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Light Subgraphs in the Family of 1-Planar Graphs with High Minimum Degree

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Abstract A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. In this paper, it is shown that each 1-planar graph with minimum degree 7 contains a copy of $K_2 \vee (K_1 \cup K_2)$ with all vertices of degree at most 12. In addition, we also prove the existence of a graph $K_1 \vee (K_1 \cup K_2)$ with relatively small degree vertices in 1-planar graphs with minimum degree at least 6.

Keywords 1-Planar graph, lightness, height, discharging

MR(2000) Subject Classification 05C10, 05C75

1 Introduction

Throughout this paper, all graphs are finite, simple and undirected. We use the standard graph terminology by [1]. In particular, by k^+-vertex (face), we mean a vertex (face) in a graph G of degree at least k. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *union* of them, defined by $G_1 \cup G_2$, is a graph $G_u = (V_u, E_u)$ with its vertex set $V_u = V_1 \cup V_2$ and its edge set $E_u = E_1 \cup E_2$; and the *join* of them, defined by $G_1 \vee G_2$, is a graph $G_j = (V_j, E_j)$ with its vertex set $V_j = V_1 \cup V_2$ and its edge set $E_j = E_1 \cup E_2 \cup \{xy|x \in V_1, y \in V_2\}$. For an integer $n \geq 1$, denote nG to be the union of n copies of a graph G. We use P_n and C_n to stand for the path and cycle of order n. By $S_n = nK_1 \vee K_1$ and $W_n = K_1 \vee C_n$, we denote an n-star and an n-wheel respectively. We say a 3-star (or $K_1 \vee (K_1 \cup K_2)$) is of type $(\leq d_1; \leq d_2, \leq d_2, \leq d_2)$ (or of type $(\leq d_1; \leq d_2; \leq d_2, \leq d_2)$) if its center vertex is of degree at most d_1 with the others being of degree at most d_2. Note that the graph $K_1 \vee (K_1 \cup K_2)$ can be obtained by adding an extra edge between two vertices of the leaves of a 3-star.

While studying the vertex-face coloring (a simultaneous coloring on both the vertex set and the face set such that each pair of adjacent/incident elements receive different colorings) of a planar graph, Ringel [2] introduced the notion of 1-*planar graph* (namely, a graph that can be drawn in the plane so that each edge is crossed by at most one other edge). In the mentioned paper, he proved that each 1-planar graph G is 7-colorable and conjectured that 6 colors are

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enough to properly color the vertices of G. Now, this conjecture was confirmed by Borodin [3, 4] using the Discharging Method which is frequently used in studying problems on planar graphs. Note that the bound 6 here is sharp because K_6 is a 1-planar graph, but for 1-planar graph with girth at least 5, the bound of the chromatic number can be improved to 5 [5]. Borodin et al. [6] also proved that each 1-planar graph is acyclically 20-colorable. In addition, the list analogue of vertex coloring of 1-planar graphs was investigated by Albertson and Mohar [7]. Wang and Lih [8] proved that each 1-planar graph is list 7-colorable. Recently, Zhang et al. showed that each 1-planar graph G with maximum degree Δ is Δ-edge-colorable provided that $\Delta \geq 10$ [9], or $\Delta \geq 9$ and G contains no chordal 5-cycles [10], or $\Delta \geq 8$ and G contains no chordal 4-cycles [11], or $\Delta \geq 7$ and G contains no 3-cycles [12]. The linear arboricity and the $(p, 1)$-total labelling of 1-planar graphs were studied in [13] and [14], respectively. Besides the coloring aspects, the research of the global and local structures of 1-planar graphs can also be found in many papers, such as [5, 15–23].

In the next, we use the notation $\mathcal{P}_\delta^1$ to denote the family of all 1-planar graphs with degree at least δ. By the fact that each 1-planar graph is 7-degenerate (the value 7 being sharp) [5], the parameter δ here should be at most 7. Note that $\mathcal{P}_6^1$ and $\mathcal{P}_7^1$ have no intersection with the family of planar graphs since each planar graph has a vertex of degree at most 5.

Let $\mathcal{G}$ be a family of graphs and H be a connected graph such that at least one member of $\mathcal{G}$ contains a subgraph isomorphic to H. Denote $\mathfrak{H}(H, \mathcal{G})$ and $\mathfrak{W}(H, \mathcal{G})$ respectively to be the smallest integers with the property that each graph $G \in \mathcal{G}$, which contains a subgraph isomorphic to H, contains also a subgraph $K \simeq H$ such that $\max_{x \in V(K)}\{d_G(x)\} \leq \mathfrak{h}(H, \mathcal{G})$ and $\sum_{x \in V(K)}\{d_G(x)\} \leq \mathfrak{W}(H, \mathcal{G})$. These two parameters $\mathfrak{h}(H, \mathcal{G})$ and $\mathfrak{W}(H, \mathcal{G})$ are called the *height* and the *weight* of H in the family $\mathcal{G}$. If they are finite, then we say that H is *light* in $\mathcal{G}$, otherwise we say that H is *heavy* in $\mathcal{G}$. By $\mathcal{L}(\mathcal{G})$, we denote the set of light graphs in the family $\mathcal{G}$. It seems to be an interesting open problem to determine the set $\mathcal{L}(\mathcal{P}_\delta^1)$, especially for large δ, since it is proved in [5, 15] that

$$\mathcal{L}(\mathcal{P}_4^1) = \{P_1, P_2, P_3\} \tag{1.1}$$

and

$$\{P_1, P_2, P_3, S_3\} \subseteq \mathcal{L}(\mathcal{P}_5^1) \subseteq \{P_1, P_2, P_3, P_4, S_3\} \tag{1.2}$$

for $\delta = 4, 5$. In [5, 18], it is shown that each 1-planar graph with minimum degree 6 contains a copy of light C_3, C_4 and S_4 while each 1-planar graph with minimum degree 7 contains a copy of light K_4, C_5, W_5, S_6 and $3K_1 \vee K_2$ (note that if $G \in \mathcal{L}(\mathcal{P}_\delta^1)$, then obviously $G' \in \mathcal{L}(\mathcal{P}_\delta^1)$ for any subgraph $G' \subseteq G$. Take S_3 for example, since $S_3 \subseteq S_4 \subseteq S_6$, we also have $S_3 \in \mathcal{L}(\mathcal{P}_\delta^1)$ for $\delta = 6, 7$ as a corollary). Besides these graphs (and their subgraphs) we list here, until now there is no other known light subgraphs in the families $\mathcal{P}_6^1$ or $\mathcal{P}_7^1$. Regarding the known results, in particular it is proved in [18] and [5] that

Result A *Each* 1*-planar graph with minimum degree* 7 *contains a copy of* $3K_1 \vee K_2$ *with all vertices of degree at most* 13.

Result B *Each* 1*-planar graph with minimum degree* 7 *contains a copy of* K_4 *with all vertices of degree at most* 13*.*

Result C *Each* 1*-planar graph with minimum degree at least* 6 *contains a copy of* S_3 *of the type* $(6; \leq 15, \leq 15, \leq 15)$ *or* $(7; \leq 9, \leq 9, \leq 9)$*.*

The aim of this paper is not only to improve the above three results but also to generalize them to larger classes of graphs (hence some new light subgraphs in the families $\mathcal{P}_6^1$ or $\mathcal{P}_7^1$ are discovered). Note that $3K_1 \vee K_2 \subset K_2 \vee (K_1 \cup K_2)$, $K_4 \subset K_2 \vee (K_1 \cup K_2)$ and $S_3 \subset K_1 \vee (K_1 \cup K_2)$. We prove the following two stronger theorems in the next two sections.

Theorem 1.1 *Each* 1*-planar graph with minimum degree* 7 *contains a copy of* $K_2 \vee (K_1 \cup K_2)$ *with all vertices of degree at most* 12*.*

Theorem 1.2 *Each* 1*-planar graph with minimum degree at least* 6 *contains a copy of* $K_1 \vee (K_1 \cup K_2)$ *of the type* $(6; \leq 14; \leq 14, \leq 14)$ *or* $(7; \leq 9; \leq 9, \leq 9)$*.*

By Theorem 1.2, we directly have the following corollary.

Corollary 1.3 *Each* 1*-planar graph with minimum degree* 7 *contains a copy of* $K_1 \vee (K_1 \cup K_2)$ *with all vertices of degree at most* 9*.*

2 Proof of Theorem 1.1

In the following two sections, we always assume that G is a 1-planar graph (with minimum degree 6 or 7) that has been drawn in a plane so that every edge is crossed by at most one another edge and the number of crossings is as small as possible. The *associated plane graph* $G^\times$ of G is the plane graph that is obtained from G by turning all crossings of G into new 4-valent vertices. A vertex in $G^\times$ is called *false* if it is of degree 4 and *true* otherwise. One can observe that any two false vertices in $G^\times$ are not adjacent in the drawing of G. By false face (or edge), we mean a face (or an edge) in $G^\times$ that is incident with at least one false vertex, and the face (or edge) in $G^\times$ incident with no false vertex is called *true*.

The proof of Theorem 1.1 is carried out by contradiction. Suppose that G is a minimal counterexample to the theorem. We can assume that G is connected. Consider the associated plane graph $G^\times$ of G. Then we have

$$\sum_{v \in V(G^\times)} (d(v) - 4) + \sum_{f \in F(G^\times)} (d(f) - 4) = -8 \tag{2.1}$$

by combining the Euler polyhedral formula $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 2$ on $G^\times$ and the well-known relation $\sum_{v \in V(G^\times)} d(v) = \sum_{f \in F(G^\times)} d(f) = 2|E(G^\times)|$. Define the initial charge w on $V(G^\times) \cup F(G^\times)$ by

$$w(x) = d(x) - 4, \quad \text{if } x \in V(G^\times) \cup F(G^\times). \tag{2.2}$$

Thus we have the total charge $\sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = -8$ by Equations (2.1) and (2.2). In order to prove Theorem 1.1, we shall design the following discharging rules so that after discharging the new charge w' of each element in $V(G^\times) \cup F(G^\times)$ is nonnegative. But our rules only move charge around, and do not affect the total charges, which leads to a contradiction in final and completes our proof. In this section, a *big* vertex denotes a vertex of degree at least 13 and an *intermediate* vertex denotes a vertex of degree between 7 and 12. By $w_i(x)$, we denote the charge of an element $x \in V(G^\times) \cup F(G^\times)$ after application of the following first i rules.

R1 Each intermediate vertex transfers $\frac{3}{7}$ to each incident 3-face.

R2 Each big vertex transfers $\frac{4}{7}$ to each incident false 3-face.

R3 Let $\alpha = [xyz]$ be a true 3-face of $G^\times$. If all of x, y, z are big, then each of them transfers $\frac{1}{3}$ to α; if x is intermediate and y, z are big, then each of y, z transfers $\frac{2}{7}$ to α; if x, y are intermediate and z is big, then z transfers $\frac{1}{7}$ to α.

R4 Let $\alpha = [xyz]$ be a 3-face of $G^\times$ having a common edge xy with a 4^+-face β. If v is a true vertex that is adjacent to x or y in $G^\times$, then v transfers $\frac{1}{7}$ to α.

R5 Let $\alpha = [xyz]$ and $\beta = [yzv]$ be two adjacent 3-faces in $G^\times$ and v be a big vertex. If both y and z are intermediate, or the edge yz is a false edge, then v transfers $\frac{1}{7}$ to α.

R6 Each 3-face with positive charge after application of R1–R5 redistributes this charge uniformly among its adjacent 3-faces which have negative charge after application of the mentioned rules.

R7 Let α be a 4^+-face of $G^\times$ incident with a false vertex x such that $vx, ux \in E(G^\times[\alpha])$. Assuming uu' crosses vv' in G at the point x, if v is intermediate and $u'v' \in E(G)$, then v transfers $\frac{1}{14}$ to the 3-face $\beta = [xu'v']$ in $G^\times$.

R8 Each 5^+-face transfers its charge equally among all adjacent 3-faces.

In the following, we will check the final charge w' of vertices and faces after the charge redistribution and prove that $w'(x) = w_8(x) \geq 0$ holds for each $x \in V(G^\times) \cup F(G^\times)$. Note that all false vertices and all 4-faces are not involved in the discharging rules, so they must all have nonnegative final charges. By R8, one can also easily check that $w_8(f) = 0$ holds for all 5^+-faces. Therefore, we shall only check the final charge of all 3-faces, intermediate vertices and big vertices below.

Case 1 Suppose that $f = [xyz]$ is a 3-face. Then it is clear that $w'(f) \geq 0$ holds for each true 3-faces in $G^\times$ by R1 and R3. In the following, we analyze in a deeper detail the final charge of the false 3-faces. Without loss of generality, we assume that x is a false vertex at which yy' is crossed by zz' in G. Denote f_1, f_2, f_3 and f_4, which are different to f, respectively to be the faces incident with xy, xz, yz and the path $y'xz'$ in $G^\times$. If at least one of y, z is big, then one can easily check $w'(v) \geq 0$ by R1 and R2. If both y and z are intermediate, then $w_3(f) = -1 + 2 \times \frac{3}{7} = -\frac{1}{7}$. In order to show $w_8(f) \geq 0$, we will prove that an additional $\geq \frac{1}{7}$ charge will be transferred to f by R4–R8, which yields that $w_8(f) \geq w_3(f) + \frac{1}{7} = 0$.

Case 1.1 Let at least one of f_1, f_2 and f_3 be a 4^+-face. If f_1 is a 4^+-face (the case when $d(f_2) \geq 4$ can be dealt with similarly), then by R4, z' shall transfer $\frac{1}{7}$ to f since z' is true. If f_3 is a 5^+-face, then by R8, f_3 shall transfer at least $\frac{5-4}{5} = \frac{1}{5} > \frac{1}{7}$ to f. If f_3 is a 4-face, then there exists a true vertex on f_3 that is adjacent to y or z, from which f shall also receive $\frac{1}{7}$ by R4 (here, note that both y and z are true and no 4-vertices is adjacent in $G^\times$).

Case 1.2 Let all of f_1, f_2 and f_3 be 3-faces. Then $f_1 = [xyz']$ and $f_2 = [xy'z]$. If y' or z' is big, then by R5, f shall receive $\frac{1}{7}$ from y' or z' since xy and xz are both false edges. So in the following, we assume that y' and z' are both intermediate vertices. Denote $f_3 = [yzv]$. By distinguishing whether v is true or not, we will divide our proof into two subcases.

Case 1.2.1 Let v be a true vertex. If v is big, then by R5, f shall receive $\frac{1}{7}$ from v since y and z are both intermediate. If v is intermediate, then $d(f_4) \geq 4$ (otherwise $y'z' \in E(G^\times)$ and we find a light copy of $K_2 \vee (K_1 \cup K_2)$ because all of v, y, z, y' and z' are intermediate). It follows that both y' and z' shall transfer $\frac{1}{14}$ to f by R7. Therefore, f will receive an additional $2 \times \frac{1}{14} = \frac{1}{7}$ by R4–R8.

Case 1.2.2 Let v be a false vertex. We assume that yy'' crosses zz'' in G at the point v (note that y'' and z'' are both true vertices, who are different to y, z, y' and z'). Denote h_1 and h_2 respectively to be the faces incident with the path $y''vz$ and the path yvz'' in $G^\times$. Suppose that h_1 and h_2 are both 4^+-faces, then by R4, f_3 will totally receive $2 \times \frac{1}{7} = \frac{2}{7}$ from y'' and z''. It follows that $w_5(f_3) \geq -1 + 2 \times \frac{3}{7} + \frac{2}{7} = \frac{1}{7}$. Note that $w_5(f) = -1 + 2 \times \frac{3}{7} = -\frac{1}{7} < 0$ and $\min\{d(h_1), d(h_2)\} \geq 4$. By R6, f_3 shall transfer to f a charge $w_5(f_3) = \frac{1}{7}$. It follows that $w_8(f) \geq w_5(f) + \frac{1}{7} = 0$. Suppose that only one of h_1 and h_2, say h_1, is a 4^+-face. Then $d(h_2) = 3$ and $yz'' \in E(G^\times)$. If z'' is big, then by R5, z'' shall transfer $\frac{1}{7}$ to f_3 since the edge yv is false. In addition, y'' shall also transfer $\frac{1}{7}$ to f_3 by R4. Therefore, $w_5(f_3) \geq -1 + 2 \times \frac{3}{7} + \frac{1}{7} + \frac{1}{7} = \frac{1}{7}$. Note that $w_5(f) = -\frac{1}{7} < 0$, $w_5(h_2) \geq -1 + \frac{3}{7} + \frac{4}{7} = 0$ and $d(h_1) \geq 4$ by R1–R5. Then by R6, f_3 will totally transfer its new charge after application of R1–R5 to f, which also yields that $w_8(f) \geq w_5(f) + \frac{1}{7} = 0$. If z'' is intermediate, then we must have $d(f_4) \geq 4$ (for otherwise the five intermediate vertices y, z, y', z' and z'' would form a light copy of $K_2 \vee (K_1 \cup K_2)$ in G). So the following argument is just the same as the one in Case 1.2.1. Finally, suppose that h_1 and h_2 are both 3-faces. It follows that $yz'', y''z \in E(G^\times)$. If at least one of y'' and z'' is intermediate, then the following argument is just the same as before. If both y'' and z'' are big vertices, then by R5, each of them shall transfer $\frac{1}{7}$ to f_3 since both yv and zv are false edges. Therefore, $w_5(f_3) \geq -1 + 2 \times \frac{3}{7} + \frac{1}{7} + \frac{1}{7} = \frac{1}{7}$. Again, note that $w_5(f) = -\frac{1}{7} < 0$ and $\min\{w_5(h_1), w_5(h_2)\} \geq -1 + \frac{3}{7} + \frac{4}{7} = 0$ by R1–R5. Hence, by R6, f_3 shall transfer a charge $\frac{1}{7}$ to just one adjacent 3-face f. Thus, it follows that $w_8(f) \geq w_5(f) + \frac{1}{7} = 0$.

Case 2 Suppose that v is an intermediate vertex. If v is incident with a 3-face, then v shall only send out $\frac{3}{7}$ by applying R1 at most once. If v is incident with a 4^+-face, then v shall send out at most $2 \times \frac{1}{7} + 2 \times \frac{1}{14} = \frac{3}{7}$ by respectively applying each of R4 and R7 at most twice. We conclude that the value of the charge transferred from v via each incident face is at most $\frac{3}{7}$; therefore, $w_8(v) \geq d(v) - 4 - \frac{3}{7}d(v) = \frac{4}{7}d(v) - 4 \geq 0$ for $d(v) \geq 7$.

Case 3 Suppose that v is a big vertex. If v is incident with a true 3-face $f = [uvw]$ with both u and w being intermediate, then v shall send out at most $\frac{1}{7} + \frac{1}{7} = \frac{2}{7}$ by applying each of R3 and R5 at most once. If v is incident with a true 3-face $f = [uvw]$ so that at least one of u and w is big, then v shall only send out at most $\max\{\frac{1}{3}, \frac{2}{7}\} = \frac{1}{3}$ by applying R3 at most once (note that in this case R5 would not be applied since the edge uw is not in the form of the mentioned rule). If v is incident with a false 3-face, then v shall send out at most $\frac{4}{7}+\frac{1}{7}=\frac{5}{7}$ by respectively applying R2 and R5 at most once. If v is incident with a 4^+-face, then v shall only send out at most $2\times\frac{1}{7}=\frac{2}{7}$ by applying R4 at most twice (note that in this case R7 would also not be applied). By all these arguments, we obtain the estimation $w_8(v)\geq d(v)-4-\max\{\frac{2}{7},\frac{1}{3},\frac{5}{7}\}\cdot d(v)=\frac{2}{7}d(v)-4\geq 0$ for $d(v)\geq 14$. Finally, if $d(v)=13$, then v can be adjacent to at most twelve false 3-faces because any two false vertices can not be adjacent. Therefore, $w_8(v)\geq 13-4-12\times\frac{5}{7}-\max\{\frac{2}{7},\frac{1}{3}\}=\frac{2}{21}>0$.

3 Proof of Theorem 1.2

For convenience, we use some specialized notations during our proof in this section. Let v be a false vertex in $G^\times$ (the associated plane graph of G) and v_1,v_2,v_3,v_4 be its neighbors in clockwise order. Define f_i to be the face incident with vv_i and vv_{i+1}, where the subtraction and addition on subscripts are taken modulo 4. By this definition, one can easily observe that if $d(f_i)=3$, then $v_iv_{i+1}\in E(G)$. In this case, let f_i' be the other face incident with the edge v_iv_{i+1}. If $d(f_i')=3$, the its third vertex, which is different with v_i and v_{i+1}, will be denoted by v_i'. Then v_i' is a false vertex if and only if f_i' is false, in which case we denote the neighbor of v_i (or v_{i+1}) in G, such that the edge connecting them in G contains the crossing point v_i', to be v_i'' (or v_{i+1}'', respectively). Namely, v_iv_i'' and $v_{i+1}v_{i+1}''$ are two edges in G crossed by each other at the point v_i'. Denote the face that is incident with the path $v_iv_i'v_{i+1}''$ (or $v_{i+1}v_i'v_i''$) in $G^\times$ by f_i^L (or f_i^R, respectively).

For a face f in $G^\times$, denote $c(f)$ to be the number of false vertices that f is incident with. Let $\delta(G)\geq 6$ and v be a vertex in $G^\times$. We call v *big* if $d(v)\geq 15$, *intermediate* if $6\leq d(v)\leq 14$, *sub-big* if $10\leq d(v)\leq 14$, and *sub-intermediate* if $6\leq d(v)\leq 9$. Thus, each true vertex in $G^\times$ is either big or intermediate; and each intermediate vertex in $G^\times$ is either sub-big or sub-intermediate.

The proof of Theorem 1.2 is also carried out by contradiction. Suppose that G is a minimal counterexample to the theorem. We can assume that G is connected. Moreover G contains no copy of $K_1\vee(K_1\cup K_2)$ either with its center vertex being of degree 6 and the others being intermediate, or its center vertex being of degree 7 and the others being sub-intermediate. The main proof follows the similar strategy (by using Discharging Method on the associated plane graph $G^\times$ of G) described in Section 2. The only difference is the form of Euler polyhedral formula we used here which is

$$\sum_{v\in V(G^\times)}(d(v)-6)+\sum_{f\in F(G^\times)}(2d(f)-6)=-12 \tag{3.1}$$

and the initial charge assigned to $x\in V(G^\times)\cup F(G^\times)$ being

$$w(x)=\begin{cases} d(x)-6, & \text{if } x\in V(G^\times);\\ 2d(x)-6, & \text{if } x\in F(G^\times).\end{cases} \tag{3.2}$$

Thus we have the total charge $\sum_{x\in V(G^\times)\cup F(G^\times)}w(x)=-12$ by Equations (3.1) and (3.2). In the next, we will check that $w'(x)\geq 0$ (the new charge after discharging) holds for all

$x \in V(G^\times) \cup F(G^\times)$ according to the following discharging rules, which leads to a contradiction.

R1 Each 4^+-face transfers $\frac{3}{4}$ to each incident false vertex.

R2 Let α be a 4^+-face having a common edge xy with a false 3-face $\beta = [xyz]$. If z is a false vertex, then α transfers $\frac{1}{4}$ to z through xy.

R3 Let α be a 4^+-face having a common edge xy with a false 3-face $\beta = [xyz]$. If x is a false vertex and yz is incident with another false 3-face $\gamma = [yzu]$, then α transfer $\frac{1}{8}$ to u through xy and yz.

R4 Let $\alpha = [xyz]$ be a true 3-face having a common edge yz with a false 3-face $\beta = [uyz]$. Then x transfers to u through yz a charge

$$\begin{cases} \frac{1}{4}, & \text{if } x \text{ is big;} \\ \frac{1}{7}, & \text{if } x \text{ is sub-big.} \end{cases}$$

R5 Let $\alpha = [xyz]$ and $\beta = [uyz]$ be two adjacent false 3-faces and z be a false vertex. Suppose that yu is incident with another false 3-face $\gamma = [yuw]$ such that yy' crosses uu' in G at the point w. Then x transfers to w through yz and yu a charge

$$\begin{cases} \frac{1}{8}, & \text{if } x \text{ is big;} \\ \frac{1}{28}, & \text{if } x \text{ is sub-big, } t(w) = 3,\ d(y) = 7,\ d(u) \geq 8, \text{ and } y' \text{ is sub-intermediate;} \\ \frac{1}{28}, & \text{if } x \text{ is sub-big, } t(w) = 3,\ d(u) = 7,\ d(u) \geq 8, \text{ and } u' \text{ is sub-intermediate;} \\ \frac{1}{28}, & \text{if } x \text{ is sub-big, } t(w) = 3 \text{ and } d(u) = d(v) = 7; \\ \frac{1}{14}, & \text{if } x \text{ is sub-big, } t(w) = 4,\ d(y) = d(u) = 7 \text{ and } \min\{d(y'), d(u')\} = 7, \end{cases}$$

where the parameter $t(w)$ above denotes the number of 3-faces incident with w in $G^\times$.

R6 Let $\alpha = [xyz]$ and $\beta = [uyz]$ be two adjacent false 3-faces and z be a false vertex. Then y transfers z a charge

$$\begin{cases} 1, & \text{if } y \text{ is big;} \\ \frac{5}{7}, & \text{if } y \text{ is sub-big;} \\ \frac{1}{2}, & \text{if } 8 \leq d(y) \leq 9; \\ \frac{2}{7}, & \text{if } d(y) = 7. \end{cases}$$

R7 Let $\alpha = [xyz]$ be a false 3-face having a common edge yz with a 4^+-face β and z be a false vertex. Then y transfers z a charge

$$\begin{cases} \frac{5}{8}, & \text{if } y \text{ is big;} \\ \frac{11}{28}, & \text{if } y \text{ is sub-big;} \\ \frac{3}{8}, & \text{if } 8 \leq d(y) \leq 9; \\ \frac{3}{14}, & \text{if } d(y) = 7. \end{cases}$$

Since all 3-faces and 6-vertices are not involved in the discharging rules, they have final nonnegative charges. In the following, we will check in detail the final charge w' of the 4^+-faces, false vertices and 7^+-vertices after the charge redistribution and prove that $w'(x) \geq 0$ holds for each of them.

Case 1 Suppose that f is a 4^+-face. Since any two false vertices are not adjacent in $G^\times$, f is incident with at most $\lfloor \frac{d(f)}{2} \rfloor$ false vertices. Namely, $c(f) \leq \lfloor \frac{d(f)}{2} \rfloor$. On the other hand, by the definitions of R2 and R3, for f, one can observe that R2 can be applied at most $d(f) - 2c(f)$ times while R3 can be applied at most $2c(f)$ times. Therefore, $w'(f) \geq 2d(f) - 6 - \frac{3}{4}c(f) - \frac{1}{4}(d(f) - 2c(f)) - \frac{1}{8} \times 2c(f) = \frac{7}{4}d(f) - \frac{1}{2}c(f) - 6 \geq \frac{7}{4}d(f) - \frac{1}{2}\lfloor \frac{d(f)}{2} \rfloor - 6 \geq \frac{3}{2}d(f) - 6 \geq 0$ for $d(f) \geq 4$.

Case 2 Suppose that v is a false vertex. Let $s(v)$ be the total charge transferred to v by R1–R7. Note that $w(v) = d(v) - 6 = -2$ and v shall not be able to send out any charge by the mentioned discharging rules, in the following, we just need to prove $s(v) \geq 2$.

Case 2.1 Suppose that v is incident with at least three 4^+-faces. Then by R1, each of them shall transfer $\frac{3}{4}$ to v. Thus, v shall totally receive a charge at least $3 \times \frac{3}{4} = \frac{9}{4} > 2$.

Case 2.2 Suppose that v is incident with just two 4^+-faces.

Case 2.2.1 Suppose $d(f_1) = d(f_2) = 3$. Then by R1, v shall receive $2 \times \frac{3}{4} = \frac{3}{2}$ from both f_3 and f_4. Furthermore, if one of v_1, v_2 or v_3 is big, then v can receive an additional 1 or $\frac{5}{8}$ via R6 or R7. Therefore, $s(v) \geq \frac{3}{2} + \frac{5}{8} = \frac{17}{8} > 2$. So suppose that each of v_i, $i = 1, 2, 3$, is intermediate. If $\min\{d(v_1), d(v_2), d(v_3)\} \geq 7$, then by R6 and R7, v shall receive, in addition, at least $\frac{2}{7} + 2 \times \frac{3}{14} = \frac{5}{7}$ from v_1, v_2 and v_3, which also implies $s(v) \geq \frac{3}{2} + \frac{5}{7} > 2$. Hence, suppose that at least one of v_1, v_2 and v_3 is a 6-vertex.

Case 2.2.1.1 Suppose that at least two of v_1, v_2 and v_3 are 6-vertices. Without loss of generality, we assume $d(v_1) = d(v_2) = d(v_3) = 6$ (the case when only two of them are 6-vertices can be dealt with similarly). Now consider the face f_1'. If it is a 4^+-face, then it will transfer $\frac{1}{4}$ to v through v_1v_2 by R2. If f_1' is a 3-face, denoted by $v_1v_2v_1'$ (note that $v_1' \neq v_3, v_4$), then v_1' must be a big vertex while f_1' is true to avoid a light copy of $K_1 \vee (K_1 \cup K_2) = [v_1; v_1'; v_2, v_3]$ with its center vertex v_1 being 6-vertex. In such a case, v_1' will transfer $\frac{1}{4}$ to v through v_1v_2 by R4. If f_1' is false, then v_1' is a false vertex. Now consider the faces f_1^L and f_2^R. If f_1^L is a 4^+-face, then it will transfer $\frac{1}{8}$ to v through v_1v_2 by R3. Otherwise f_1^L must be a false 3-face, denoted by $v_1v_1'v_2''$ (note that $v_2'' \neq v_3, v_4$). Note that $[v_1; v_2''; v_2, v_3]$ is a copy of $K_1 \vee (K_1 \cup K_2)$ with its center vertex v_1 being 6-vertex and v_2, v_3 being intermediate, which yields that v_2'' must be big. Then by R5, v_2'' would transfer $\frac{1}{8}$ to v through v_1v_2. By a similar discussion on f_1^R, v will receive another $\frac{1}{8}$ from f_1^R or v_1'' through v_1v_2 (here, note that $v_1'' \neq v_3$, and if $v_1'' = v_4$, then f_1^R must be a 4^+-face). Consequently, using the transfers through v_1v_2, v receives totally $2 \times \frac{1}{8} = \frac{1}{4}$. Similarly, we can also prove that v shall also receive another $\frac{1}{4}$ by using the transfers through v_2v_3. Therefore, $s(v) \geq \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$.

Case 2.2.1.2 Suppose that only one of v_1, v_2 and v_3 is 6-vertex. Without loss of generality, we assume $d(v_2) = 6$ (other cases can be dealt with similarly). By R6 and R7, v_1 and v_3 would respectively transfer at least $\frac{3}{14}$ to v. Moreover, by a similar argument as in Case 2.2.1.1, a total charge $2 \times \frac{1}{4} = \frac{1}{2}$ shall be also transferred to v through v_1v_2 and v_2v_3. It follows that $s(v) \geq \frac{3}{2} + 2 \times \frac{3}{14} + \frac{1}{2} > 2$.

Case 2.2.2 Suppose that $d(f_1) = d(f_3) = 3$. Then by R1, each of f_2 and f_4 shall transfer $\frac{3}{4}$ to v. In addition, if one of v_i, $i = 1, 2, 3, 4$, is big, then by R7, it shall transfer an additional $\frac{5}{8}$ to v; in this case, $s(v) \geq 2 \times \frac{3}{4} + \frac{5}{8} = \frac{17}{8} > 2$. So assume that all the neighbors of v in $G^\times$ are intermediate. Then by a similar argument as in Case 2.2.1.1, one can also observe that if v_1 or v_2 (v_3 or v_4, resp.) is 6-vertex, then v shall receive a charge $\frac{1}{4}$ by using the transfers through v_1v_2 (v_3v_4, resp.). Hence, if at least three of v_i, $i = 1, 2, 3, 4$, are 6-vertices, then $s(v) \geq \frac{3}{2} + 2 \times \frac{1}{4} = 2$; and if one or two of them are 6-vertices (namely, three or two of them are 7^+-vertices, from whom v shall totally receive at least $2 \times \frac{3}{14} = \frac{3}{7}$ by R7), then $s(v) \geq \frac{3}{2} + \frac{1}{4} + \frac{3}{7} > 2$; and if none of them are 6-vertices, then by R7, each of them shall transfer $\frac{3}{14}$ to v, which also follows that $s(v) \geq \frac{3}{2} + 4 \times \frac{3}{14} > 2$.

Case 2.3 Suppose that v is incident with only one 4^+-face, say f_4. Then by R1, f_4 shall transfer $\frac{3}{4}$ to v. If at least two of v_i, $i = 1, 2, 3, 4$, are big, then by R6 and R7, v shall receive at least $2 \times \frac{5}{8} = \frac{5}{4}$ from its big neighbors in $G^\times$. It follows that $s(v) \geq \frac{3}{4} + \frac{5}{4} = 2$. So we assume that at most one of v_i, $i = 1, 2, 3, 4$, is big. Firstly, suppose that v_2 is big (the case when v_3 is big can be dealt with by symmetry), from whom v shall receive 1 by R6. If $d(v_3) \geq 7$, then also by R6, v_3 shall transfer $\frac{2}{7}$ to v, which yields that $s(v) \geq \frac{3}{4} + 1 + \frac{2}{7} > 2$. If $d(v_3) = 6$, then by a similar argument as in Case 2.2.1.1 (note that both v_1 and v_4 are intermediate here), v shall receive a charge $\frac{1}{4}$ by using the transfers through v_3v_4. So we still have $s(v) \geq \frac{3}{4} + 1 + \frac{1}{4} = 2$. Secondly, suppose that v_1 is big (the case when v_4 is big can also be dealt with by symmetry), from whom v shall receive $\frac{5}{8}$ by R7. If both v_2 and v_3 are 6-vertices, then by using the transfers through v_1v_2, v_2v_3 and v_3v_4, v shall receive in total $3 \times \frac{1}{4} = \frac{3}{4}$. It implies that $s(v) \geq \frac{3}{4} + \frac{5}{8} + \frac{3}{4} > 2$. If one of v_2 and v_3 is 6-vertex and the other is 7^+-vertex, then one can also obtain $s(v) \geq \frac{3}{4} + \frac{5}{8} + \frac{2}{7} + 2 \times \frac{1}{4} > 2$ (note that this 7^+-vertex shall transfer $\frac{2}{7}$ to v by R6 and this 6-vertex implies the existence of a double transfers of $\frac{1}{4}$ through its incident edges). So we assume that both v_2 and v_3 are 7^+-vertices. In this case, by a similar argument as above, we can also deduce that $s(v) \geq \frac{3}{4} + \frac{5}{8} + 2 \times \frac{2}{7} + \frac{3}{14} > 2$ if $d(v_4) \geq 7$, and $s(v) \geq \frac{3}{4} + \frac{5}{8} + 2 \times \frac{2}{7} + \frac{1}{4} > 2$ if $d(v_4) = 6$. So in the following, we assume that all of v_i, $i = 1, 2, 3, 4$, are intermediate vertices. Now, we must have $\min\{d(v_2), d(v_3)\} \geq 7$, for otherwise a light copy of $K_1 \vee (K_1 \cup K_2)$ with the mentioned property in Theorem 1.2 would appear in G.

Case 2.3.1 Suppose that at least three of v_i, $i = 1, 2, 3, 4$, are sub-big vertices. Then by R6 and R7, we have $s(v) \geq \frac{3}{4} + 2 \times \frac{11}{28} + \frac{5}{7} > 2$.

Case 2.3.2 Suppose that only two of v_i, $i = 1, 2, 3, 4$, are sub-big vertices. If v_2 and v_3 are both sub-big vertices, then by R6, they transfer in total $2 \times \frac{5}{7} = \frac{10}{7}$ to v, which implies $s(v) \geq \frac{3}{4} + \frac{10}{7} > 2$. If v_1 and v_4 are both sub-big vertices, then by R7, they contribute in total $2 \times \frac{11}{28} = \frac{11}{14}$ to v. On the other hand, each of v_2 and v_3 shall transfer at least $\frac{2}{7}$ to v by R6 (recall that $\min\{d(v_2), d(v_3)\} \geq 7$). Therefore, $s(v) \geq \frac{3}{4} + \frac{11}{14} + 2 \times \frac{2}{7} > 2$. If only one of v_2 and v_3 is sub-big, then by R6 and R7, a total charge $\frac{5}{7} + \frac{11}{28} = \frac{31}{28}$ would be transferred to v. Note that $\min\{d(v_2), d(v_3)\} \geq 7$, then by R6, we still have $s(v) \geq \frac{3}{4} + \frac{31}{28} + \frac{2}{7} > 2$.

Case 2.3.3 Suppose that v_1 is sub-big but v_2, v_3, v_4 are sub-intermediate (the case when v_4 is sub-big can be dealt with by symmetry). Then v_1 shall transfer $\frac{11}{28}$ to v by R7 while v_2 (or v_3) shall transfer $\frac{2}{7}$ to v if $d(v_2) = 7$ (or $d(v_3) = 7$) and at least $\frac{1}{2}$ to v if $d(v_2) \geq 8$ (or $d(v_2) \geq 8$). Firstly, we suppose $d(v_4) = 6$, then v shall also receive another $\frac{1}{4}$ via the transfer through v_3v_4 by a similar argument as in Case 2.2.1.1. Therefore, $s(v) \geq \frac{3}{4} + \frac{11}{28} + \frac{1}{2} + \frac{2}{7} + \frac{1}{4} > 2$ when at least one of v_2 and v_3 is a 8^+-vertex. So we assume $d(v_2) = d(v_3) = 7$. Now consider f_i' (or f_i^L and f_i^R), $i = 1, 2$, by R2–R5, we can also prove that v can respectively receive at least $\min\{\frac{1}{4}, 2 \times \frac{1}{8}, \frac{1}{7}, 2 \times \frac{1}{28}, \frac{1}{8} + \frac{1}{28}\} = \frac{1}{14}$ through the edges v_1v_2 and v_2v_3 by using a similar discussion as in Case 2.2.1.1 (note that the essential principle we use here is to avoid a light copy of $K_1 \vee (K_1 \cup K_2)$ with its center vertex being of degree 7 and the others being sub-intermediate). Therefore, we also have $s(v) \geq \frac{3}{4} + \frac{11}{28} + 2 \times \frac{2}{7} + 2 \times \frac{1}{14} + \frac{1}{4} > 2$. Secondly, we suppose $d(v_4) \geq 7$. If at least one of v_2, v_3 and v_4 is 8^+-vertex, then by R6 and R7, we have $s(v) \geq \frac{3}{4} + \frac{11}{28} + 2 \times \frac{2}{7} + \frac{3}{8} > 2$. So we shall assume $d(v_2) = d(v_3) = d(v_4) = 7$. In such a case at least $3 \times \frac{1}{14} = \frac{3}{14}$ would be transferred to v through v_1v_2, v_2v_3 and v_3v_4 by the same argument as above. Hence, by R6 and R7, we still have $s(v) \geq \frac{3}{4} + \frac{11}{28} + 2 \times \frac{2}{7} + \frac{3}{14} + \frac{3}{14} > 2$.

Case 2.3.4 Suppose that v_2 is sub-big but v_1, v_3, v_4 are sub-intermediate (the case when v_3 is sub-big can be dealt with by symmetry). Then by R6, v_2 shall transfer $\frac{5}{7}$ to v. Suppose that at least one of v_1 and v_4, say v_1, is a 6-vertex. Then, using the transfer through v_1v_2, v shall receive $\frac{1}{4}$. Consequently, $s(v) \geq \frac{3}{4} + \frac{5}{7} + \frac{2}{7} + \frac{1}{4} = 2$ (recall that $d(v_3) \geq 7$). So we assume $\min\{d(v_1), d(v_4)\} \geq 7$. In such a case, by R6 and R7, we still have $s(v) \geq \frac{3}{4} + \frac{5}{7} + \frac{2}{7} + 2 \times \frac{3}{14} > 2$.

Case 2.3.5 Suppose that none of v_i, $i = 1, 2, 3, 4$, is sub-big. Namely, all of them are sub-intermediate, which implies that $\min\{d(v_2), d(v_3)\} \geq 8$ (for otherwise we would find a copy of $K_1 \vee (K_1 \cup K_2)$ in G with its center vertex being of degree 7 and the others being sub-intermediate). Then by R6, both v_2 and v_3 shall respectively transfer at least $\frac{1}{2}$ to v. Similarly as in the proof in Case 2.3.4, we can also show that $s(v) \geq \frac{3}{4} + 2 \times \frac{1}{2} + \frac{1}{4} = 2$ if at least one of v_1 and v_4 is a 6-vertex and $s(v) \geq \frac{3}{4} + 2 \times \frac{1}{2} + 2 \times \frac{3}{14} > 2$ if v_1 and v_4 are both 7^+-vertices.

Case 2.4 Suppose that v is incident with four 3-faces. If at least two of v_i, $i = 1, 2, 3, 4$, are big vertices, then by R6, each of the big neighbor of v shall transfer 1 to v, which implies $s(v) \geq 2 \times 1 = 2$. If only one of the neighbors of v, say v_1, is a big vertex, then v_1 shall transfer 1 to v by R6. In this case (note that all of v_2, v_3 and v_4 are intermediate), if all of v_2, v_3 and v_4 are 6-vertices, then by a same argument as in Case 2.2.1.1, v shall receive a total charge $4 \times \frac{1}{4} = 1$ via using the transfers through v_1v_2, v_2v_3, v_3v_4 and v_4v_1. It follows that $s(v) = 1 + 1 = 2$. If two of v_2, v_3 and v_4 are 6-vertices (without loss of generality, say v_2 and v_3), then using the transfers through v_1v_2, v_2v_3 and v_3v_4, v receives $3 \times \frac{1}{4} = \frac{3}{4}$. On the other hand, by R6, v_4 shall also transfer at least $\frac{2}{7}$ to v since $d(v_4) \geq 7$. Therefore, $s(v) \geq 1 + \frac{3}{4} + \frac{2}{7} > 2$. If only one of v_2, v_3 and v_4, say v_2, is a 6-vertex, then v shall receive a charge $2 \times \frac{1}{4} = \frac{1}{2}$ through the edges

v_1v_2 and v_2v_3 and another $2 \times \frac{2}{7} = \frac{4}{7}$ from v_3 and v_4 by R6 since $\min\{d(v_3), d(v_4)\} \geq 7$. Hence, we also have $s(v) \geq 1 + \frac{1}{2} + \frac{4}{7} > 2$. So we assume $\min\{d(v_2), d(v_3), d(v_4)\} \geq 7$. If at least one of v_2, v_3 and v_4, say v_2, is an 8^+-vertex, then by R6, v_2 shall transfer at least $\frac{1}{2}$ while both v_3 and v_4 shall respectively transfer at least $\frac{2}{7}$ to v. It follows that $s(v) \geq 1 + \frac{1}{2} + 2 \times \frac{2}{7} > 2$. So we just need to consider the case when $d(v_2) = d(v_3) = d(v_4) = 7$ in detail. By a similar discussion as in Case 2.3.3, we can prove that v shall respectively receive at least $\frac{1}{14}$ from each of the edges v_1v_2, v_2v_3, v_3v_4 and v_4v_1. On the other hand, by R6, each of v_2, v_3 and v_4 shall also transfer $\frac{2}{7}$ to v. Thus, we again have $s(v) \geq 1 + 4 \times \frac{1}{14} + 3 \times \frac{2}{7} > 2$. So in the following, we assume that all of v_i, $i = 1, 2, 3, 4$, are intermediate vertices. Furthermore, all of them should also be 7^+-vertices, for otherwise a light copy of $K_1 \vee (K_1 \cup K_2)$ with the property mentioned in Theorem 1.2 would appear in G.

Case 2.4.1 Suppose that at least two of v_i, $i = 1, 2, 3, 4$, are sub-big. Then by R6, each of the sub-big neighbors of v shall transfer $\frac{5}{7}$ while each of the remaining neighbors of v shall transfer at least $\frac{2}{7}$ to v. This leads to $s(v) \geq \min\{4 \times \frac{5}{7}, 3 \times \frac{5}{7} + \frac{2}{7}, 2 \times \frac{5}{7} + 2 \times \frac{2}{7}\} = 2$.

Case 2.4.2 Suppose that only one of v_i, $i = 1, 2, 3, 4$, say v_1, is sub-big and the others are sub-intermediate. If at least two of v_2, v_3 and v_4 are 8^+-vertices, then by R6, each 8^+-vertex of them shall transfer at least $\frac{1}{2}$ while each 7-vertex of them shall transfer at least $\frac{2}{7}$ to v. Therefore, $s(v) \geq \min\{3 \times \frac{1}{2} + \frac{5}{7}, 2 \times \frac{1}{2} + \frac{2}{7} + \frac{5}{7}\} = 2$ since v_1 should also contribute $\frac{5}{7}$ to v by R6. So at least two of v_2, v_3 and v_4 are 7-vertices. Firstly, we suppose that only two of them are 7-vertices. Without loss of generality, we assume $d(v_2) = d(v_3) = 7$ and $d(v_4) \geq 8$. In such a case, by R6, v shall receive a total charge $\frac{5}{7} + 2 \times \frac{2}{7} + \frac{1}{2} = \frac{25}{14}$ from all the neighbors of v. On the other hand, using the transfers through v_1v_2, v_2v_3 and v_3v_4, v shall also receive another $3 \times \frac{1}{14} = \frac{3}{14}$ by a similar argument as in Case 2.3.3. Therefore, we also have $s(v) \geq \frac{25}{14} + \frac{3}{14} = 2$. Finally, we suppose that all of v_2, v_3 and v_4 are 7-vertices. Then from the neighbors of v, v receives at least $\frac{5}{7} + 3 \times \frac{2}{7} = \frac{11}{7}$ by R6. By a similar argument as in Case 2.3.3, one can prove that v shall respectively receive at least $\min\{\frac{1}{4}, 2 \times \frac{1}{8}, \frac{1}{7}, 2 \times \frac{1}{14}, \frac{1}{8} + \frac{1}{14}\} = \frac{1}{7}$ by using the transfers through v_2v_3 and v_3v_4 (here, note that once when we consider f_i, $i = 2, 3$, if f_i' and f_i^L (or f_i^R) are both false 3-faces, then v_{i+1}'' (or v_i'') should either be sub-big or big. Furthermore, if v_{i+1}'' (or v_i'') is sub-big, then by R5, it shall transfer $\frac{1}{14}$ (not $\frac{1}{28}$ now) to v because $t(v) = 4$ and $d(v_2) = d(v_3) = d(v_4) = 7$). On the other hand, using the transfers through v_1v_2 and v_4v_1, v shall receive another $2 \times \frac{1}{14} = \frac{1}{7}$ by the same discussion as in Case 2.3.3. Consequently, we have $s(v) \geq \frac{11}{7} + 2 \times \frac{1}{7} + \frac{1}{7} = 2$ in final.

Case 2.4.3 Suppose that none of v_i, $i = 1, 2, 3, 4$, is sub-big. Namely, they are all sub-intermediate, which yields that $\min\{d(v_1), d(v_2), d(v_3), d(v_4)\} \geq 8$ for otherwise, we would find a light copy of $K_1 \vee (K_1 \cup K_2)$ in G with its center vertex being of degree 7 and the others being sub-intermediate, a contradiction. Therefore, by R6, each of v_i, $i = 1, 2, 3, 4$, shall transfer at least $\frac{1}{2}$ to v, which yields $s(v) \geq 4 \times \frac{1}{2} = 2$.

Case 3 Suppose that v is a 7^+-vertex. Let $F(v)$ denote the subgraph induced by the faces which are incident with v. Then $F(v)$ can be decomposed into many parts, each of which is one of the five clusters in Figure 1, and any two parts of which are adjacent only if they have a common edge vw such that w is a true vertex. The hollow vertices in Figure 1 are all false and the solid ones are true, and all the faces are marked by f_i are 4^+-faces. Denote $n_i(v)$ to be the number of i-clusters contained in $F(v)$ and $m_i(v)$ to be the charges sent out from v through an i-cluster. Then by their definitions, one can easily observe that $2n_1(v)+2n_2(v)+n_3(v)+3n_4(v)+n_5(v) \leq d(v)$ and $w'(v) = d(v) - \sum_{i=1}^{5} n_i(v)m_i(v) - 6$.

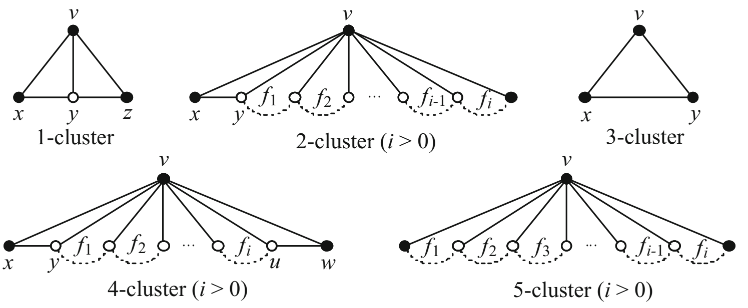

Figure 1 $F(v)$ can be decomposed into the combination of the above five clusters

Case 3.1 Suppose $7 \leq d(v) \leq 9$. If $d(v) = 7$, then we have $m_1(v) = \frac{2}{7}$ by applying R6 once, $m_2(v) = \frac{3}{14}$ by applying R7 once, $m_4(v) = 2 \times \frac{3}{14} = \frac{3}{7}$ by applying R7 twice and $m_3(v) = m_5(v) = 0$. Therefore, $w'(v) = d(v) - \frac{2}{7}n_1(v) - \frac{3}{14}n_2(v) - \frac{3}{7}n_4(v) - 6 = 1 - \frac{1}{7}(2n_1(v) + 2n_2(v) + n_3(v) + 3n_4(v) + n_5(v)) + \frac{1}{14}n_2(v) + \frac{1}{7}n_3(v) + \frac{1}{7}n_5(v) \geq 1 - \frac{1}{7} \times 7 = 0$. If $8 \leq d(v) \leq 9$, using a similar argument as above, we also have $w'(v) = d(v) - \frac{1}{2}n_1(v) - \frac{3}{8}n_2(v) - \frac{3}{4}n_4(v) - 6$ $= d(v) - 6 - \frac{1}{4}(2n_1(v) + 2n_2(v) + n_3(v) + 3n_4(v) + n_5(v)) + \frac{1}{8}n_2(v) + \frac{1}{4}n_3(v) + \frac{1}{4}n_5(v) \geq d(v) - 6 - \frac{1}{4}d(v) = \frac{3}{4}d(v) - 6 \geq 0$ by applying the mentioned rules.

Case 3.2 Suppose $10 \leq d(v) \leq 14$ (namely, v is a sub-big vertex). Note that now, not only R6 and R7 but also R4 and R5 are involved in the charge transformations of v. If $F(v)$ contains a 2-cluster, then by R7, v shall transfer $\frac{11}{28}$ to y (see Figure 1). Besides, another (at most) $\frac{1}{28}$ should also be sent out from v through xy by applying R5 (note that $t(y) \leq 3$). It follows that $m_2(v) \leq \frac{11}{28} + \frac{1}{28} = \frac{3}{7}$. If $F(v)$ contains a 3-cluster, then v shall send out at most $\frac{1}{7}$ through xy by R4. So $m_3(v) \leq \frac{1}{7}$. Similarly, we can prove that $m_4(v) \leq 2 \times \frac{11}{28} + 2 \times \frac{1}{28} = \frac{6}{7}$ by applying R7 twice and R5 at most twice, and $m_5(v) = 0$. The last and most complicated case is when $F(v)$ contains a copy of 1-cluster. In such a case, v shall firstly transfer a charge $\frac{5}{7}$ to y by R6. Secondly, if $t(y) \leq 3$, then by applying R5 at most twice, v shall also totally transfer at most $2 \times \frac{1}{28} = \frac{1}{14}$ through xy and yz. In the next, we claim that R5 can only be applied at most once for v when $t(y) = 4$ (see Figure 1). Denote w to be the fourth neighbor of y (namely, vw and xz are both false edges in G containing a crossing point y). If now R5 has been applied twice for v, then by the definition of the mentioned rule, $wx, wz \in E(G^\times)$ and $d(x) = d(z) = d(w) = 7$. Furthermore, xw must be contained in a 3-cycle in G, say xww', such that $w' \neq z$ and $d(w') = 7$. Therefore, the graph $[w; z; x, w']$ forms a copy of $K_1 \vee (K_1 \cup K_2)$ in G with all its incident vertices being of degree 7, a contradiction. So v can only transfer just one $\frac{1}{14}$ through either xy or yz by applying R5 at most once. Thus in either case, we uniformly have

$m_1(v) \le \frac{5}{7} + \frac{1}{14} = \frac{11}{14}$. Hence, we have $w'(v) \ge d(v) - \frac{11}{14}n_1(v) - \frac{3}{7}n_2(v) - \frac{1}{7}n_3(v) - \frac{6}{7}n_4(v) - 6 = d(v) - 6 - \frac{11}{28}(2n_1(v) + 2n_2(v) + n_3(v) + 3n_4(v) + n_5(v)) + \frac{5}{14}n_2(v) + \frac{1}{4}n_3(v) + \frac{9}{28}n_4(v) + \frac{11}{28}n_5(v) \ge d(v) - 6 - \frac{11}{28}d(v) = \frac{17}{28}d(v) - 6 > 0$ for $d(v) \ge 10$.

Case 3.3 Suppose $d(v) \ge 15$ (namely, v is a big vertex). Then by a similar argument as in Case 3.2, we can show that $m_1(v) \le 1 + 2 \times \frac{1}{8} = \frac{5}{4}$ by applying R6 once and R5 at most twice (note that here v is allowed to transfer two copies of $\frac{1}{8}$ through both xy and yz even when $t(y) = 4$, which is different to the discussion in Case 3.2); $m_2(v) \le \frac{5}{8} + \frac{1}{8} = \frac{3}{4}$ by respectively applying R7 and R5 at most once; $m_3(v) \le \frac{1}{4}$ by applying R4 at most once; and $m_4(v) \le 2 \times \frac{5}{8} + 2 \times \frac{1}{8} = \frac{3}{2}$ by respectively applying R7 and R5 at most twice. For 5-clusters, it is still trivial that $m_5(v) = 0$. Therefore,

$$\begin{aligned}
w'(v) &\ge d(v) - \frac{5}{4}n_1(v) - \frac{3}{4}n_2(v) - \frac{1}{4}n_3(v) - \frac{3}{2}n_4(v) - 6 \\
&= d(v) - 6 - \frac{5}{8}(2n_1(v) + 2n_2(v) + n_3(v) + 3n_4(v) + n_5(v)) + \frac{3}{8}(n_2(v) + n_3(v) \\
&\quad + n_4(v) + n_5(v)) + \frac{1}{8}n_2(v) + \frac{1}{4}n_5(v) \\
&\ge d(v) - 6 - \frac{5}{8}d(v) + \frac{3}{8}(n_2(v) + n_3(v) + n_4(v) + n_5(v)) \\
&= \frac{3}{8}d(v) - 6 + \frac{3}{8}(n_2(v) + n_3(v) + n_4(v) + n_5(v)) \ge 0
\end{aligned}$$

for $d(v) \ge 16$. Finally, if $d(v) = 15$, then there is at least one of i-cluster $(2 \le i \le 5)$ in $F(v)$ since any two false vertices can not be adjacent in $G^{\times}$. So we have $n_2(v) + n_3(v) + n_4(v) + n_5(v) \ge 1$, which yields that

$$\begin{aligned}
w'(v) &\ge \frac{3}{8}d(v) - 6 + \frac{3}{8}(n_2(v) + n_3(v) + n_4(v) + n_5(v)) \\
&\ge \frac{3}{8} \times 15 - 6 + \frac{3}{8} = 0
\end{aligned}$$

for $d(v) = 15$ in final.

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