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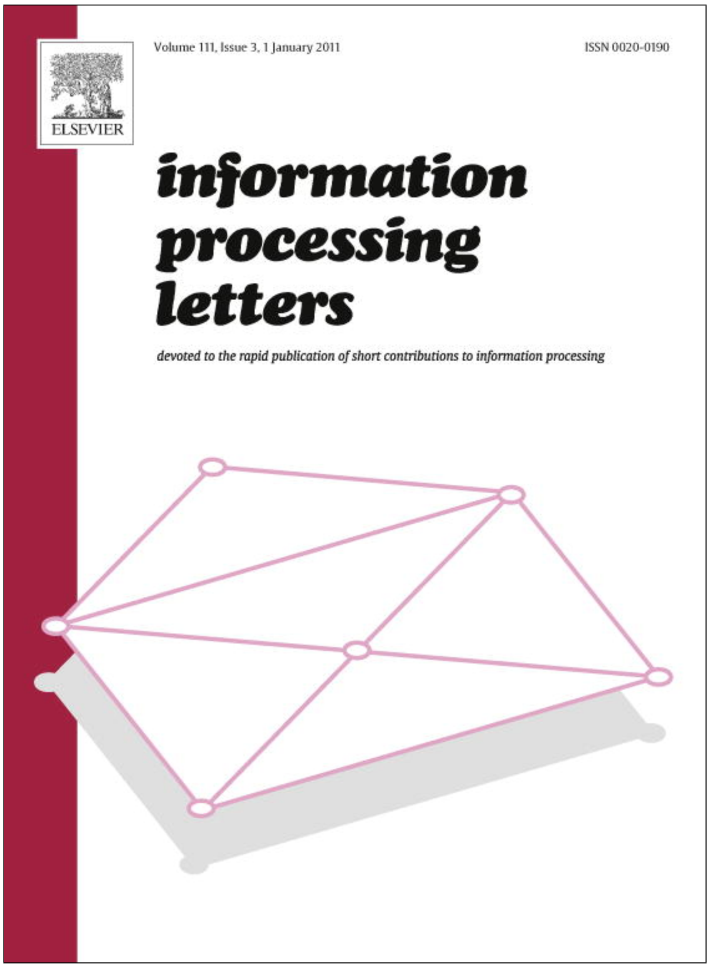

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On edge colorings of 1-planar graphs without adjacent triangles ☆

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ABSTRACT

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is proved that every 1-planar graph without adjacent triangles and with maximum degree $\Delta \geqslant 8$ can be edge-colored with Δ colors.

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1. Introduction

We consider only finite, simple and undirected graphs in this paper unless stated otherwise. For a plane graph G, which is a particular drawing of a planar graph in the Euclidean plane, we use $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$ to denote the set of vertices, the set of edges, the set of faces, the minimum degree and the maximum degree of G, respectively. For an element $x \in V(G) \cup F(G)$, $d_G(x)$ denotes the degree of x in G. Throughout this paper, a k-, $\geqslant k$- and $\leqslant k$-vertex (resp. face) is a vertex (resp. face) of degree k, at least k and at most k. For other undefined notations, we refer the readers to the reference of West [8].

A *proper edge coloring* of a graph is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. The smallest number of colors needed in a proper edge coloring of a graph G is the *chromatic index*, denoted by $\chi'(G)$. The well-known Vizing's theorem tells us that $\chi'(G)$ equals to either $\Delta(G)$ or $\Delta(G) + 1$ for every graph G. This theorem divides all graphs into two classes: *Class 1* graphs have $\chi'(G) = \Delta(G)$ and *Class 2* graphs have $\chi'(G) = \Delta(G) + 1$. For planar graphs, in [6], Vizing presented examples of planar graphs of Class 2 with maximum degree Δ for each $\Delta \in \{2, 3, 4, 5\}$, proved that every planar graph with maximum degree $\Delta \geqslant 8$ is of Class 1, and conjectured that the conclusion holds for the cases $\Delta \in \{6, 7\}$, which is known as Vizing's Planar Graph Conjecture. This conjecture was verified for $\Delta = 7$ by Sanders and Zhao [5] and Zhang [9], independently. It remains open for $\Delta = 6$. However, the case $\Delta = 6$ has been settled for some special graphs. For example, Bu and Wang [1] showed that every planar graph without adjacent triangles and with maximum degree 6 is of Class 1.

In this paper, we focus on 1-planar graphs. A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. This notion of 1-planar graphs was introduced by Ringel [4] while trying to simultaneously color the vertices and faces of a plane graph G such that any pair of adjacent/incident elements receive different colors. The first result concerning the edge colorings of 1-planar graphs is due to Zhang and Wu [11], who proved that every 1-planar graph with maximum degree $\Delta \geqslant 10$ is of Class 1. Recently, Zhang and Liu [12] (also in [10]) constructed Class 2 1-planar graphs with maximum degree 6 or 7. This fact along with Vizing's results stated above imply the following proposition.

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Proposition 1. *There are* 1*-planar graphs of Class* 2 *with maximum degree* Δ *for each* $\Delta \leqslant 7$.

In many papers such as [2], it was shown that every 1-planar graph contains a vertex of degree at most 7. This fact along with Proposition 1 motivate us to do the following conjecture.

Conjecture 2. *Every* 1*-planar graphs with maximum degree at least* 8 *is of Class* 1.

In this paper, we confirm Conjecture 2 for 1-planar graphs without adjacent triangles.

2. Main results and their proofs

From now on, for any 1-planar graph G, we always assume that G has been embedded on a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible; thus we consider G being a *1-plane graph*. The *associated plane graph* $G^\times$ of a 1-plane graph G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is called *false* if it is not a vertex of G and *true* otherwise. We call a 3-face in $G^\times$ *false* or *true* according to whether it is incident with a false vertex or not. In [11], the authors showed some basic properties on a 1-plane graph G and its associated plane graph $G^\times$.

Lemma 3. *(See [11].) Let G be a* 1*-plane graph and let $G^\times$ be its associated plane graph. Then the following hold*:

1. *For any two false vertices u and v in $G^\times$, $uv \notin E(G^\times)$.*
2. *If there is a* 3*-face $uvwu$ in $G^\times$ such that $d_G(v) = 2$, then u and w are both true vertices.*
3. *If $d_G(u) = 3$ and v is a false vertex in $G^\times$, then either $uv \notin E(G^\times)$ or uv is not incident with two 3-faces.*
4. *If a 3-vertex v in G is incident with two 3-faces and adjacent to two false vertices in $G^\times$, then v must also be incident with a ($\geqslant 5$)-face.*
5. *For any* 4*-vertex u in G, u is incident with at most three false* 3*-faces.*

Now we turn our attention to 1-planar graphs without adjacent triangles and prove the following lemma.

Lemma 4. *Let G be a* 1*-plane graph without adjacent triangles and let $G^\times$ be its associated plane graph. For every vertex $v \in V(G)$, if $d_G(v) \geqslant 5$, then v is incident with at most $\lfloor \frac{4}{5} d_G(v) \rfloor$* 3*-faces in $G^\times$.*

Proof. If $d_G(v) = 5$ and v is incident with five 3-faces in $G^\times$, then one can easily find two adjacent triangles in G. So we assume that $d_G(v) \geqslant 6$. In the following, we just need to prove that there are no five consecutive 3-faces that are incident with v. Otherwise, consider such five 3-faces vv_iv_{i+1} in $G^\times$, where $1 \leqslant i \leqslant 5$. We first suppose that v_1 is a true vertex in $G^\times$. If v_2 is true now, then v_3 shall be false since otherwise vv_1v_2 and vv_2v_3 are two adjacent triangles in G. But by (1) of Lemma 3, v_4 would be true once v_3 is false. This implies that vv_1v_2 and vv_2v_4 are two adjacent triangles in G, a contradiction. So v_2 must be false and then v_3 shall be true. By a same argument as above, this is also impossible. So we have to assume that v_1 is false. Similarly we can show that v_2 is a false vertex too. However, since $v_1v_2 \in E(G^\times)$, v_2 must be true by (1) of Lemma 3. This contradiction completes the proof. □

Before proving our main result, we introduce some lemmas on Δ-critical graphs. Recall that G is *Δ-critical* if G is a graph with maximum degree Δ and G is of Class 2, but $G - e$ is of Class 1 for every edge $e \in E(G)$. In the following, we use $N_G(x)$ to denote the set of all neighbors of x in G. For $x, y \in V(G)$, $N_G(x, y) = N_G(x) \cup N_G(y)$ and in general, for any set $S \subseteq V(G)$, let $N_G(S) = \bigcup_{v \in S} N_G(v)$.

Lemma 5 *(Vizing's Adjacency Lemma). (See [7].) Let G be a Δ-critical graph and let v, w be adjacent vertices of G with $d_G(v) = k$. Then*

1. *If $k < \Delta$, then w is adjacent to at least $(\Delta - k + 1)$ Δ-vertices.*
2. *If $k = \Delta$, then w is adjacent to at least two Δ-vertices.*

Lemma 6. *(See [5,9].) Let G be a Δ-critical graph and let xy be an edge in G with $d_G(x) + d_G(y) = \Delta + 2$. Then the following hold*:

1. *Every vertex of $N_G(x, y) \backslash \{x, y\}$ is a Δ-vertex.*
2. *Every vertex of $N_G(N_G(x, y)) \backslash \{x, y\}$ is of degree at least $\Delta - 1$.*
3. *If $\max\{d_G(x), d_G(y)\} < \Delta$, then every vertex of $N_G \times (N_G(x, y)) \backslash \{x, y\}$ is a Δ-vertex.*

Lemma 7. *(See [3].) Let G be a Δ-critical graph with $\Delta \geqslant 6$ and let x be a* 4*-vertex. Then the following hold*:

1. *If x is adjacent to a $(\Delta - 2)$-vertex, say y, then every vertex of $N_G(N_G(x)) \backslash \{x, y\}$ is a Δ-vertex.*
2. *If x is not adjacent to any $(\Delta - 2)$-vertex and if one of the neighbors y of x is adjacent to $d_G(y) - (\Delta - 3) \leqslant (\Delta - 2)$-vertices, then each of the other three neighbors of x is adjacent to only one $\leqslant (\Delta - 2)$-vertex, which is x.*
3. *If x is adjacent to a $(\Delta - 1)$-vertex, then there are at least two Δ-vertices in $N_G(x)$ which are adjacent to at most two $\leqslant (\Delta - 2)$-vertices. Moreover, if x is adjacent to two $(\Delta - 1)$-vertices, then each of the two Δ-neighbors of adjacent to exactly one $\leqslant (\Delta - 2)$-vertex, which is x.*

Theorem 8. *Let G be a* 1*-planar graph without adjacent triangles. If $\Delta(G) \geqslant 8$, then $\chi'(G) = \Delta(G)$.*

Proof. Since every 1-planar graph with maximum degree $\Delta \geqslant 10$ has chromatic index Δ (see Theorem 7 of [11]), we assume that $8 \leqslant \Delta(G) \leqslant 9$ in the following proof. Suppose that G is a counterexample to the theorem with the smallest number of edges. Then G is a $\Delta(G)$-critical 1-plane graph. By Vizing's Adjacency Lemma (VAL for short), we have $\delta(G) \geqslant 2$. Now we assign an initial charge c

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on $V(G) \cup F(G^\times)$ by letting $c(v) = d_G(v) - 4$ for every $v \in V(G)$ and $c(f) = d_{G^\times}(f) - 4$ for every $f \in F(G^\times)$. Note that $G^\times$ is a planar graph, by Euler's formula, we can easily deduce that

$$\begin{aligned}\sum_{x \in V(G)\cup F(G^\times)} c(x) &= \sum_{v\in V(G)} \big(d_G(v) - 4\big) + \sum_{f \in F(G^\times)} \big(d_{G^\times}(f) - 4\big) \\ &= \sum_{v\in V(G^\times)} \big(d_{G^\times}(v) - 4\big) + \sum_{f \in F(G^\times)} \big(d_{G^\times}(f) - 4\big) \\ &= -8 \end{aligned} \tag{2.1}$$

Using Lemmas from 3 to 7, we redistribute the charges of vertices in G and faces in $G^\times$ according to Rules 1–9 below and also check that the final charge c' on each vertex and each face is nonnegative. Since our rules only move charge around and do not affect the total charges, this leads to a contradiction to (2.1) in final and completes our proof. Let c_i denote the charge after applying Rules 1–i. So we have $c'(x) = c_9(x)$ for every element $x \in V(G) \cup F(G^\times)$. If $x \in V(G)$, we denote the degree of the neighbors of x in G as $\delta_1(x) \leqslant \delta_2(x) \leqslant \cdots \leqslant \delta_{d_G(x)}(x)$. By V_i, we denote the set of all i-vertices in G. In the following, we use Δ instead of $\Delta(G)$ for simplicity and say u is a *k-neighbor* of v if $uv \in E(G)$ and $d_G(u) = k$.

2.1. Final charges of faces and vertices $v \in \bigcup_{i=2}^{7} V_i$

Rule 1. Let f be a false 3-face in $G^\times$. Then f receives $\frac{1}{2}$ from each of its incident true vertices.

Rule 2. Let f be a true 3-face in $G^\times$. Then f receives $\frac{1}{2}$ from each of its incident ($\geqslant 5$)-vertices.

Rule 3. Let f be a ($\geqslant 5$)-face in $G^\times$ and let t the number of 3-vertices incident with f. Then each 3-vertex incident with f receives $\frac{d_{G^\times}(f)-4}{t}$ from f.

Note that 4-faces are not involved in the above rules, so $c_3(f) = c(f) = 0$ for every 4-face in $G^\times$. By (1) of Lemma 3 and VAL, every false 3-face in $G^\times$ is incident with two true vertices and every true 3-face in $G^\times$ is incident with at least two $\geqslant 5$-vertices. So by Rules 1 and 2, we have $c_2(f) \geqslant 0$ for every 3-face f in $G^\times$. By Rule 3, one can easily see that $c_3(f) = 0$ for every $\geqslant 5$-face. Since faces in $G^\times$ participate only in Rules 1–3 (the readers can make themselves sure of that), $c'(f) \geqslant 0$ for every face $f \in F(G^\times)$.

Rule 4. Let v be a 2-vertex in G. Then v receives 1 from each of its neighbors in G.

By (2) of Lemma 3, v is incident with no false 3-faces in $G^\times$. So v sends out none by Rules 1–4 and thus $c_4(v) = -2 + 2 \times 1 = 0$. Since 2-vertices participate only in Rule 4, $c'(v) = c_4(v) = 0$ for every vertex $v \in V_2$.

Rule 5. Let v be a 3-vertex in G. Then v receives $\frac{1}{2}$ from each of its neighbors in G.

By (3) and (4) of Lemma 3, if v is incident with two false 3-faces in $G^\times$, then v shall also be incident with a $\geqslant 5$-face f, from which v receives at least $\frac{d_{G^\times}(f)-4}{t} \geqslant \frac{d_{G^\times}(f)-4}{\lfloor d_{G^\times}(f)/2\rfloor} \geqslant \frac{1}{2}$ by Rule 3, because no two 3-vertices are adjacent in G by VAL. So we have $c_5(v) \geqslant -1 - \max\{\frac{1}{2}, 2 \times \frac{1}{2} - \frac{1}{2}\} + 3 \times \frac{1}{2} = 0$. Since 3-vertices participate only in Rules 1, 3 and 5, $c'(v) = c_5(v) \geqslant 0$ for every vertex $v \in V_3$.

Rule 6. Let v be a 4-vertex in G. Then we divide this rule into six independent parts.

Rule 6.1. If $\delta_1(v) = \Delta - 2$, then v receives $\frac{1}{2}$ from each of its Δ-neighbors in G.

By (5) of Lemma 3, v is incident with at most three false 3-faces. Since v has a $(\Delta - 2)$-neighbor here, another three neighbors of v must be Δ-vertices by (1) of Lemma 6. So we have $c_6(v) \geqslant 0 - 3 \times \frac{1}{2} + 3 \times \frac{1}{2} = 0$ by Rules 1 and 6.1.

Rule 6.2. If $\delta_1(v) = \Delta - 1$ and v is adjacent to two $(\Delta - 1)$-vertices, then v receives $\frac{2}{3}$ from each of its Δ-neighbors and $\frac{1}{12}$ from each of its $(\Delta - 1)$-neighbors in G.

If this rule is called, then by VAL, v has exactly two $(\Delta - 1)$-neighbors and two Δ-neighbors in G. So $c_6(v) \geqslant 0 - 3 \times \frac{1}{2} + 2 \times \frac{2}{3} + 2 \times \frac{1}{12} = 0$.

Rule 6.3. If $\delta_1(v) = \Delta - 1$, v is adjacent to exactly one $(\Delta - 1)$-vertex and some Δ-neighbor y of v is adjacent to three $\leqslant (\Delta - 2)$-vertices, then v receives $\frac{1}{3}$ from y and $\frac{2}{3}$ from each of its another two Δ-neighbors in G.

If this rule is called, then by (2) of Lemma 7, each Δ-neighbor of v except y is adjacent to only one $\leqslant (\Delta - 2)$-vertex, which is v. So this rule is defined properly and thus we have $c_6(v) \geqslant 0 - 3 \times \frac{1}{2} + \frac{1}{3} + 2 \times \frac{2}{3} > 0$.

Rule 6.4. If $\delta_1(v) = \Delta - 1$, v is adjacent to exactly one $(\Delta - 1)$-vertex and every Δ-neighbor of v is adjacent to at most two $\leqslant (\Delta - 2)$-vertices, then v receives $\frac{5}{12}$ from each of its Δ-neighbors and $\frac{1}{4}$ from each of its $(\Delta - 1)$-neighbors in G.

In this case, v has three Δ-neighbors and one $(\Delta - 1)$-neighbor. So $c_6(v) \geqslant 0 - 3 \times \frac{1}{2} + 3 \times \frac{5}{12} + \frac{1}{4} = 0$.

Rule 6.5. If $\delta_1(v) = \Delta$ and some Δ-neighbor y of v is adjacent to three $\leqslant (\Delta - 2)$-vertices, then v receives $\frac{2}{3}$ from each of its Δ-neighbors in G except y.

By a similar argument as in Rule 6.3, there is only one Δ-neighbor of v which has three $\leqslant (\Delta - 2)$-neighbors. So the definition of this rule is also proper and thus $c_6(v) \geqslant 0 - 3 \times \frac{1}{2} + 3 \times \frac{2}{3} > 0$.

Rule 6.6. If $\delta_1(v) = \Delta$ and every Δ-neighbor of v is adjacent to at most two $\leqslant (\Delta - 2)$-vertices, then v receives $\frac{5}{12}$ from each of its Δ-neighbors in G.

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In this case, it is clear that $c_6(v) \geqslant 0 - 3 \times \frac{1}{2} + 4 \times \frac{5}{12} > 0$. By VAL, one can easily find that every 4-vertex in G would satisfy (only) one of the conditions among those stated in the above six subrules. Note that 4-vertices participate only in Rules 1 and 6, we have $c'(v) = c_6(v) \geqslant 0$ for every vertex $v \in V_4$.

Rule 7. Let v be a 5-vertex in G. Then v receives $\frac{1}{4}$ from each of its $\geqslant (\Delta - 2)$-neighbors in G.

By Lemma 4 and VAL, v is incident with at most four 3-faces in $G^\times$ and is adjacent to at least four $\geqslant (\Delta - 2)$-vertices in G. So by Rules 1, 2 and 7, $c_7(v) \geqslant 1 - 4 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0$. Since 5-vertices participate only in above three rules, we have $c'(v) = c_7(v) \geqslant 0$ for every vertex $v \in V_5$.

Rule 8. Let v be a 6-vertex in G. Then v receives $\frac{1}{8}$ from each of its Δ-neighbors in G.

By Lemma 4, v is incident with at most four 3-faces in $G^\times$. Note that v may send charges to its adjacent vertices in G only by Rule 7. If $\delta_1(v) \leqslant 5$, then by VAL, v is adjacent to at most two 5-vertices and is adjacent to at least four Δ-vertices in G, which implies that $c_8(v) \geqslant 2 - 4 \times \frac{1}{2} - 2 \times \frac{1}{4} + 4 \times \frac{1}{8} = 0$ by Rules 1, 2, 7 and 8. If $\delta_1(v) \geqslant 6$, then it is trivial that $c_8(v) \geqslant 2 - 4 \times \frac{1}{2} = 0$ by Rules 1 and 2. Since 6-vertices will not participate in the following Rule 9, $c'(v) = c_8(v) \geqslant 0$ for every vertex $v \in V_6$.

Rule 9. Let v be a 7-vertex in G. Then v receives $\frac{1}{6}$ from each of its Δ-neighbors in G.

By Rules 5, 6 and 7, v may send at most $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ to each of its adjacent 3-vertices, 4-vertices and 5-vertices, respectively. By Lemma 4 and VAL, v is incident with at most five 3-faces in $G^\times$ and is adjacent to at most i ($\leqslant 5$)-vertices in G if $\delta_1(v) = i + 2$, where $1 \leqslant i \leqslant 3$. On the other hand, v is also adjacent to at least two Δ-vertices in G by VAL, from which v receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ by Rule 9. So by Rules 1, 2, 5, 6 and 7, $c'(v) = c_9(v) \geqslant 3 - 5 \times \frac{1}{2} - \max\{\frac{1}{2}, 2 \times \frac{1}{4}, 3 \times \frac{1}{4}\} + \frac{1}{3} > 0$. This implies that $c'(v) \geqslant 0$ for every vertex $v \in V_7$.

2.2. Final charges of vertices $v \in \bigcup_{i=8}^{\Delta} V_i$

Claim. *The final charge of every* 8*-vertex is nonnegative.*

Proof. Note that a 8-vertex v may be Δ-vertex here. So v may participate in all of the above nine discharging rules. Without loss of generality, we just assume that $\Delta = 8$ here, because the case $\Delta = 9$ can be dealt with much more easily. By Lemma 4, v is incident with at most six 3-faces in $G^\times$. So we have $c_3(v) \geqslant 4 - 6 \times \frac{1}{2} = 1$. If $\delta_1(v) = 2$, then by VAL, v is adjacent to seven Δ-vertices in G. So v participates just in Rule 4 among the last six rules. This implies that $c'(v) = c_3(v) - 1 \geqslant 0$. If $\delta_1(v) \geqslant 5$, then by VAL, Rules 7, 8 and 9 we can also obtain $c'(v) \geqslant c_3(v) - \max\{4 \times \frac{1}{4}, \frac{1}{8} + 4 \times \frac{1}{6}, 6 \times \frac{1}{6}\} \geqslant 0$. So we leave only two cases here: one is $\delta_1(v) = 3$ and the other is $\delta_1(v) = 4$.

Case 1. $\delta_1(v) = 3$.

By VAL, v is adjacent to at least six Δ-vertices now. If $\delta_2(v) = 3$ or $\delta_2(v) \geqslant 5$, then by Rules 5, 7, 8 and 9, we still have $c'(v) \geqslant c_3(v) - 2 \times \frac{1}{2} \geqslant 0$. So we shall assume that $\delta_2(v) = 4$ here. Let u and w be the 3-vertex and the 4-vertex adjacent to v in G, respectively. If v sends some charge to w by Rule 6.1 or Rule 6.2, then by (3) of Lemma 6 or (3) of Lemma 7, all neighbors of v except w shall be of degree at least $\Delta - 1$, a contradiction to $d_G(u) = 3$ and $uv \in E(G)$. If v sends some charge to w by Rule 6.3 (note that v is adjacent to only two $\leqslant (\Delta - 2)$-vertices in G), then by (2) of Lemma 7, v is adjacent to only one $\leqslant (\Delta - 2)$-vertex in G, which is w, a contradiction. Similarly, if v sends some charge to w by Rule 6.5, then we would also obtain a same contradiction. Thus v may only send at most $\frac{5}{12}$ to w by Rule 6.4 or Rule 6.6. This implies that $c'(v) \geqslant c_3(v) - \frac{1}{2} - \frac{5}{12} > 0$ by Rule 5.

Case 2. $\delta_1(v) = 4$.

By VAL, v is adjacent to at least five Δ-vertices now. Let u be a 4-vertex that is adjacent to v in G. If v sends some charge to u by Rule 6.1, then by (3) of Lemma 6, v is adjacent to seven Δ-vertices in G, which implies that $c'(v) \geqslant c_3(v) - \frac{1}{2} > 0$. If v sends some charge to u by Rule 6.2, then by (3) of Lemma 7, we have $\delta_2(v) \geqslant \Delta - 1 = 7$. So by Rule 9, we have $c'(v) \geqslant c_3(v) - \frac{2}{3} - 2 \times \frac{1}{6} \geqslant 0$. If v sends some charge to u by Rule 6.3, then we will consider two subcases. First, suppose that v is adjacent to three $\leqslant (\Delta - 2)$-vertices in G. Then v will send $\frac{1}{3}$ to u. If now v is also adjacent to another 4-vertex w in G, then by a same argument as in Case 1, v will not send any charge to w by one of Rules 6.1, 6.2, 6.3 and 6.5. However, since v is adjacent to three $\leqslant (\Delta - 2)$-vertices in G, v will also not send any charge to w by Rule 6.4 or Rule 6.6. That is to say, v sends no charges to its 4-neighbors except u. So by Rules 7, 8 and 9, we have $c'(v) \geqslant c_3(v) - \frac{1}{3} - 2 \times \frac{1}{4} > 0$. Second, suppose that v is adjacent to at most two $\leqslant (\Delta - 2)$-vertices in G. Then v will send $\frac{2}{3}$ to u and by (2) of Lemma 7 we shall also assume that $\delta_2(v) \geqslant \Delta - 1 = 7$. This implies that $c'(v) \geqslant c_3(v) - \frac{2}{3} - 2 \times \frac{1}{6} \geqslant 0$ by Rule 9. If v sends some charge to u by Rule 6.4, then by the condition of that rule, v is adjacent to at most two $\leqslant (\Delta - 2)$-vertices in G. Suppose v is adjacent to another 4-vertex w in G. Then by a same argument as in Case 1, v can send at most $\frac{5}{12}$ to w only by Rule 6.4 or Rule 6.6. This implies that $c'(v) \geqslant c_3(v) - 2 \times \frac{5}{12} - \frac{1}{6} \geqslant 0$ by Rule 9. On the other hand, we still have $c'(v) \geqslant c_3(v) - \frac{5}{12} - \frac{1}{4} - \frac{1}{6} > 0$ by Rules 7, 8 and 9 if $\delta_2(v) \geqslant 5$. If v sends some charge to u by Rule 6.5, then by (2) of Lemma 7, v is adjacent to only one $\leqslant (\Delta - 2)$-vertex in G, which is u. This implies that $c'(v) \geqslant c_3(v) - \frac{2}{3} - 2 \times \frac{1}{6} \geqslant 0$ by Rules 6.5 and 9. If v sends some charge to u by Rule 6.6, then by a same argument as in the case that v sends some charge to u by Rule 6.4, we can also obtain $c'(v) \geqslant 0$. So at last we assume that v will not send any charge to its adjacent 4-vertices. This easiest case implies that $c'(v) \geqslant c_3(v) - 2 \times \frac{1}{4} > 0$ by Rules 7, 8 and 9 in final. □

Note that we have already completed the proof of the theorem for the case $\Delta = 8$. In fact, if $\Delta = 9$, then we can

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prove that $c'(v) \geqslant 0$ for every $v \in V_9$ by a same argument as in the above claim. Much more easily, we can also show that the final charge of every 8-vertex in G is nonnegative. Hence the proof of the theorem completes here. □

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