathcal{C}(G)$ be the set of crossed chords in G.

Theorem 3.1. *Let G be a Hamiltonian pseudo-outerplanar diagram and let C be the Hamiltonian boundary of this diagram. If $y \in V(C)$ and $yx, yz \in E(C)$, then there exists a linear forest T in G such that $E(T) \subseteq \mathcal{C}(G)$, $d_T(y) = 0$, $\max\{d_T(x), d_T(z)\} \leq 1$, and $G - E(T)$ is an outerplanar diagram.*

Proof. We proceed by induction on the order of G. One can see that this claim holds for $|G| \leq 4$, since the case $G = K_4$ is trivial. Hence, we may assume that $|G| \geq 5$ and the three vertices x, y, z occur on C in a clockwise sequence.

First, we consider the case when $d_G(y) = 2$. If the edge xz already exists in G, then let $G' = G - y$ and $C' = C - y$; otherwise, let $G' = (G - y) + xz$ and $C' = (C - y) + xz$. It is easy to see that G' is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary C'. Let $x' \neq z$ be a vertex such that $xx' \in E(C')$ (x' exists because $|V(G)| \geq 5$). By induction on (G', C', x', x, z) (as (G, C, x, y, z), respectively), there exists a linear forest T' in G' such that $E(T') \subseteq \mathcal{C}(G')$, $d_{T'}(x) = 0$, $\max\{d_{T'}(x'), d_{T'}(z)\} \leq 1$, and $G' - E(T')$ is an outerplanar diagram. Let $T = T'$. Since $\mathcal{C}(G') = \mathcal{C}(G)$, we have $E(T) \subseteq \mathcal{C}(G)$, $d_T(x) = d_T(y) = 0$, and $d_T(z) \leq 1$. Furthermore, one can easily see that $G - E(T)$ is an outerplanar diagram.

If $d_G(y) = 3$ and $xz \in E(G)$, then the edge xz is crossed by another edge yw. Assume first that $\mathcal{V}(z, w) = \emptyset$. We then immediately have $zw \in E(C)$. Let $G' = G[\mathcal{V}[w, x]] + wx$ and let C' be the cycle consisting of the edge xw and the clockwise subpath around C from w to x. We assume that $N_{C'}(x) \setminus \{w\} \neq \emptyset$, because otherwise G would have less than five vertices, a contradiction. Let $x' \neq w$ be a vertex such that $xx' \in E(C')$ (see 1st graph of Fig. 3). Note that G' is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary C'. By induction on (G', C', x', x, w), there exists a linear forest T' in G' such that $E(T') \subseteq \mathcal{C}(G')$, $d_{T'}(x) = 0$, $\max\{d_{T'}(x'), d_{T'}(w)\} \leq 1$, and $G' - E(T')$ is an outerplanar diagram. Let $T = T' + xz$. One can easily check that $E(T) \subseteq \mathcal{C}(G)$, $d_T(y) = 0$, $d_T(x) = d_T(z) = 1$, and $G - E(T)$ is an outerplanar diagram. Thus, a linear forest T as required can be constructed. In the following, we assume that $\mathcal{V}(z, w) \neq \emptyset$ and $\mathcal{V}(w, x) \neq \emptyset$. Let $z' \neq y, w$ be a vertex such that $zz' \in E(C_1)$ and let $x' \neq y, w$ be a vertex such that $xx' \in E(C)$ (see 2nd graph of Fig. 3). Let $G_1 = G[\mathcal{V}[z, w]] + zw$ and $G_2 = G[\mathcal{V}[w, x]] + wx$. By C_1 and C_2, we respectively denote the cycle that consists of the edge wz and the clockwise subpath around C from z to w, and that consists of the edge xw and the clockwise subpath around C from w to x. For $i = 1, 2$, G_i is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary C_i. By inductions on (G_1, C_1, w, z, z') and (G_2, C_2, w, x, x'), there exist a linear forest T_1 in G_1 with

$E(T_1) \in \mathcal{C}(G_1)$, $d_{T_1}(z) = 0$, $\max\{d_{T_1}(w), d_{T_1}(z')\} \leq 1$, and $G_1 - E(T_1)$ being an outerplanar diagram, and a linear forest T_2 in G_2 with $E(T_2) \in \mathcal{C}(G_2)$, $d_{T_2}(x) = 0$, $\max\{d_{T_2}(w), d_{T_2}(x')\} \leq 1$, and $G_2 - E(T_2)$ being an outerplanar diagram. Let $T = T_1 \cup T_2 \cup \{xz\}$. One can easily see that $E(T) \subseteq \mathcal{C}(G)$, $d_T(y) = 0$, $d_T(x) = d_T(z) = 1$, $d_T(w) \leq 2$, and $G - E(T)$ is an outerplanar diagram. Since T_1 and T_2 intersect on at most one vertex, w, of degree at most one in each forest and there is no edge between $V(T_1) \setminus \{w\}$ and $V(T_2) \setminus \{w\}$, $T_1 \cup T_2$ is a linear forest. Furthermore, since x, y and z have degree 0 in $T_1 \cup T_2$, $T_1 \cup T_2 \cup \{xz\}$ is the required linear forest.

The last case is when $d_G(y) \geq 3$ and $xz \notin E(G)$. We label the neighbors of y by $y_1, y_2, \ldots, y_k$ in a clockwise sequence on C, where $y_1 = z$, $y_k = x$ and $k \geq 3$. If yy_2 is not a crossed chord in G, then let $G_1 = G[\mathcal{V}[y, y_2]]$ and $G_2 = G[\mathcal{V}[y_2, y]]$. Denote by C_1 (resp. C_2) the cycle consisting of the edge yy_2 and the clockwise subpath around C from y to y_2 (resp. from y_2 to y). For $i = 1, 2$, G_i is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary C_i. By inductions on (G_1, C_1, y_2, y, z) and (G_2, C_2, y_2, y, x), it is easy to construct a linear forest as required. Hence, we may assume that yy_2 is crossed by another edge $y_2^Ly_2^R$ in G, where y_2^L, y_2, y_2^R are labeled clockwise. Since there is no edge between $\mathcal{V}(y, y_2^L)$ and $\mathcal{V}(y_2^L, y)$, or between $\mathcal{V}(y, y_2^R)$ and $\mathcal{V}(y_2^R, y)$, we can add two edges yy_2^L and yy_2^R to G if they do not really exist so that they do not generate new crossings in G, and thus the resulting graph is still pseudo-outerplanar (see the 3rd graph of Fig. 3). By C_1, C_2 and C_3, we respectively denote the cycle that consists of the edge y_2^Ly and the clockwise subpath around C from y to y_2^L, and that consists of the path $y_2^Ryy_2^L$ and the clockwise subpath around C from y_2^L to y_2^R, and that consists of the edge yy_2^R and the clockwise subpath around C from y_2^R to y. Let G_i be the subgraph of G contained in the closed disk of C_i for $i = 1, 2, 3$. Here one should be careful that if $y_2^L = y_1$ (resp. $y_2^R = y_k$), then C_1 (resp. C_3) is not a cycle and G_1 (resp. G_3) is defined to be a null graph. However, G_1 and G_3 cannot simultaneously be null graphs, since $y_1y_k \notin E(G)$. Hence any of G_i for $i = 1, 2, 3$ is a subgraph of G with smaller order. Moreover, every non-null graph G_i is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary C_i. Without loss of generality, we assume that none of G_i for $i = 1, 2, 3$ is a null graph. By inductions on $(G_1, C_1, y_1, y, y_2^L)$, $(G_2, C_2, y_2^R, y, y_2^L)$ and $(G_3, C_3, y_k, y, y_2^R)$, there exist a linear forest T_i in G_i for $i = 1, 2, 3$ such that $E(T_i) \in \mathcal{C}(G_i)$, $d_{T_i}(y) = 0$, and $G_i - E(T_i)$ is an outerplanar diagram. Meanwhile, we have $\max\{d_{T_1}(y_1), d_{T_1}(y_2^L), d_{T_2}(y_2^L), d_{T_2}(y_2^R), d_{T_3}(y_2^R), d_{T_3}(y_k)\} \leq 1$. Let $T = T_1 \cup T_2 \cup T_3$. Since there is no edge whose end points are belong to different parts of the vertex partition $[\mathcal{V}(y, y_2^L), \mathcal{V}(y_2^L, y_2^R), \mathcal{V}(y_2^R, y)]$ (because otherwise either yy_2 or $y_2^Ly_2^R$ may be crossed twice), T is a forest. Since $d_T(y_2^R) \leq d_{T_2}(y_2^R) + d_{T_3}(y_2^R) \leq 2$ and $d_T(y_2^L) \leq d_{T_1}(y_2^L) + d_{T_2}(y_2^L) \leq 2$, $\Delta(T) \leq 2$ and thus T is a linear forest. Since $\mathcal{C}(G_i) \subseteq \mathcal{C}(G)$ for $i = 1, 2, 3$, $E(T) = E(T_1) \cup E(T_2) \cup E(T_3) \in \mathcal{C}(G_1) \cup \mathcal{C}(G_1) \cup \mathcal{C}(G_3) \in \mathcal{C}(G)$. Meanwhile, $d_T(y) = d_{T_1}(y) + d_{T_2}(y) + d_{T_3}(y) = 0$, $d_T(x) = d_T(y_k) = d_{T_3}(y_k) \leq 1$, and $d_T(z) = d_T(y_1) = d_{T_1}(y_1) \leq 1$. Since $G - E(T) \subseteq \bigcup_{i=1}^{3}(G_i - E(T_i))$, $G - E(T)$ is an outerplanar diagram. Hence we construct a linear forest T as required in G and completes the proof of the theorem. □

A *star forest* is a graph in which every component is a star. The *root* of a star is the vertex of maximum degree. Note that K_2 has two roots. The *roots* of a star forest is the union of the root of each star component. The following Theorem 3.2 is an analog of Theorem 3.1 (note that the condition $\max\{d_T(x), d_T(z)\} \leq 1$ in Theorem 3.1 is equivalent to that x or z is a vertex of T if and only if x or z is a leaf of T), whose proof is almost the same with that of Theorem 3.1. Actually, we can still proceed by induction on the order of G and split the proofs into three cases: the first is $d_G(y) = 2$, the second is $d_G(y) = 3$ and $xz \in E(G)$, and the last is $d_G(y) \geq 3$ and $xz \notin E(G)$. In each case we can construct a star forest T as required by the same way as in the proof of Theorem 3.1. The detailed proof of Theorem 3.2 is left to the readers.

Theorem 3.2. *Let G be a Hamiltonian pseudo-outerplanar diagram and let C be the Hamiltonian boundary of this diagram. If $y \in V(C)$ and $yx, yz \in E(C)$, then there exists a star forest T in G such that $E(T) \in \mathcal{C}(G)$, $d_T(y) = 0$, x or z is a vertex of T if and only if x or z is a root of T, and $G - E(T)$ is an outerplanar diagram.*

Corollary 3.3. *Each pseudo-outerplanar graph can be decomposed into an outerplanar graph and a linear forest, or an outerplanar graph and a star forest.*

Proof. Without loss of generality, let G be a quasi-Hamiltonian pseudo-outerplanar diagram (otherwise we can add some edges to close the circumferential boundary of each block). In what follows, we proceed by induction on the number of blocks, $\omega(G)$, in G. The base case when $\omega(G) = 1$ follows directly from Theorems 3.1 and 3.2, so we assume that $\omega(G) \geq 2$. Choose a block B of G that contains only one cut vertex y (i.e. B is an end-block). By Theorems 3.1 and 3.2, B can be decomposed into an outerplanar graph H_1 and a linear forest T_1 with $d_{T_1}(y) = 0$, or an outerplanar graph H_2 and a star forest T_2 with $d_{T_2}(y) = 0$. Meanwhile, by the induction hypothesis, $G - B$ can also be decomposed into an outerplanar graph H_3 and a linear forest T_3, or an outerplanar graph H_4 and a star forest T_4. Therefore, G can be covered by the linear forest $T = T_1 \cup T_3$ and the outerplanar graph $H = H_1 \cup H_3$, or the star forest $T = T_2 \cup T_4$ and the outerplanar graph $H = H_2 \cup H_4$. □

Theorem 3.4. *For every integer $n \geq 12$, there exists a 2-connected pseudo-outerplanar graph with order n that cannot be decomposed into an outerplanar graph and a matching.*

Proof. We show the last graph G in Fig. 3 is a graph that cannot be decomposed into an outerplanar graph and a matching. Otherwise, we may assume that $E(G) = E(H) \cup E(M)$, where H is an outerplanar and M is matching. Set $S_i = \{v_iv_{i+1}, v_iv_{i+2}, v_iv_{i+3}, v_{i+1}v_{i+3}, v_{i+2}v_{i+3}\}$ (mod 12) for $i = 1, 4, 7, 10$. We now prove that there exists an edge set S_i that is contained in $E(H)$.

If this claim does not hold, then we may meet one of the following two cases. If $v_1v_2 \in E(M)$, then $v_1v_k \in E(H)$ for $k = 3, 4, 7, 10, 11, 12$ and exactly one of $v_{10}v_{11}$ and $v_{10}v_{12}$ should be contained in $E(M)$, say $v_{10}v_{11}$. It follows that $v_kv_{10} \in E(H)$ for $k = 4, 7, 12$. However, the five vertices $v_1, v_4, v_7, v_{10}, v_{12}$ and the three disjoint paths $v_1v_4v_{10}, v_1v_7v_{10}, v_1v_{12}v_{10}$ form a copy of $K_{2,3}$ in H; this is a contradiction. If $v_1v_4 \in E(M)$, then $v_1v_2, v_1v_3, v_1v_7, v_2v_4, v_3v_4, v_4v_7 \in E(H)$ and the graph induced by $\{v_1, v_2, v_3, v_4, v_7\}$ is a $K_{2,3}$, which is impossible in an outerplanar graph.

Hence in the following, we may assume that $S_1 \subseteq E(H)$. If $\{v_1v_7, v_4v_7\} \subseteq E(H)$, then the five vertices v_1, v_2, v_3, v_4, v_7 and the three disjoint paths $v_1v_2v_4, v_1v_3v_4, v_1v_7v_4$ form a copy of $K_{2,3}$ in H, a contradiction. Thus, exactly one of v_1v_7 and v_4v_7 shall be contained in $E(M)$, say v_1v_7. Similarly, we can prove that $\{v_1v_{10}, v_4v_{10}\} \nsubseteq E(H)$. Thus, we have $v_1v_{10} \in E(H)$, $v_4v_{10} \in E(M)$, and $v_7v_{10} \in E(H)$. Now the six vertices $v_1, v_2, v_3, v_4, v_7, v_{10}$ and the three disjoint paths $v_1v_3v_4, v_1v_2v_4, v_1v_{10}v_7v_4$ form a $K_{2,3}$-minor in H. This contradiction completes the proof of this theorem. □

Theorem 3.5. *Every maximal pseudo-outerplanar graph G is obtained from a maximal pseudo-outerplanar diagram H by gluing a K_3 or a K_4 along a boundary edge of H.*

Proof. Without loss of generality, we assume that G is a 2-connected maximal pseudo-outerplanar diagram. Since G is maximal, G is Hamiltonian and has at least one chord. Let C be the Hamiltonian boundary of the diagram of G with vertices $v_1, v_2, \ldots, v_{|G|}$. We split the proof into two cases.

Case 1. *There exists a crossed chord in G.*

Let v_iv_j be a chord in G that crosses another chord v_kv_l for $1 \le i < k < j < l \le |G|$. Actually, we can properly choose i and j such that there is no pair of mutually crossed chords in $\mathcal{C}[v_i, v_l] \setminus \{v_iv_j, v_kv_l\}$, because otherwise we can change the value of i or j to meet this condition.

Assume first that there is no non-crossed chord in $\mathcal{C}[v_i, v_l] \setminus \{v_iv_l\}$. In this case, we shall have $k = i + 1$. Otherwise, since $v_iv_k \notin E(G)$, we can add v_iv_k to G so that G is still pseudo-outerplanar, contradicting the fact that G is maximal. Similarly, by the maximality of G, we have $j = k + 1$, $l = j + 1$, and $v_iv_l \in E(G)$. Furthermore, $d(v_k) = d(v_j) = 3$. Remove the vertices v_k and v_j from G and denote the resulting graph by H. Actually, H is a maximal pseudo-outerplanar diagram, because otherwise we can add an edge $e = v_av_b \notin E(H)$, where $a, b \neq k$ or j, to H so that $H + e$ is pseudo-outerplanar, and thus $G + e$ is pseudo-outerplanar, since $e \notin E(G)$, contradicting the fact that G is maximal. At this stage, one can easily see that G is obtained from H by gluing a K_4 along the boundary edge v_iv_l of H.

Second, assume that there is a non-crossed chord v_rv_s in $\mathcal{C}[v_i, v_l] \setminus \{v_iv_l\}$. Since there is no crossed chord in $\mathcal{C}[v_r, v_s]$, we can properly choose r and s such that $\mathcal{C}[v_r, v_s] \setminus \{v_rv_s\} = \emptyset$. By the maximality of G, we have $s = r + 2$, otherwise we can add an edge v_rv_{r+2} to G so that the resulting graph is still pseudo-outerplanar, a contradiction. Since v_rv_s is a non-crossed chord, $d(v_{r+1}) = 2$. Remove the vertex v_{r+1} from G and denote the resulting graph by H'. By a similar argument as above, one can prove that H' is a maximal pseudo-outerplanar diagram. Furthermore, one can easily see that G is obtained from H' by gluing a K_3 along the boundary edge v_rv_{r+2} of H.

Case 2. *There exists a non-crossed chord in G.*

Let v_iv_j for $1 \le i < j \le |G|$ be a non-crossed chord in G. In this case, we shall assume that there is no crossed chord in $\mathcal{C}[v_i, v_j]$, because otherwise we are in Case 1. We choose i and j such that $\mathcal{C}[v_i, v_j] \setminus \{v_iv_j\} = \emptyset$, and then we are in the second subcase of Case 1, where we can set $r := i$ and $s := j$. □

Corollary 3.6. *Each pseudo-outerplanar graph can be decomposed into two forests and a matching.*

Proof. Let G be a pseudo-outerplanar graph. In the following, we proceed by induction on the size of G and assume that G is a maximal pseudo-outerplanar diagram. By Theorem 3.5, there respectively exists a K_3 with vertices x, y and z or a K_4 with vertices x, y, u and v contained in G such that $H = G - \{xz, yz\}$ or $H = G - \{xu, xv, yu, yv, uv\}$ is a maximal pseudo-outerplanar graph with xy being its boundary edge. By induction on H, there exists two forests F_1, F_2 and a matching M such that $E(H) = E(F_1) \cup E(F_2) \cup E(M)$. In the former case, let $F_1' = F_1 + xz$, $F_2' = F_2 + yz$, and $M' = M$; and in the latter case, let $F_1' = F_1 + \{xu, xv\}$, $F_2' = F_2 + \{yu, yv\}$, and $M' = M + uv$. One can easily check that the two forests F_1', F_2' and the matching M' is the desired decomposition of G. □

Theorem 3.7. *For every integer $n \ge 6$, there exists a 2-connected pseudo-outerplanar graph with order n that cannot be decomposed into two forests.*

Proof. Let C be a cycle with n vertices $v_1, \ldots, v_n$, where $n \ge 6$. Add edges v_1v_i for all $3 \le i \le n - 1$ and edges $v_{2i}v_{2i+2}$ for all $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$. One can easily check that the resulting graph G_n is a 2-connected pseudo-outerplanar graph with order n and size $\lfloor \frac{5}{2}n \rfloor - 4$. If G_n can be decomposed into two forests F_1 and F_2, then $|E(G_n)| = |E(F_1)| + |E(F_2)| \le |V(F_1)| + |V(F_2)| - 2 \le 2n - 2$. However, for $n \ge 6$, $|E(G_n)| = \lfloor \frac{5}{2}n \rfloor - 4 > 2n - 2$. Hence, the graph G_n for $n \ge 6$ cannot be covered by two forests. □

From Corollary 3.6 and Theorem 3.7, we directly have the following two corollaries.

Corollary 3.8. *Every pseudo-outerplanar graph is* (2, 1)*-coverable; and the two parameters given here are best possible.*

Corollary 3.9. *The arboricity of a pseudo-outerplanar graph is at most* 3*; and this bound is sharp.*

4. Unavoidable structures

In this section, a vertex set $\mathcal{V}[v_i, v_j]$ is a *non-edge* if $j = i + 1$ and $v_iv_j \notin E(G)$, is a *path* if $v_kv_{k+1} \in E(G)$ for all $i \le k < j$, and is a *subpath* if $j > i+1$ and some edges in the form v_kv_{k+1} for $i \le k < j$ are missing. We say that a chord v_kv_l is contained in a chord v_iv_j if $i \le k \le l \le j$. In any figure of this section, the solid vertices have no edges of G incident with them other than those shown.

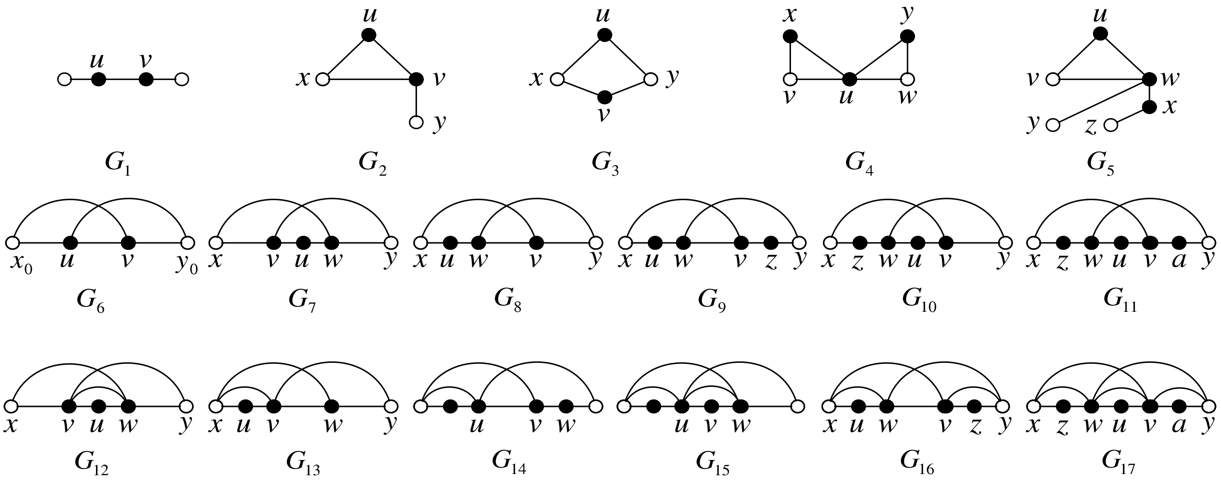
Lemma 4.1 ([12]). *If G is a 2-connected outerplanar graph, then*

(1) *G has two adjacent 2-vertices u and v, or*
(2) *G has a 3-cycle $uwxu$ such that $d(u) = 2$ and $d(w) = 3$, or*
(3) *G has a 4-vertex w, where $N(w) = \{u, v, x, y\}$, such that $d(u) = d(v) = 2$, $N(u) = \{w, x\}$ and $N(v) = \{w, y\}$.*

For the class of pseudo-outerplanar graphs, we have a similar structural theorem as Lemma 4.1.

Theorem 4.2. *If G is a pseudo-outerplanar diagram with $\delta(G) \ge 2$, then G contains one of the following configurations G_1–G_{17}. Moreover,*

(a) *if G contains some configuration among G_6–G_{17}, then the drawing of this configuration in the figure is a part of the diagram of G with its bending edges corresponding to the chords;*
(b) *if G contains the configuration G_3 and $xy \notin E(G)$, where x and y are the vertices of G_3 as described in the figure, then we can properly add an edge xy to G so that the resulting diagram is still pseudo-outerplanar.*



Proof. We first consider the case when G is a 2-connected pseudo-outerplanar diagram. Let $v_1, v_2, \ldots, v_{|G|}$ be the vertices of the diagram lying in a clockwise sequence. If there is no crossing in G, then G is an outerplanar graph, and thus G satisfies this claim by Lemma 4.1. Otherwise, we can properly choose one chord v_iv_j such that

(1) v_iv_j crosses v_kv_l in G;
(2) v_i, v_k, v_j and v_l are lying in a clockwise sequence;
(3) besides v_iv_j and v_kv_l, there is no crossed chord in $\mathcal{C}[v_i, v_l]$.

The condition (3) can be easily fulfilled, because otherwise we could change the values of i and j to meet this condition (note that the values of k and l are determined by i and j). Without loss of generality, assume that $1 \le i < k < j < l \le |G|$, because otherwise we can adjust the labellings of the vertices in G to meet it.

Claim 1. *$\mathcal{V}[v_i, v_k]$ is either non-edge or path, and so do $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$.*

We only need to prove that $\mathcal{V}[v_i, v_k]$ cannot be subpath. Otherwise, there exists two vertices v_m and v_{m+1}, where $i \le m \le k - 1$, such that $v_mv_{m+1} \notin E(G)$. If there are chords in the form v_av_{m+1} such that $i \le a \le m - 1$, then we choose one among them such that a is maximum. One can see that v_a is a vertex cut of G, because there is no edge between $\mathcal{V}[v_{a+1}, v_m]$ and $\mathcal{V}[v_{m+1}, v_{a-1}]$, by the choice of a and by (3). This contradicts the fact that G is 2-connected. Thus, there is no chord in the form v_av_{m+1} such that $i \le a \le m - 1$. Similarly, there is no chord in the form v_mv_b such that $m+2 \le b \le k$. Let $p = \max\{n | v_{m+1}v_n \in E(G), m + 1 < n \le k\}$ and $q = \min\{n | v_nv_m \in E(G), i \le n < m\}$. Since $\mathcal{V}[v_i, v_k]$ is neither non-edge nor path, we have $k - i \ge 2$, and thus at least one of the integers p and q exists. Without loss of generality, suppose that p exists. In this case, v_p is a vertex cut of G, because there is no edge between $\mathcal{V}[v_{m+1}, v_{p-1}]$ and $\mathcal{V}[v_{p+1}, v_m]$, by the choices of m, p and by (3). This contradiction completes the proof of Claim 1.

Claim 2. *If $\mathcal{V}[v_i, v_k]$ is a path and $k - i \ge 3$, then G has a subgraph isomorphic to one of the configurations G_1, G_2 and G_4. This result also holds for $\mathcal{V}[v_k, v_j]$ or $\mathcal{V}[v_j, v_l]$ when $j - k \ge 3$ or $l - j \ge 3$, respectively.*

If there is no other chord except v_iv_k (if exists) in $\mathcal{V}[v_i, v_k]$, then the configuration G_1 occurs in G, since $k - i \geq 3$. Hence we may assume that $S := \mathcal{C}[v_i, v_k] \setminus \{v_iv_k\} \neq \emptyset$.

We now prove that there exists at least one chord in S that contains at least one other chord. If such a chord does not exist, then we first choose a chord $v_mv_n \in S$ for $m < n$. Without loss of generality, assume that $n \neq k$. If $n - m \geq 3$, then we can easily find a copy of G_1 in G, since v_mv_n contains no other chord by our assumption. If $n - m = 2$, then it is trivial to see that $d(v_{m+1}) = 2$. In this case, if $\min\{d(v_m), d(v_n)\} \leq 3$, then a copy of G_2 would be found in G. Hence we may assume that $\min\{d(v_m), d(v_n)\} \geq 4$. Since v_mv_n cannot be contained in a chord in the form v_qv_n for $q < n$ by the assumption, there exists another chord v_nv_p for $n < p$ in S. If $p - n \neq 2$ or $d(v_{n+1}) \neq 2$, then the configuration G_1 would be found in G. If $p - n = 2$ and $d(v_{n+1}) = 2$, then $d(v_n) = 4$, because otherwise there would be chord in S that contains either v_mv_n or v_nv_p, a contradiction. At this stage, one can see that the graph induced by $\mathcal{V}[v_m, v_p]$ contains the configuration G_4.

By the above arguments, we can choose one chord $v_av_b \in S$ for $a < b$ such that v_av_b contains at least one chord, and moreover, every chord contained in v_av_b contains no other chords (this condition can be easily fulfilled by properly changing the values of a and b if necessary). Let v_mv_n for $m < n$ be the chord contained in v_av_b. By a similar argument as above, we only need to consider the case when $n - m = 2$, $d(v_{m+1}) = 2$, and $\min\{d(v_m), d(v_n)\} \geq 4$. Without loss of generality, assume that $n \neq b$. Since $d(v_n) \geq 4$ and v_mv_n cannot be contained in a chord in the form v_qv_n for $q < n$ by the choices of a and b, there is a chord v_nv_p for $n < p \leq b$ in S. If G contains no copies of G_1 or G_2, then $p - n = 2$ and $d(v_{n+1}) = 2$. Furthermore, by the choices of a and b, one can similarly prove that $d(v_n) = 4$. Thus, we would find a copy of G_4 in the graph induced by $\mathcal{V}[v_m, v_p]$.

Claim 3. *At most one of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ can be non-edge.*

If $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_k, v_j]$ are non-edge, then v_l is a vertex cut of G, contradicting the fact that G is 2-connected. If $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_j, v_l]$ are non-edge, then we can adjust the drawing of G by replacing the vertices order $v_i, v_k, v_{k+1}, \ldots, v_{j-1}, v_j, v_l$ with $v_i, v_j, v_{j-1}, \ldots, v_{k+1}, v_k, v_l$. This operation can reduce the number of crossings in the drawing of G by one, contradicting the assumption that this diagram minimizes the number of crossings.

Claim 4. *If one of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is non-edge, then G has a subgraph isomorphic to one of the configurations G_1, G_2 and G_3.*

Suppose first that $\mathcal{V}[v_i, v_k]$ is a non-edge. By Claims 1–3, both $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are paths with $1 \leq j - k \leq 2$ and $1 \leq l - j \leq 2$. If $j - k = 2$ and $v_kv_j \in E(G)$, then it is clear that $d(v_k) = 3$ and $d(v_{k+1}) = 2$, implying that the configuration G_2 occurs in G. If $j - k = 2$ and $v_kv_j \notin E(G)$, then $d(v_k) = d(v_{k+1}) = 2$, implying that the configuration G_1 occurs in G. Hence we may assume that $j = k + 1$. If $l = j + 2$, then $d(v_{j+1}) = 2$ whenever v_jv_l is a chord or not. In this case, the configuration G_3 occurs in G, since $d(v_k) = 2$, and moreover, $G + v_jv_l$ is still pseudo-outerplanar if $v_jv_l \notin E(G)$. Thus, we shall assume that $l = j + 1$. Since v_k, v_j, v_l form a triangle satisfying $d(v_k) = 2$ and $d(v_j) = 3$, the configuration G_2 occurs in G. The case when $\mathcal{V}[v_j, v_l]$ is a non-edge can be dealt with similarly, so we omit it here.

Second, suppose that $\mathcal{V}[v_k, v_j]$ is a non-edge. By Claims 1–3, both $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_j, v_l]$ are paths with $1 \leq k - i \leq 2$ and $1 \leq l - j \leq 2$. If $k - i = 2$ or $j - l = 2$, then by a similar argument as before, we either have $d(v_{k-1}) = d(v_k) = 2$ or $d(v_j) = d(v_{j+1}) = 2$, implying that the configuration G_1 occurs in G. If $k - i = l - j = 1$, then the four vertices v_i, v_j, v_l and v_k form a quadrilateral with $d(v_i) = d(v_k) = 2$, which implies that the configuration G_3 occurs in G, and moreover, $G + v_iv_l$ is still pseudo-outerplanar if $v_iv_l \notin E(G)$.

In the following, we assume that $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are all paths, where $\max\{k - i, j - k, l - j\} \leq 2$. Set $X = \mathcal{C}[v_i, v_l] \setminus \{v_iv_j, v_kv_l\}$ and let $x = |X|$. It is clear that $x \leq 3$.

Claim 5. *If $x = 0$, then G has a subgraph isomorphic to one of the configurations G_6–G_{11}; if $x = 1$, then G has a subgraph isomorphic one of the configurations G_5, G_{12}, G_{13} and G_{14}; if $x = 2$, then G has a subgraph isomorphic to one of the configurations G_5, G_{15} and G_{16}; and if $x = 3$, then G has a subgraph isomorphic to the configuration G_{17}.*

We prove for the case when $x = 2$ and $v_kv_j, v_jv_l \in X$ for example, and leave the discussions on other cases to the readers, since they are quite similar. In fact, if $k - i = 1$ (resp. $k - i = 2$), then the configuration G_{15} (resp. G_5) would occurs in G, since $d(v_k) = 4$ and $d(v_{i+1}) = d(v_{k+1}) = d(v_{j+1}) = 2$, and moreover, the drawing of the configuration G_{15} (resp. G_5) in the figure is just a part of the diagram of G with its bending edges corresponding to the chords.

Claims 1–5 complete the proof of this theorem for the case when G is 2-connected. We now suppose that G is a counterexample that has at least two blocks. Let B be an end block and let $v_1, v_2, \ldots, v_{|B|}$ be the vertices of B that lies in a clockwise sequence. Without loss of generality, let v_1 be the unique cut vertex of B.

Claim 6. *B is an outerplanar graph.*

We prove that there is no crossing in B. Suppose, to the contrary, that there is a chord v_iv_j that crosses another chord v_kv_l, where $1 \leq i < k < j < l$. Note that the chord v_iv_j satisfies (1) and (2) now. If it does not fulfill (3) at this stage, then there must be at least one pair of mutually crossed chords contained in either $\mathcal{C}[v_i, v_k]$, or $\mathcal{C}[v_k, v_j]$, or $\mathcal{C}[v_j, v_l]$. We choose one pair v_av_b and v_cv_d among them, where $a < c < b < d$, such that there is no other crossed chord in $\mathcal{C}[v_a, v_d]$ besides v_av_b and v_cv_d, and then set $i := a, j := b,\ k := c$ and $l := d$. In any case, we can find a pair of mutually crossed chords

v_iv_j and v_kv_l, where $1 \le i < k < j < l$, such that the three conditions at the beginning of the proof are fulfilled. Since B is a 2-connected pseudo-outerplanar diagram, we can set $v_i,\ v_j,\ v_k,\ v_l$ as we did in the 2-connected case. Recall the proofs of Claims 1–5, every time we find a copy of some configuration the vertices v_i and v_l cannot be the solid vertices (i.e. the degrees of them in the configuration shall not necessarily to be confirmed). For a vertex $v \in V(B) \setminus \{v_1\}$, its degree in B is equal to its degree in G, since B is an end block and v_1 is the unique cut vertex of the B. Among the vertices in $\mathcal{V}[v_i, v_l]$, only v_i may be the cut vertex, since $1 \le i < k < j < l$. Therefore, the proofs of Claims 1–5 are also valid for this claim, and then the same results would be obtained.

Claim 7. *B has a subgraph isomorphic to one of the configurations G_1, G_2 and G_4 in such a way that v_1 is not a solid vertex.*

Since B is a 2-connected outerplanar graph, B is Hamiltonian and $\mathcal{V}[v_1, v_{|B|}]$ is a path. The proof of Claim 2 implies that if $\mathcal{V}[v_i, v_k]$, where $k - i \ge 3$, is a path such that there is no crossed edge in $\mathcal{C}[v_i, v_k]$ and no edge between $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$, then G contains one of G_1, G_2 and G_4 in such a way that v_i and v_k are not the solid vertices. In this claim, if $|B| \ge 4$, then we set $i := 1$, $k := |B|$ and come back to the proof of Claim 2. If $|B| \le 3$, then it is trivial to see that G_1 would appear in G. This contradiction completes the proof of the theorem for the case when G has cut vertices. □

The following is a straightforward corollary of Theorem 4.2.

Corollary 4.3. *Each pseudo-outerplanar graph contains a vertex of degree at most* 3.

5. Edge chromatic number and linear arboricity

In this section, we consider the problems of covering a pseudo-outerplanar graph G with matchings or linear forests. A graph G is *χ'-critical* if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) \le \Delta(G)$ for any proper subgraph $H \subset G$, and is *la-critical* if $\mathrm{la}(G) > \lceil \frac{\Delta(G)}{2} \rceil$ and $\mathrm{la}(H) \le \lceil \frac{\Delta(G)}{2} \rceil$ for any proper subgraph $H \subset G$.

Lemma 5.1. *If G is χ'-critical and $uv \in E(G)$, then $d(u) + d(v) \ge \Delta(G) + 2$.*

Lemma 5.2. *If G is la-critical and $uv \in E(G)$, then $d(u) + d(v) \ge 2\lceil \frac{\Delta(G)}{2} \rceil + 2$.*

The above two lemmas are very classic and useful; their proofs can be found in [3,14], respectively. Given a coloring φ of G, $c_j(v)$ denotes the number of edges incident with v colored by j. Let $C_\varphi^i(v) = \{j | c_j(v) = i\}$ for $i = 0, 1, 2$. If φ is a proper edge k-coloring, then $C_\varphi^0(v) \cup C_\varphi^1(v) = \{1, 2, \ldots, k\}$; if φ is a tree k-coloring, then $C_\varphi^0(v) \cup C_\varphi^1(v) \cup C_\varphi^2(v) = \{1, 2, \ldots, k\}$. For brevity, we use the notion *k-coloring* to replace the statements of proper edge k-coloring or tree k-coloring, and use the notion *PO-graph* to replace the statement of pseudo-outerplanar graph. For a graph G and two distinct vertices $u, v \in V(G)$, denote by $G + xy$ the graph obtained from G by adding an new edge xy if $xy \notin E(G)$, or G itself if $xy \in E(G)$.

Theorem 5.3. *Let G be a pseudo-outerplanar graph. If $\Delta(G) \ge 4$, then $\chi'(G) = \Delta(G)$.*

Proof. Suppose, for a contradiction, that there exists a minimal (in terms of the size) pseudo-outerplanar diagram G with $\Delta(G) \ge 4$ that has no $\Delta(G)$-colorings. One can easily see that G is 2-connected and χ'-critical. By Theorem 4.2 and Lemma 5.1, G contains at least one of the configurations $G_3, G_4, G_5, G_6, G_{12}, G_{13}, G_{16}$ and G_{17}. Set $S = \{1, 2, \ldots, \Delta(G)\}$.

If $G \supseteq G_3$, then the pseudo-outerplanar graph $G' = G \setminus \{u, v\}$ admits a $\Delta(G)$-coloring ϕ, by the minimality of G (when $\Delta(G') = \Delta(G)$) or Vizing's Theorem (when $\Delta(G') \le \Delta(G) - 1$). Construct a $\Delta(G)$-coloring φ of G as follows. If $C_\phi^1(x) = C_\phi^1(y) := L$ (note that $|L| = \Delta(G) - 2$ by Lemma 5.1), then let $\varphi(ux) = \varphi(yv) \in S \setminus L$ and $\varphi(uy) = \varphi(xv) \in S \setminus (L \cup \{\varphi(ux)\})$. If $C_\phi^1(x) \ne C_\phi^1(y)$, then $(S \setminus C_\phi^1(x)) \cap C_\phi^1(y) \ne \emptyset$, since $d(x) = d(y) = \Delta(G)$ by Lemma 5.1. Let $\varphi(ux) \in (S \setminus C_\phi^1(x)) \cap C_\phi^1(y)$, $\varphi(xv) \in S \setminus (C_\phi^1(x) \cup \{\varphi(ux)\})$, $\varphi(vy) \in S \setminus (C_\phi^1(y) \cup \{\varphi(xv)\})$, and $\varphi(uy) \in S \setminus (C_\phi^1(y) \cup \{\varphi(yv)\})$. In each case, we color the remaining edges of G by the same colors used in ϕ. Thus, we have constructed a $\Delta(G)$-coloring φ of G from the $\Delta(G)$-coloring ϕ of G'. In the discussions of the next seven cases, while constructing a coloring φ of G from the coloring ϕ of G', we only give the colorings for the edges in $E(G) \setminus E(G')$, since for every edge $e \in E(G) \cap E(G')$, we always let $\varphi(e) = \phi(e)$.

If $G \supseteq G_4$, then we shall assume that $d(v) = d(w) = \Delta(G) = 4$ because of Lemma 5.1. By the minimality of G, the PO-graph $G' = G \setminus \{x, y, u\}$ admits a 4-coloring ϕ. Construct a 4-coloring φ of G as follows, where two cases are considered without loss of generality (wlog. for short). If $C_\phi^1(v) = C_\phi^1(w) = \{1, 2\}$, then let $\varphi(uy) = 1$, $\varphi(ux) = 2$, $\varphi(uw) = \varphi(vx) = 3$, and $\varphi(uv) = \varphi(wy) = 4$. If $C_\phi^1(v) = \{1, 2\}$, $1 \notin C_\phi^1(w)$ and $3 \in C_\phi^1(w)$, then let $\varphi(uw) = 1$, $\varphi(ux) = 2$, $\varphi(xv) = \varphi(uy) = 3$, $\varphi(uv) = 4$, and $\varphi(wy) \in \{2, 3, 4\} \setminus C_\phi^1(w)$.

If $G \supseteq G_5$, then we shall assume that $d(v) = \Delta(G) = 4$ because of Lemma 5.1. By the minimality of G, the PO-graph $G' = G \setminus \{u\}$ admits a 4-coloring ϕ. One can easily see that $(C_\phi^1(v) \cap C_\phi^1(w)) \setminus \{\phi(vw)\} \ne \emptyset$, because otherwise vw would be incident with four colors under ϕ. Assume that $C_\phi^1(v) = \{1, 2, 3\}$ and $\phi(vw) = 3$ wlog. If $C_\phi^1(w) \ne C_\phi^1(v)$, then assume that $C_\phi^1(w) = \{1, 3, 4\}$ wlog. Whereafter, we extend ϕ to a 4-coloring of φ of G by taking $\varphi(uv) = 4$ and $\varphi(uw) = 2$. If $C_\phi^1(w) = C_\phi^1(v)$, then we consider two subcases. If $\phi(xz) = 4$, then construct a 4-coloring of G by recoloring wx and wv with

3 and 4, and coloring uv and uw with 3 and 2, respectively. If $\phi(xz) \neq 4$, then construct a 4-coloring of G by recoloring wx with 4 and coloring uv and uw with 4 and 2, respectively.

If $G \supseteq G_6$, then we shall assume that $\min\{d(x_0), d(y_0)\} \geq 3$ and $\Delta(G) = 4$ by Lemma 5.1. Assume first that $d(x_0) = d(y_0) = 4$. If $x_0y_0 \notin E(G)$, then let $N(x_0) = \{u, v, x_1, x_2\}$ and $N(y_0) = \{u, v, y_1, y_2\}$. Let $G' = G \setminus \{u, v\} + x_0y_0$. By Theorem 4.2, the configuration G_6 is a part of the pseudo-outerplanar diagram of G, so G' is a PO-graph that admits a 4-coloring ϕ by the minimality of G. Set $M = \{\phi(x_0x_1), \phi(x_0x_2), \phi(y_0y_1), \phi(y_0y_2)\}$ and let $m = |M|$. Since the colors used in ϕ is at most four and $x_0y_0 \in E(G')$, $m \leq 3$ (otherwise the edge x_0y_0 cannot be colored under ϕ, because it is already incident with four colored edges). If $m = 3$, then assume that $\phi(x_0x_1) = \phi(y_0y_1) = 1$, $\phi(x_0x_2) = 2$ and $\phi(y_0y_2) = 3$ wlog. Now we extend ϕ to a 4-coloring φ of G by taking $\varphi(uv) = 1$, $\varphi(vy_0) = 2$, $\varphi(ux_0) = 3$, and $\varphi(vx_0) = \varphi(uy_0) = 4$. If $m \leq 2$, then assume that $\phi(x_0x_1) = \phi(y_0y_1) = 1$ and $\phi(x_0x_2) = \phi(y_0y_2) = 2$ wlog. Now we extend ϕ to a 4-coloring φ of G by taking $\varphi(uv) = 1$, $\varphi(vy_0) = \varphi(ux_0) = 3$, and $\varphi(vx_0) = \varphi(uy_0) = 4$. On the other hand, if $x_0y_0 \in E(G)$, then let $N(x_0) = \{u, v, y_0, x_1\}$ and $N(y) = \{u, v, x_0, y_1\}$. One can see that $x_1 \neq y_1$, because otherwise we have $G \simeq G[\{u, v, x_0, y_0, x_1\}]$, by the 2-connectivity of G, and thus G can be 4-colorable. Consider the graph $G' = G \setminus \{u, v\} - x_0y_0$, which admits a 4-coloring ϕ by the minimality of G. If $\phi(x_0x_1) = \phi(y_0y_1) = 1$, then let $\varphi(uv) = 1$, $\varphi(x_0y_0) = 2$, $\varphi(ux_0) = \varphi(vy_0) = 3$, and $\varphi(vx_0) = \varphi(uy_0) = 4$. If $\phi(x_0x_1) = 1$ and $\phi(y_0y_1) = 2$, then let $\varphi(vy_0) = 1$, $\varphi(ux_0) = 2$, $\varphi(uv) = \varphi(x_0y_0) = 3$, and $\varphi(vx_0) = \varphi(uy_0) = 4$. Second, assume that one of x_0 and y_0 has degree three. Assume that $d(x_0) = 3$ wlog. Let $N(x_0) = \{u, v, w\}$. Consider the PO-graph $G' = G - ux_0$. By the minimality of G, G' has a 4-coloring ϕ. If $A := S \setminus \{\phi(vx_0), \phi(wx_0), \phi(uv), \phi(uy_0)\} \neq \emptyset$ (recall that $S = \{1, 2, 3, 4\}$), then let $\varphi(ux_0) \in A$. Otherwise, assume that $\phi(vx_0) = 1$, $\phi(wx_0) = 2$, $\phi(uv) = 3$, and $\phi(uy_0) = 4$ wlog. Since $d(v) = 3$, $\phi(uy_0) = 4$, and $vy_0 \in E(G')$, v is not incident with the color 4 under ϕ. Thus, we can extend ϕ to a 4-coloring of G by recoloring vx_0 with 4 and coloring ux_0 with 1.

If $G \supseteq G_{12}$, then we shall assume that $\Delta(G) = 4$ because of Lemma 5.1. Assume first that $d(x) = d(y) = 4$. If $xy \notin E(G)$, then let $N(x) = \{v, w, x_1, x_2\}$ and $N(y) = \{v, w, y_1, y_2\}$. Consider the graph $G' = G \setminus \{v, w\} + xy + ux + uy$. Since the configuration G_{12} is a part of the pseudo-outerplanar diagram of G by Theorem 4.2, we can properly add three edges xy, ux and uy to $G \setminus \{v, w\}$ so that G' is still a PO-graph. By the minimality of G, G' admits a 4-coloring ϕ. One can see that $\{\phi(xx_1), \phi(xx_2)\} \neq \{\phi(yy_1), \phi(yy_2)\}$ (otherwise we cannot properly color the triangle uxy under ϕ) and $\{\phi(xx_1), \phi(xx_2)\} \cap \{\phi(yy_1), \phi(yy_2)\} \neq \emptyset$ (otherwise we cannot color the edge xy under ϕ). Assume that $\phi(xx_1) = 1$, $\phi(xx_2) = \phi(yy_1) = 2$, and $\phi(yy_2) = 3$ wlog. We now construct a 4-coloring φ of G by taking $\varphi(uv) = \varphi(wy) = 1$, $\varphi(vw) = 2$, $\varphi(uw) = \varphi(vx) = 3$, and $\varphi(wx) = \varphi(vy) = 4$. If $xy \in E(G)$, then let $N(x) = \{v, w, y, x_1\}$ and $N(y) = \{v, w, x, y_1\}$. We shall also assume that $x_1 \neq y_1$, because otherwise $G \simeq G[\{u, v, w, x, y, x_1\}]$, by the 2-connectivity of G, and thus G admits a 4-coloring. Now we remove u, v and w from the diagram of G. Denote by G'' the resulting diagram. One can see that G'' is a PO-graph with x and y being of degree 2 in G''. Since the diagram of G minimizes the number of crossings, xx_1 does not cross yy_1 in G (and thus in G''). Denote by G' the graph obtained from G'' by contracting the edge xy. From the above arguments, one can see that G' is a PO-graph with $E(G) \setminus E(G') = \{uv, uw, vw, vx, wx, vy, wy, xy\}$. Furthermore, by the minimality of G, G' admits a 4-coloring ϕ with $\phi(xx_1) \neq \phi(yy_1)$. Suppose that $\phi(xx_1) = 1$ and $\phi(yy_1) = 2$. We construct a 4-coloring φ of G by taking $\varphi(uw) = \varphi(vy) = 1$, $\varphi(uv) = \varphi(wx) = 2$, $\varphi(vw) = \varphi(xy) = 3$, and $\varphi(vx) = \varphi(wy) = 4$. Second, assume that one of x and y, say x wlog., has degree at most three. If $d(x) \leq 2$, then it is easy to see that $G \simeq G[\{u, v, w, x, y\}]$, by the 2-connectivity of G, and thus G admits a 4-coloring. If $d(x) = 3$, then let $N(x) = \{v, w, x_1\}$. Consider the PO-graph $G' = G - uv$, which admits a 4-coloring ϕ by the minimality of G. If $A := S \setminus \{\phi(uw), \phi(vw), \phi(vy), \phi(vx)\} \neq \emptyset$ (recall that $S = \{1, 2, 3, 4\}$), then let $\varphi(uv) \in A$. Otherwise, assume that $\phi(uw) = 1$, $\phi(vw) = 2$, $\phi(vy) = 3$, and $\phi(vx) = 4$ wlog. It follows that $\phi(wx) = 3$ and $\phi(wy) = 4$. If $\phi(xx_1) = 1$, then we construct a 4-coloring of G by recoloring vx and uw with 2, recoloring vw with 1 and coloring uv with 4. If $\phi(xx_1) = 2$, then we construct a 4-coloring of G by recoloring vx with 1 and coloring uv with 4.

If $G \supseteq G_{13}$, then we shall assume that $d(x) = \Delta(G) = 4$ by Lemma 5.1. Denote the fourth neighbor of x by x_1 and assume that $d(y) = 4$ and $N(y) = \{v, w, y_1, y_2\}$ wlog. By the minimality of G, the PO-graph $G' = G \setminus \{u, v, w\}$ admits a 4-coloring ϕ. Assume that $\phi(xx_1) = 1$ wlog. Construct a 4-coloring φ of G as follows. If $1 \in C_\phi^1(y)$ (suppose $\phi(yy_1) = 1$ and $\phi(yy_2) = 2$ wlog.), then let $\varphi(vw) = 1, \varphi(uv) = \varphi(wx) = 2, \varphi(vx) = \varphi(wy) = 3$, and $\varphi(ux) = \varphi(vy) = 4$. If $1 \notin C_\phi^1(y)$ (suppose $\phi(yy_1) = 2$ and $\phi(yy_2) = 3$ wlog.), then let $\varphi(vy) = 1$, $\varphi(ux) = \varphi(vw) = 2$, $\varphi(uv) = \varphi(wx) = 3$, and $\varphi(vx) = \varphi(wy) = 4$.

If $G \supseteq G_{16}$, then we shall assume that $d(x) = d(y) = \Delta(G) = 4$ by Lemma 5.1. Denote the fourth neighbor of x and y by x_1 and y_1, respectively. By the minimality of G, the PO-graph $G' = G \setminus \{u, v, w, z\}$ admits a 4-coloring ϕ. Construct a 4-coloring φ of G as follows. If $\phi(xx_1) = \phi(yy_1) = 1$, then let $\varphi(vw) = 1, \varphi(ux) = \varphi(vz) = \varphi(wy) = 2$, $\varphi(wx) = \varphi(vy) = 3$, and $\varphi(uw) = \varphi(vx) = \varphi(yz) = 4$. If $1 = \phi(xx_1) \neq \phi(yy_1) = 2$, then let $\varphi(vz) = \varphi(wy) = 1$, $\varphi(ux) = \varphi(wy) = \varphi(vz) = 2$, $\varphi(wx) = \varphi(vy) = 3$, and $\varphi(uw) = \varphi(vx) = 4$.

If $G \supseteq G_{17}$, then we shall assume that $d(x) = d(y) = \Delta(G) = 5$ by Lemma 5.1. By the minimality of G, the PO-graph $G' = G \setminus \{u, v, w, z, a\}$ admits a 5-coloring ϕ. Construct a 5-coloring φ of G as follows. If $C_\phi^1(x) = C_\phi^1(y) = \{1, 2\}$, then let $\varphi(uw) = \varphi(av) = 1, \varphi(wz) = \varphi(uv) = 2, \varphi(xz) = \varphi(vw) = \varphi(ay) = 3$, $\varphi(wx) = \varphi(vy) = 4$, and $\varphi(vx) = \varphi(wy) = 5$. If $|C_\phi^1(x) \cap C_\phi^1(y)| = 1$ (suppose $C_\phi^1(x) = \{1, 2\}$ and $C_\phi^1(y) = \{1, 3\}$ wlog.), then let $\varphi(vw) = 1$, $\varphi(wy) = \varphi(av) = 2, \varphi(wz) = \varphi(vx) = 3$, $\varphi(wx) = \varphi(uv) = \varphi(ay) = 4$, and $\varphi(xz) = \varphi(uw) = \varphi(vy) = 5$. If $|C_\phi^1(x) \cap C_\phi^1(y)| = 0$ (suppose $C_\phi^1(x) = \{1, 2\}$

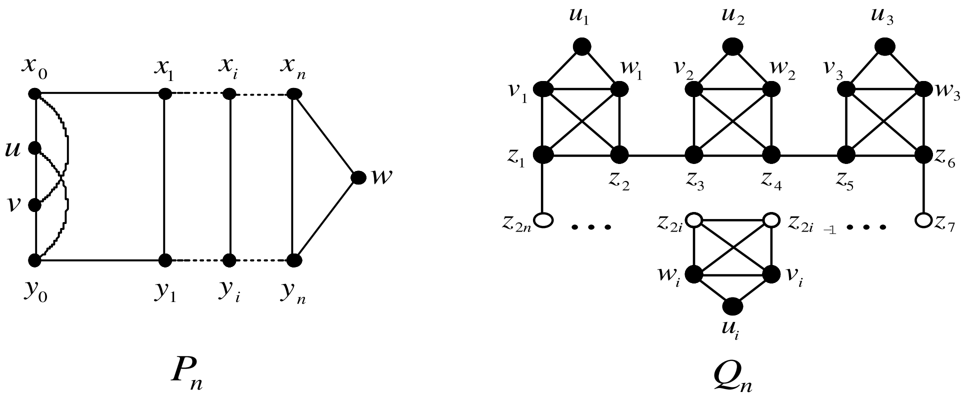


Fig. 4. Special pseudo-outerplanar graphs.

and $C^1_\phi(y) = \{3, 4\}$ wlog.), then let $\varphi(vw) = \varphi(ay) = 1, \varphi(wz) = \varphi(vy) = 2,\ \varphi(vx) = \varphi(uw) = 3,\ \varphi(wx) = \varphi(av) = 4$, and $\varphi(xz) = \varphi(uv) = \varphi(wy) = 5$. $\square$

Theorem 5.4. *For each integer $n \geq 1$, there exists a 2-connected pseudo-outerplanar G with order $2n+5$ and $\Delta(G) = 3$ so that $\chi'(G) = \Delta(G) + 1$.*

Proof. Let $x_0, \ldots, x_n w y_n, \ldots, y_0 v u x_0$ be a cycle denoted by C. Add edges $x_i y_i$ for all $1 \leq i \leq n$ and add another two edges $x_0 v$ and $y_0 u$ to C. Denote the resulting graph by P_n (see Fig. 4). One can easily check that P_n is a 2-connected pseudo-outerplanar graph with $|P_n| = 2n+5$ and $\Delta(P_n) = 3$. If P_n has a 3-coloring ϕ, then we shall have $\phi(x_0 v) = \phi(y_0 u)$ and $\phi(x_0 u) = \phi(y_0 v)$ (otherwise we cannot color uv properly). Thereby we would deduce that $\phi(x_i x_{i+1}) = \phi(y_i y_{i+1})$ for all $0 \leq i \leq n-1$ and $\phi(x_n w) = \phi(y_n w)$. This contradiction implies that $\chi'(P_n) = \Delta(P_n) + 1 = 4$. $\square$

Theorem 5.5. *Let G be a pseudo-outerplanar graph. If $\Delta(G) = 3$ or $\Delta(G) \geq 5$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. Since Conjecture 1.1 has already been proved for planar graphs and every PO-graph is planar (cf. Section 1), this claim holds trivially when $\Delta(G)$ is odd. Hence in the following we may assume that $\Delta(G)$ is an even not less than 6. For brevity, we write $k = \frac{\Delta(G)}{2}$. Suppose, for a contradiction, that there exists a minimal (in terms of the size) pseudo-outerplanar graph G that has no k-colorings. One can easily observe that G is 2-connected and la-critical. By Theorem 4.2 and Lemma 5.2, G contains the configuration G_3.

If $xy \notin E(G)$, then by (b) of Theorem 4.2, $G' = G \setminus \{v\} + xy$ is still a PO-graph. By the minimality of G, G' admits a k-coloring ϕ. We now construct a k-coloring φ of G by taking $\varphi(vx) = \varphi(vy) = \phi(xy)$ and $\varphi(e) = \phi(e)$ for every $e \in E(G) \cap E(G')$.

If $xy \in E(G)$, then consider the PO-graph $G' = G \setminus \{v\}$, which has a k-coloring ϕ by the minimality of G. It is easy to see that $|C^1_\phi(x)| = |C^1_\phi(y)| = 1$, since $d(x) = d(y) = \Delta(G) = 2k$ by Lemma 5.2. We now construct a coloring φ of G by taking $\varphi(vx) \in C^1_\phi(x)$, $\varphi(vy) \in C^1_\phi(y)$, and $\varphi(e) = \phi(e)$ for every $e \in E(G) \cap E(G')$. If $C^1_\phi(x) \neq C^1_\phi(y)$, then it is easy to see that φ is a k-coloring. If $C^1_\phi(x) = C^1_\phi(y)$, then $\varphi(vx) = \varphi(vy)$ and φ is also a k-coloring unless $\varphi(xy) = \varphi(vx)$ or $\varphi(ux) = \varphi(uy) = \varphi(vx)$. If $\varphi(xy) = \varphi(vx)$, then $\varphi(vx) \notin \{\varphi(ux), \varphi(uy)\}$, and thus we can exchange the colors on ux and vx. One can easy to check that the resulting coloring of G is a k-coloring. If $\varphi(ux) = \varphi(uy) = \varphi(vx)$, then we recolor xy with $\varphi(vx)$ and recolor both vx and uy with $\varphi(xy)$. The resulting coloring of G is also a k-coloring. $\square$

Theorem 5.6. *For each integer $m \geq 1$, there exists a 2-connected pseudo-outerplanar G with order $10m+5$ and $\Delta(G) = 4$ so that $la(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$.*

Proof. Let $z_1, \ldots, z_{2n} z_1$ be a cycle denoted by T_0 and let $u_i v_i w_i u_i$ be a triangle denoted by T_i for each $1 \leq i \leq n$. Assume that any two of $T_0, \ldots, T_n$ are vertex-disjoint. For each $1 \leq i \leq n$, add fours edges $v_i z_{2i-1}$, $v_i z_{2i}$, $w_i z_{2i-1}$ and $w_i z_{2i}$. Denote the resulting graphs by Q_n (see Fig. 4). One can easily check that Q_n is a 2-connected pseudo-outerplanar graph with $\Delta(Q_n) = 4$. Consider the graph Q_{2m+1} for $m \geq 1$. It is trivial that $|Q_{2m+1}| = 10m+5$ and $\mathrm{la}(Q_{2m+1}) \leq 3$ by Lemma 5.2. If Q_{2m+1} has a 2-coloring ϕ, then we shall have $\phi(z_{2i-2} z_{2i-1}) \neq \phi(z_{2i} z_{2i+1})$ for all $1 \leq i \leq 2m+1$, where $z_0 = z_{4m+2}$ and $z_{4m+3} = z_1$ (otherwise we cannot properly color the set of edges $\{u_i v_i, v_i w_i, w_i u_i, v_i z_{2i-1}, v_i z_{2i}, w_i z_{2i-1}, w_i z_{2i}\}$ for some i). However, the size of the set $\{z_2 z_3, z_4 z_5, \ldots, z_{4m+2} z_1\}$ is $2m+1$, which is odd, but there are only two colors that can be used in ϕ. This contradiction implies that $\mathrm{la}(Q_{2m+1}) = \lceil \frac{\Delta(Q_{2m+1})}{2} \rceil + 1 = 3$. $\square$

Acknowledgments

The authors thank the referees for many helpful comments and suggestions, which have greatly improved the presentation of the results in this paper, and acknowledge the editors for pointers to relevant literature and phraseological comments. The work of the first author is also under the financial support from The Chinese Ministry of Education Prize for Academic Doctoral Fellows.

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