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On r-acyclic edge colorings of planar graphs

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ABSTRACT

A proper edge coloring of G is r-acyclic if every cycle C contained in G is colored with at least $\min\{|C|, r\}$ colors. The r-acyclic chromatic index of a graph, denoted by $a'_r(G)$, is the minimum number of colors required to produce an r-acyclic edge coloring. In this paper, we study 4-acyclic edge colorings by proving that $a'_4(G) \leq 37\Delta(G)$ for every planar graph, $a'_4(G) \leq \max\{2\Delta(G), 3\Delta(G) - 4\}$ for every series–parallel graph and $a'_4(G) \leq 2\Delta(G)$ for every outerplanar graph. In addition, we prove that every planar graph with maximum degree at least r and girth at least $5r + 1$ has $a'_r(G) = \Delta(G)$ for every $r \geq 4$.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G. The order $|G|$ of G is the value of $|V(G)|$ and the size of G is the value of $|E(G)|$. Let $N_G(v)$ be the set of neighbors of v and let $d_G(v)$ be the degree of v in G. Sometimes we write $N(v)$ and $d(v)$ instead of $N_G(v)$ and $d_G(v)$ for short. A k-vertex is a vertex of degree k. A k-cycle or a k^+-cycle is a cycle with length k or at least k. For a subset $U \subseteq V(G)$, $G[U]$ denotes the subgraph induced by U. For other undefined concepts we refer the readers to [4].

In this paper we aim to study generalized acyclic edge colorings of graphs. We begin by reviewing some separate concepts. A mapping c from $E(G)$ to the set of colors $\{1, \ldots, k\}$ is a *proper edge k-coloring* of G if any two adjacent edges receive different colors. A proper edge coloring of G is *r-acyclic* if every cycle C contained in G is colored with at least $\min\{|C|, r\}$ colors. The *r-acyclic chromatic index* of a graph G, denoted by $a'_r(G)$, is the minimum number of colors required to produce an r-acyclic edge coloring. Clearly we have $a'_r(G) \geq \Delta(G)$ for every graph G.

A 2-acyclic edge coloring coincides with a proper edge coloring, thus by the well-known Vizing's Theorem, $\Delta(G) \leq a'_2(G) \leq \Delta(G) + 1$ for every graph G. For a planar graph G, Sanders and Zhao [19], and Zhang [21], independently proved that $a'_2(G) = \Delta(G)$ if $\Delta(G) \geq 7$. This bound of 7 for $\Delta(G)$ can be improved to 3 (being sharp) if we restrict G to be a series–parallel graph [15].

A 3-acyclic edge coloring, which is also known as an acyclic edge coloring in the literature, was introduced by Alon et al. [1], who proved for every graph G that $a'_3(G) \leq 64\Delta(G)$ using the Local lemma. This upper bound was later improved to $16\Delta(G)$ by Molloy and Reed [16] and to $9.62\Delta(G)$ by Ndreca et al. [18]. In 2001, Alon, Sudakov and Zaks conjectured a stronger bound of $\Delta(G) + 2$ for $a'_3(G)$. For a planar graph G, Fiedorowicz, Haluszczak and Narayanan [9] proved an upper

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bound of $2\Delta(G)+29$ for $a'_3(G)$ which was later improved to $\max\{2\Delta(G)-2, \Delta(G)+22\}$ by Hou et al. [13], to $\Delta(G)+22$ by Cohen, Havet and Müller [7], and to $\Delta(G)+12$ by Basavaraju et al. [3]. Hou et al. also proved that $a'_3(G) \le \Delta(G)+1$ for every series–parallel graph G [13] and $a'_3(G) = \Delta(G)$ for every outerplanar graph with maximum degree at least 5 [14].

In view of the results mentioned in the previous two paragraphs, we can see that $a'_r(G) = O(\Delta(G))$ for every $r \le 3$. However, the r-acyclic chromatic index of G may not be linear in $\Delta(G)$ for some $r \ge 4$. For example, the complete bipartite graph $H := K_{m,n}$ satisfies $a'_r(H) = mn$ for every $r \ge 4$, since every edge of G is contained in a 4-cycle and the edges in each 4-cycle should be colored all distinctly. In general, there exist Δ-regular graphs G with $a'_r(G) \ge c_r \Delta^{\lfloor \Delta/2 \rfloor}$, where c_r depends on r but is constant with respect to Δ, for $r \ge 6$ by Greenhill and Pikhurko [12], and for $r \ge 4$ by Aravind and Subramanian [2]. It is natural to guess that the graph with its r-acyclic chromatic index superlinear in $\Delta(G)$ may have very special structures. The question of finding the classes of graphs with $a'_r(G) = \Delta(G)$ for some (or all) $r \ge 4$ is also interesting. Until now, the only two known classes of graphs with linear bounds for the r-acyclic chromatic index are the random regular graphs [10] and the graphs with large enough girth [11].

In this paper, we mainly work with 4-acyclic edge colorings by proving that $a'_4(G) \le 37\Delta(G)$ for every planar graph, $a'_4(G) \le \max\{2\Delta(G), 3\Delta(G)-4\}$ for every series–parallel graph and $a'_4(G) \le 2\Delta(G)$ for every outerplanar graph. Moreover, we show that the r-acyclic chromatic index of every planar graph with maximum degree at least r and girth at least $5r+1$ is exactly $\Delta(G)$ for every $r \ge 4$.

2. Planar graphs

A graph is *planar* if it can be drawn in the plane so that there are no crossed edges. It is well-known that each planar graph contains a vertex of degree of at most 5. In [5], Borodin et al. established a more general structural theorem, which is quite useful for dealing with coloring problems for planar graphs. For example, Cohen et al. [7] have used this "configuration existence" theorem in combination with minimum counterexample arguments to show that $a'_3(G) \le \Delta(G)+22$ for every planar graph G. Before stating this theorem, we recall two important definitions in [7,5].

A vertex v in a planar graph G is *good* if it is of degree at most 5 and there is a vertex $w \in N(v)$ such that $d(u) \le 25$ for all $u \in N(v) \setminus \{w\}$ and $\sum_{u \in N(v)\setminus\{w\}} d(u) \le 38$. We say G has a *bunch* of length $m \ge 3$ with *poles* p and q if G contains a sequence of paths $P_1, \dots, P_m$, joining p with q, which satisfy the following properties. First, each P_i has length 1 or 2. Furthermore, for each $1 \le i \le m-1$ the cycle formed by P_i and P_{i+1} is not separating in G, i.e., there is no vertex of G inside this cycle. Moreover, this sequence of paths is maximal in the sense that there is no path P_0 or P_{m+1} that can be added to the bunch preserving the above properties. A path $P_i = pq$ of length 1 in the bunch will be referred to as a *parental edge*. If a path $P_i = pv_iq$ in the bunch has length 2, then the middle vertex is called *bunch vertex*. If the cycle bounded by P_1 and P_m separates G, then the vertices v_1 and v_m (if they exist) are called *end vertices* of the bunch. The vertex v_i in the bunch is *interior* if $2 \le i \le m-1$ and *strictly interior* if $3 \le i \le m-2$. Note that each strictly interior vertex has degree 2, 3 or 4 and is adjacent to the poles and possibly to one or two interior vertices, whose degrees are at most 4. For an example of a bunch, see Fig. 1.

The "configuration existence" theorem of Borodin et al. is as follows.

Theorem 1 ([5]). *Every planar graph contains either a good vertex or a vertex v of degree $d(v) \ge 26$ that is a pole for a bunch of length at least $d(v)/5$.*

We say two bunches are *internally disjoint* if they do not share any bunch vertices. The following corollary of Theorem 1 is due to Cohen et al. [7].

Theorem 2 ([7]). *Every planar graph contains either a good vertex or a vertex v that is a pole for $1 \le k \le 6$ internally disjoint bunches of length at least 6 and has at most $24-4k$ neighbors that do not belong to these bunches.*

In what follows, we use Theorem 2 to prove one of the main results in this paper.

Theorem 3. *The 4-acyclic chromatic index of every planar graph G is at most $37\Delta(G)$.*

Proof. Suppose that G is a counterexample to the theorem with the smallest number of vertices. First of all, one can verify that $\delta(G) \ge 2$, because otherwise there is a 1-vertex v in G and a 4-acyclic edge coloring of $G-v$ can be easily extended to G by coloring the edge incident with v properly. By Theorem 2, we split our proofs into two cases.

First, we assume that G contains a good vertex v with its neighbors $v_1, \dots, v_d$ satisfying $\max_{2\le i\le d} d(v_i) \le 25$ and $\sum_{i=2}^{d} d(v_i) \le 38$, where $2 \le d := d(v) \le 5$. By the minimality of G, $G' = G - v$ admits a 4-acyclic edge coloring ϕ using 37Δ colors (here and in the following, we use Δ instead of $\Delta(G)$ for brevity). Let C be the color set of ϕ. For a vertex u in G' define $\Phi(u) = \{\phi(uw) \mid w \in N_{G'}(u)\}$ and for a vertex set $S \in V(G')$ define $\Phi(S) = \bigcup_{v\in S} \Phi(v)$. We first color vv_1 by a color $\phi(vv_1) \in C \setminus F(vv_1)$, where

$$F(vv_1) = \bigcup_{i=2}^{d} \Phi(N_{G'}(v_i)) \cup \Phi(v_1).$$

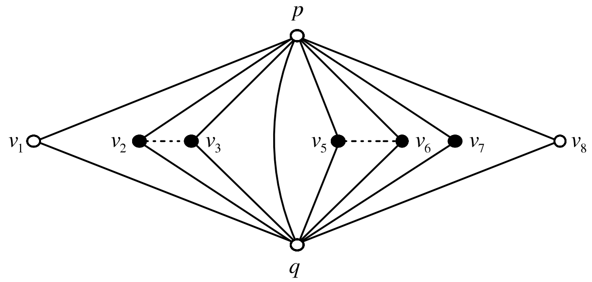

Fig. 1. A bunch of length 8 with one parental edge pq, seven bunch vertices $v_1, v_2, v_3, v_5, v_6, v_7, v_8$, two end vertices v_1, v_8, five interior vertices v_2, v_3, v_5, v_6, v_7, three strictly interior vertices v_3, v_5, v_6.

Since $\sum_{i=2}^{d} d_G(v_i) \le 38$ and $\max_{2\le i\le d} d(v_i) \le 25$, $|F(vv_1)| \le (38-d+1)\Delta + \Delta - 1 \le 37\Delta - 1$ if $d \ge 3$ and $|F(vv_1)| \le (25-1)\Delta + \Delta - 1 \le 25\Delta - 1 < 37\Delta - 1$ if $d = 2$. Thus $|C \setminus F(vv_1)| \ge 1$ and hence the coloring of vv_1 chosen above does exist. We now color $vv_2, \dots, vv_d$ greedily by assigning to vv_i $(2 \le i \le d)$ a color $\phi(vv_i) \in C \setminus F(vv_i)$, where

$$F(vv_i) = \bigcup_{i=2}^{d} \Phi(N_{G'}(v_i)) \cup \bigcup_{j=1}^{i-1} \{\phi(vv_j)\}.$$

One can also verify that $|F(vv_i)| \le (38-d+1)\Delta + i - 1 \le 36\Delta + i - 1$ here. Thus $|C \setminus F(vv_i)| \ge \Delta - (i-1) \ge 1$ since $\Delta \ge d_G(v) \ge i$, and the edges $vv_2, \dots, vv_d$ can be colored properly. Since $\bigcup_{i=1}^{d}\{\phi(vv_i)\} \cap \bigcup_{i=2}^{d} \Phi(N_{G'}(v_i)) = \emptyset$, every cycle C through v in G must be colored with at least $\min\{|C|, 4\}$ colors under the extended coloring ϕ of G. Hence ϕ is exactly a 4-acyclic edge coloring of G.

Second, we assume that G contains a vertex v that is a pole for $1 \le k \le 6$ internally disjoint bunches $B_1, \dots, B_k$ of length at least 6 and has at most $24 - 4k$ neighbors that do not belong to these bunches. For each bunch B_i incident with v, denote the other pole of it by u_i and the end vertices of it by x_i and y_i (here note that B_i may have only one end vertex). Let S be the set of neighbors of v that do not belong to these k bunches. Now consider the bunch B_1. Since its length is at least 6, there is at least one strictly interior vertex, say w, contained in B_1. Without loss of generality, assume that $d(w) = 4$. This implies that w has two interior vertices as its neighbors, say w_l and w_r. By the minimality of G, $G' = G - w$ has a 4-acyclic edge coloring ϕ using 37Δ colors. Now we arbitrarily choose four different colors from $A = C \setminus F$ to color the edges wv, wu_1, ww_l and ww_r, where

$$F = \begin{cases} \Phi(u_1) \cup \Phi(N_{G'}(w_l)) \cup \Phi(N_{G'}(w_r)) \cup \Phi(S) \cup \bigcup_{i=2}^{k} \Phi(\{x_i, y_i, u_i\}), & \text{if } k \ge 2; \\ \Phi(u_1) \cup \Phi(N_{G'}(w_l)) \cup \Phi(N_{G'}(w_r)) \cup \Phi(S), & \text{if } k = 1. \end{cases}$$

Since $d_{G'}(w_l) \le 3$, $d_{G'}(w_r) \le 3$, $|S| \le 24 - 4k$ and $1 \le k \le 6$, we have $|F| \le \Delta + 3\Delta + 3\Delta + (24-4k)\Delta + 3(k-1)\Delta = (28-k)\Delta \le 27\Delta$. Thus, $|A| \ge 37\Delta - 27\Delta > 4$. This implies that the colorings of wv, wu_1, ww_l and ww_r as assigned above do exist. Now we check that the extended coloring of G is still 4-acyclic. Since $A \cap (\Phi(N_{G'}(w_l)) \cup \Phi(N_{G'}(w_r)) \cup \Phi(S)) = \emptyset$, every 4^+-cycle through the edge ww_l or ww_r or through the vertex in S, and every 4^+-cycle contained in B_1 is colored with at least 4 colors under the final coloring. Moreover, if there is a 4^+-cycle C that goes through two bunches B_1 and B_i, then C must walk through either the end vertices of B_i or the other pole of B_i. But by the choice of A, we have assumed that $A \cap \bigcup_{i=2}^{k} \Phi(\{x_i, y_i, u_i\}) = \emptyset$ when $k \ge 2$. This implies that such a cycle C shall also be colored with at least 4 colors. Thus, every cycle of length at least 4 through w in G is colored with at least 4 colors under the final coloring. At last, it is easy to see that the edges of every 3-cycle in G are colored all distinctly. So we have obtained a 4-acyclic edge coloring of G. This contradiction completes the proof of the theorem. □

Now we focus on planar graphs with large girth. Let us begin with a useful structural lemma that was proved by Nesetril et al. [17].

Lemma 4. *Every planar graph G with minimum degree at least 2 and girth at least $5r+1$ contains an induced path $P = v_1 \cdots v_r$ such that each vertex v_i has degree 2 in G.*

The remaining content of this section is dedicated to the following Theorem 5. Note that in this theorem the bound r for $\Delta(G)$ cannot be weakened because every r^+-cycle in G should be colored with at least r colors under any r-acyclic edge coloring of G.

Theorem 5. *Every planar graph G with $g(G) \ge 5r + 1$ and $\Delta(G) \ge r$ has $a'_r(G) = \Delta(G)$ for every $r \ge 4$.*

In order to prove the above theorem, we show a slightly stronger result as follows. Indeed, Theorem 6 is only a technical strengthening of Theorem 5, without which we would get complications when considering a subgraph $G' \subset G$ such that $\Delta(G') < \Delta(G)$. Of course, the interesting case of Theorem 6 is when $M = \Delta(G)$.

Theorem 6. *Every planar graph G with $g(G) \geq 5r + 1$ and $\Delta(G) \leq M$ has $a'_r(G) \leq M$, where $M \geq r \geq 4$.*

Proof. Suppose that G is a counterexample to the theorem with the smallest number of vertices. By a similar argument as in Theorem 3, we have $\delta(G) \geq 2$. So by Lemma 4, G contains an induced path $P = v_1 \cdots v_r$ such that each vertex v_i has degree 2 in G. Denote the other neighbor of v_1 and v_r by v_0 and v_{r+1}, respectively. By the minimality of G, $G' = G - \{v_2, \ldots, v_{r-2}\}$ has an r-acyclic edge coloring ϕ using M colors. Let C be the color set of ϕ and set $A = C \setminus \{\phi(v_0v_1), \phi(v_{r-1}v_r), \phi(v_rv_{r+1})\}$. If $\{\phi(v_0v_1)\} \cap \{\phi(v_{r-1}v_r), \phi(v_rv_{r+1})\} = \emptyset$, then $|A| = M - 3 \geq r - 3$. Now color the edge $v_{r-2}v_{r-1}$ with the color $\phi(v_0v_1)$ and color the remaining $r - 3$ uncolored edges with different colors in A. If $\{\phi(v_0v_1)\} \cap \{\phi(v_{r-1}v_r), \phi(v_rv_{r+1})\} \neq \emptyset$, then $|A| = M - 2 \geq r - 2$ and therefore we can color the $r - 2$ edges deleted from G with different colors in A. In each case one can easily verify that every cycle through the path P is colored with at least r colors. Thus an r-acyclic edge coloring of G using M colors has been constructed. This contradiction completes the proof of the theorem. $\square$

3. Series–parallel graphs

A graph is *series–parallel* if it contains no subgraphs homeomorphic to K_4. Duffin [8] showed that a connected series–parallel graph can be obtained from a K_2 by repeatedly applying one of the following operation: insert a vertex into an edge, or duplicate an edge by a path of length 2. With this definition one can see that the connectivity of any series–parallel graph is at most two.

Given a 2-connected series–parallel graph G, let $\{u, v\}$ be a 2-vertex cut of G and let H be a component of $G - \{u, v\}$. Set $G[H, u] = G[V(H) \cup \{u\}]$, $G[H, v] = G[V(H) \cup \{v\}]$ and $G[H, u, v] = G[V(H) \cup \{u, v\}]$. We call H a *cycle-component* of $G - \{u, v\}$ if either $G[H, u]$ or $G[H, v]$ contains a cycle; otherwise we call H a *path-component*. It is easy to see that if H is a path-component then $G[H, u, v]$ is a $u - v$ *chord* of G, that is, an induced $u - v$ path of G with all of its internal vertices being of degree 2. A cycle component H of $G - \{u, v\}$ is called an $\infty -$ *component* if $G[H, u, v]$ has a unique cut vertex w such that $G[H, u, v]$ consists of p $u - w$ chords and q $w - v$ chords, where $p \geq 2$ and $q \geq 1$. For our purposes the following structural lemma for 2-connected series–parallel graphs, which was proved by Wu and Wu [20], is a useful starting point.

Lemma 7 (*[20]*)**.** *Every 2-connected series–parallel graph G with $\Delta(G) \geq 3$ contains a 2-vertex cut $\{u, v\}$ such that*

(1) *every component of $G - \{u, v\}$ is path component, or*
(2) *$G - \{u, v\}$ has at least one ∞-component and at most one component which is neither path-component nor ∞-component.*

We now modify the above lemma to obtain the following one.

Lemma 8. *If G is a 2-connected series–parallel graph with $\Delta(G) \geq 3$ such that the neighbors of every 2-vertex contained in G are adjacent, then either G is isomorphic to the graph obtained from a complete bipartite graph $K_{2,\Delta(G)-1}$ by adding an edge between the two vertices with maximum degree (see (a) of Fig. 2) or G contains the configuration (b) of Fig. 2, where $\{u, v\}$ is a 2-vertex cut of G, $d(w) = p + q + 2$, $p \geq 2$ and $q \geq 1$.*

Proof. By Lemma 7, there is a 2-vertex cut $\{u, v\}$ of G such that either every component of $G - \{u, v\}$ is path component or $G - \{u, v\}$ has at least one ∞-component. Since the neighbors of every 2-vertex contained in G are adjacent, G contains no adjacent 2-vertices because otherwise there is a cut vertex in G. Thus, every $u - v$ chord contained in G contains at most one 2-vertex. If the former case stated above occurs, then G is isomorphic to the graph (a) of Fig. 2. If the latter case appears, then one can easily find an ∞-component as described in (b) of Fig. 2, where $p \geq 2$ and $q \geq 1$. Moreover, we must have $N(w) = \{x_1, \ldots, x_p, y_1, \ldots, y_q, u, v\}$ by the definition of ∞-component. Hence, $d(w) = p + q + 2$. $\square$

Now we apply Lemma 8 to prove the following theorem.

Theorem 9. *The 4-acyclic chromatic index of every series–parallel graph G is at most* $\max\{2\Delta(G), 3\Delta(G) - 4\}$.

Proof. Suppose that G is a counterexample to the theorem with the smallest value of $|V(G)| + |E(G)|$. One can easily see that G is 2-connected. In the following, we assume that $\Delta(G) \geq 3$, since the case when $\Delta(G) = 2$ is very easy to verify. Let u be a 2-vertex of G and let $N(u) = \{v, w\}$. If $vw \notin E(G)$, then consider the graph $G' = G - u + vw$, which has a 4-acyclic edge coloring ϕ by the minimality of G. Color uv with $\phi(vw)$ and color uw with a color that is different from the used colors incident with v and w (note that there are at most $2\Delta(G) - 1$ used colors). One can check that such an extension of ϕ does exist and the resulting coloring of G is a still 4-acyclic, a contradiction. So we conclude that the neighbors of every 2-vertex contained in G are adjacent. This implies that either G is isomorphic to the graph (a) in Fig. 2 or G contains the configuration (b) of Fig. 2 by Lemma 8. Note that the 4-acyclic chromatic index of the graph (a) is exactly $2\Delta(G) - 1$ and the maximum degree of the graph (b) is at least $p + q + 2 \geq 5$. We conclude that $a'_4(G) \leq 2\Delta(G)$ if $2 \leq \Delta(G) \leq 4$ and hence we will only consider the latter case that G contains (b). By the minimality of G, $G' = G - x_1$ admits a 4-acyclic edge coloring ϕ using $3\Delta(G) - 4$ colors. In the following, we use Δ instead of $\Delta(G)$ for convenience and let C be the color set of ϕ. For any vertex $u \in V(G')$ define $\Phi(u) = \{\phi(uv) \mid v \in N_{G'}(u)\}$ and for a vertex set $S \in V(G')$ define $\Phi(S) = \bigcup_{v \in S} \Phi(v)$. We extend ϕ to G by coloring ux_1 with $\phi(ux_1) \in C \setminus F_1$ and coloring wx_1 with $\phi(wx_1) \in C \setminus F_2$, where $F_1 = \Phi(u) \cup \Phi(v) \cup \bigcup_{i=2}^{p}\{\phi(wx_i)\}$ and $F_2 = \Phi(v) \cup \Phi(w) \cup \bigcup_{i=1}^{p}\{\phi(ux_i)\}$. Since $\Delta \geq d(w) = p + q + 2$, we have

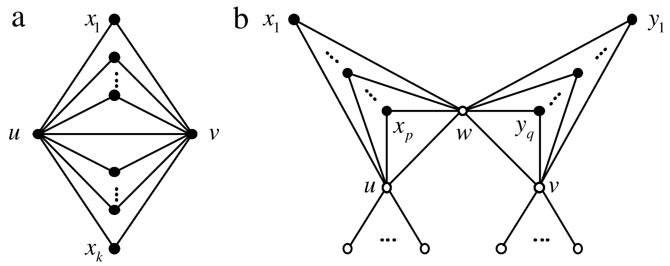

Fig. 2. Two special configurations of a series–parallel graph.

$|C \setminus F_1| \geq 3\Delta - 4 - (\Delta - 1) - \Delta - (p-1) = \Delta - p - 2 \geq q \geq 1$ and $|C \setminus F_2| \geq 3\Delta - 4 - \Delta - (p+q) - p = 2\Delta - 4 - 2p - q \geq q \geq 1$, which implies that such an extension of ϕ does exist. Since every 4^+-cycle through x_1 in G would go through either v or one of the vertices in $\{x_2, \ldots, x_p\}$, and the above extension satisfies $\{\phi(ux_1), \phi(wx_1)\} \cap \Phi(\{v, x_2, \ldots, x_p\}) = \emptyset$, the resulting coloring of G is still 4-acyclic. Hence the proof is complete. □

Note that the complete bipartite graph $K_{2,\Delta}$ is a series–parallel graph satisfying $a'_4(K_{2,\Delta}) = 2\Delta$. So the bound for $a'_4(G)$ in Theorem 9 is sharp when $2 \leq \Delta(G) \leq 4$. We make the following conjecture to end this section.

Conjecture 10. *The 4-acyclic chromatic index of every series–parallel graph G is at most $2\Delta(G)$.*

4. Outerplanar graphs

A graph is *outerplanar* if it can be drawn in the plane so that all vertices lie on the outside face. In this section, we confirm Conjecture 10 for outerplanar graphs, a subclass of series–parallel graphs. For this, we need the following structural result.

Lemma 11 ([6]). *Every outerplanar graph with minimum degree at least two contains one of the following configurations:*

(a) *two adjacent 2-vertices u and v;*
(b) *a 3-cycle uvw with $d(u) = 2$ and $d(v) = 3$;*
(c) *two intersecting 3-cycles uvw and xyw with $d(u) = d(x) = 2$ and $d(w) = 4$.*

Theorem 12. *The 4-acyclic chromatic index of every outerplanar graph G is at most $2\Delta(G)$.*

Proof. Suppose that G is a counterexample to the theorem with the smallest number of vertices. It is easy to see that $\delta(G) \geq 2$. Thus by Lemma 11, G contains one of the three configurations among (a)–(c). In the following arguments, we use Δ instead of $\Delta(G)$ for brevity.

Suppose that G contains (a). Denote the other neighbor of u and v by w and z, respectively. By the minimality of G, the graph $G' = G - u$ has a 4-acyclic edge coloring ϕ using 2Δ colors. For a vertex x in G' define $\Phi(x) = \{\phi(xy) \mid y \in N_{G'}(x)\}$ and for a vertex set $S \in V(G')$ define $\Phi(S) = \bigcup_{x \in S} \Phi(x)$. Let C be the color set of ϕ and let $A_1 = C \setminus (\Phi(w) \cup \Phi(z))$. Since $|A_1| \geq 2\Delta - (\Delta - 1 + \Delta) \geq 1$, we can color uw with a color $\phi(uw) \in A_1$. Let $B_1 = C \setminus (\{\phi(uw)\} \cup \Phi(z))$. Since $|B_1| \geq 2\Delta - (1 + \Delta) = \Delta - 1 \geq 1$, we can color uv with a color $\phi(uv) \in B_1$. Since $\{\phi(uw), \phi(uv)\} \cap \Phi(z) = \emptyset$, every 4^+-cycle through u would be colored with at least four colors. Thus a 4-acyclic edge coloring of G has been constructed.

Suppose that G contains (b). Denote the other neighbor of v by z. By the minimality of G, $G' = G - u$ has a 4-acyclic edge coloring ϕ using 2Δ colors. By C we denote the color set of ϕ. As in the previous case, we first color uw with a color $\phi(uw) \in A_2 = C \setminus (\Phi(w) \cup \Phi(z)) \neq \emptyset$. Let $B_2 = C \setminus (\{\phi(uw), \phi(wv)\} \cup \Phi(z))$. Since $|B_2| \geq 2\Delta - 2 - \Delta = \Delta - 2 \geq 1$ (note that $\Delta \geq 3$), we can color uv with a color $\phi(uv) \in B_2$. Similarly, we can verify that the extended coloring of G is 4-acyclic.

Suppose that G contains (c). We consider the graph $G' = G - u$, which admits a 4-acyclic edge coloring ϕ using 2Δ colors by the minimality of G. Let C be the color set of ϕ and let $A_3 = C \setminus (\Phi(v) \cup \Phi(y))$. As in the previous cases, we can prove that $|A_3| \geq 1$ and thus we can color uv with a color $\phi(uv) \in A_3$. Let $B_3 = C \setminus (\{\phi(uv), \phi(vw), \phi(wx)\} \cup \Phi(y))$. Since $|B_3| \geq 2\Delta - 3 - \Delta = \Delta - 3 \geq 1$ (note that $\Delta \geq 4$), we can color uw with a color $\phi(uw) \in B_3$. Since every 4^+-cycle through u would go through the vertex y and $\{\phi(uw), \phi(uv)\} \cap \Phi(y) = \emptyset$, the resulting coloring of G is still 4-acyclic. This completes the proof of the theorem. □

5. Concluding remarks

In this paper we have considered r-acyclic edge colorings of planar graphs, series–parallel graphs and outerplanar graphs, however, here the special case $r = 4$ is the main topic. For these graph classes, we established various upper bounds on the 4-acyclic chromatic index, which are all linear in the maximum degree of the graph. Recalling the proofs of Theorems 3, 9 and 12, we respectively use three "configuration existence" theorems from the literature in combination with minimum counterexample arguments. However, these arguments fail to give linear bounds for higher (fixed) values of r. Let us discuss why this is so.

Given a graph G and a fixed integer r, our task is to establish a linear upper bound (in terms of $\Delta(G)$) for $a'_r(G)$. While trying to prove the validity of such a linear bound, we prefer to use minimum counterexample arguments. Actually, we first find (or prove) a "configuration existence" theorem and then we prove that the configurations in that theorem are reducible. Recalling Theorem 2, Lemmas 8 and 11, one can see that the diameter of every configuration involved in those results is bounded by a constant that is independent of r. This makes it difficult to extend our results to a higher (fixed) r.

Indeed, if we attempt to prove the reducibility of a given configuration that has constant diameter, we would need to (completely or partially) remove this configuration from G and denote the resulting graph by G'. By the minimality of G, G' has an r-acyclic edge coloring ϕ. The remaining work would then be to extend the coloring ϕ to G. Let $v_1 \cdots v_m$ be the longest path in the deleted configuration, where m is a constant since the diameter of this configuration is a constant. Now imagine one unlucky case: there is a bicolored path of length $r - m + 1$ in G' from v_1 to v_m under the coloring ϕ. In this case, we would have an r-cycle in G that is colored by at most $m + 1$ colors no matter how we color the path $v_1 \cdots v_m$. But every r-cycle should be colored with at least r colors if the coloring is r-acyclic. Therefore, it would be impossible to extend ϕ to an r-acyclic edge coloring of G. Recalling the proof of Theorem 6, where we have proved the reducibility of an induced path $P = v_1 \cdots v_r$ such that each vertex v_i has degree 2 in G. This is possible because the configuration induced by P has a diameter that is bounded in term of r.

To end this paper, we leave an interesting problem for further research.

Problem 13. For a fixed integer $r \geq 5$, are there linear upper bounds for the r-acyclic chromatic index in terms of the maximum degree for planar graph, or for series–parallel graph, or for outerplanar graph?

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