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RESEARCH ARTICLE

List edge and list total coloring of 1-planar graphs

Xin ZHANG, Jianliang WU, Guizhen LIU

School of Mathematics, Shandong University, Jinan 250100, China

Abstract A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is proved that each 1-planar graph with maximum degree Δ is $(\Delta+1)$-edge-choosable and $(\Delta+2)$-total-choosable if $\Delta \geqslant 16$, and is Δ-edge-choosable and $(\Delta+1)$-total-choosable if $\Delta \geqslant 21$. The second conclusion confirms the list coloring conjecture for the class of 1-planar graphs with large maximum degree.

Keywords 1-planar graph, list coloring conjecture, discharging
MSC 05C10, 05C15

1 Introduction

In this paper, all graphs are finite, simple, and undirected. For a graph G, denote by $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$ the vertex set, the edge set, the minimum degree, and the maximum degree of G, respectively. Let $e(G) = |E(G)|$ and $v(G) = |V(G)|$. For plane graphs, we use $F(G)$ to denote the face set of G and let $f(G) = |F(G)|$. A vertex (resp. face) of degree k is called a *k-vertex* (resp. *k-face*) while a vertex (resp. face) of degree at least k is called a *k^+-vertex* (resp. *k^+-face*). For undefined concepts we refer the reader to [2].

A graph G is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planar graph was introduced by Ringel [16] while studying the simultaneous vertex-face coloring of plane graphs. Indeed, once we are given a plane graph G, a 1-planar graph G' can be constructed as follows. First of all, let $V(G') = V(G) \cup F(G)$. For any two vertices $x, y \in V(G')$, if their corresponding elements are adjacent or incident in G, then we add an edge xy in G'. For example, the graph G in Fig. 1 is a plane graph with six vertices and five faces, and the graph G', which contains eleven vertices, is a 1-planar graph constructed by the above definition. Note

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Corresponding author: Guizhen LIU, E-mail: gzliu@sdu.edu.cn

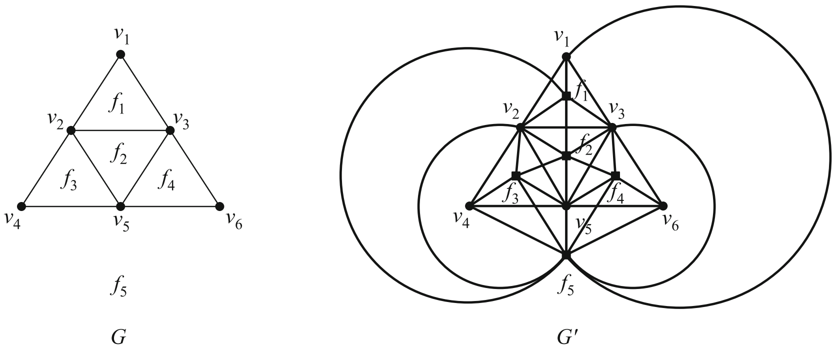

Fig. 1 1-planar graph G' is constructed from plane graph G

that in G the face f_1 is incident with three vertices v_1, v_2, v_3 and adjacent to two faces f_2, f_5, so in G' the vertex f_1 is of degree five with its neighbors being v_1, v_2, v_3, f_2 and f_5. One can find that the vertex-face coloring of the plane graph G is just equivalent to the vertex coloring of the 1-planar graph G'.

Ringle [16] proved that seven colors suffice both to simultaneously color the vertices and faces of each plane graph and thus to color the vertices of each 1-planar graph, and conjectured that six colors are enough and showed that this bound is best possible if it is true. This conjecture was confirmed by Borodin [3,5] and a list analogue of vertex coloring of 1-planar graphs was investigated by Albertson and Mohar [1], and by Wang and Lih [19]. Borodin [7] also proved that each 1-planar graph is (list) acyclically 20-colorable. Zhang et al. showed that each 1-planar graph G with maximum degree Δ is edge Δ-colorable if $\Delta \geqslant 10$ [27], or $\Delta \geqslant 9$ and G contains no chordal 5-cycles [22], or $\Delta \geqslant 8$ and G contains no chordal 4-cycles [21], or $\Delta \geqslant 7$ and G contains no 3-cycles [23]. Zhang et al. also showed that the $(p, 1)$-total labelling number of each 1-planar graph G is at most $\Delta(G) + 2p - 2$ if $\Delta(G) \geqslant 8p + 4$ [29], and the linear arboricity of each 1-planar graph G is exactly $\lceil \Delta(G)/2 \rceil$ if $\Delta(G) \geqslant 33$ [25]. On the other hand, the local structures (including the girth, the lightness, and the embedding, etc.) of 1-planar graphs were extensively studied by many authors including [6,9,12,17,24,26,28]. However, comparing to the family of planar graphs, the family of 1-planar graphs is still little explored.

Let

$$f\colon\ E(G) \cup V(G) \to \mathbb{N}$$

be a function into positive integers. We say that G is *total-f-choosable* if, whenever we are given a list A_x of colors with $|A_x| = f(x)$ for each $x \in E(G) \cup V(G)$, we can choose a color from A_x for each element x such that no two adjacent (incident) elements receive the same color. The *list total chromatic index* $\chi''_l(G)$ of G is the smallest integer k such that G is *total-f-choosable* when we assign $f(x) = k$ for each $x \in E(G) \cup V(G)$. The *list edge chromatic index* $\chi'_l(G)$ can be defined similarly. As in [2], we use $\chi'(G)$ and $\chi''(G)$ to denote the ordinary edge and total coloring of a graph G, respectively. Obviously, it holds that

$$\chi'_l(G) \geqslant \chi'(G) \geqslant \Delta(G), \quad \chi''_l(G) \geqslant \chi''(G) \geqslant \Delta(G) + 1.$$

As far as list edge and list total coloring are concerned, Vizing, Gupta, Abertson and Collins, and Bollobás and Harris (see [13] for details) independently possed part (a) of the conjecture as follows, which is well known as *List Edge Coloring Conjecture.* Part (b) of the following conjecture, also known as *List Total Coloring Conjecture*, was posed by Borodin et al. [8].

Conjecture 1 (List Coloring Conjecture) For any graph G,

(a) $\chi'_l(G) = \chi'(G)$;

(b) $\chi''_l(G) = \chi''(G)$.

Although list coloring conjecture has been proved for a few special cases such as outerplanar graphs [18] and planar graphs with maximum degree at least 12 [8], this conjecture remains open.

Combining the well-known Vizing's Theorem and Total Coloring Conjecture as follows with List Coloring Conjecture, Conjecture 3 below is natural but still interesting.

Conjecture 2 (Total Coloring Conjecture) For any graph G,

$$\Delta(G) + 1 \leqslant \chi''(G) \leqslant \Delta(G) + 2.$$

Conjecture 3 For any graph G,

(a) $\chi'_l(G) \leqslant \Delta(G) + 1$;

(b) $\chi''_l(G) \leqslant \Delta(G) + 2$.

An earlier result of Harris showed that $\chi'_l(G) \leqslant 2\Delta(G) - 2$ if G is a graph with $\Delta(G) \geqslant 3$ [10]. This result implies (a) of Conjecture 3 for the case $\Delta(G) = 3$. Juvan et al. settled the case of $\Delta(G) = 4$ [15]. For planar graphs, Borodin [4] confirmed (a) of Conjecture 3 for the case $\Delta(G) \geqslant 9$.

Part (b) of Conjecture 3 has been verified for graphs with maximum degree $\Delta(G) \leqslant 3$ and bipartite graphs [14]. Hou, Liu and Wu [11] also showed that it holds for planar graphs with $\Delta(G) \geqslant 9$.

Motivated by the above three conjectures, we consider the case when G is a 1-planar graph with large maximum degree in this paper. The next section is devoted to verify Total Coloring Conjecture and Conjecture 3 for 1-planar graphs with maximum degree at least 16, and to confirm List Coloring Conjecture for 1-planar graphs with maximum degree at least 21.

2 Main results and their proofs

In this section, for a 1-planar graph G, we always assume that G has been drawn on a plane so that

(1) every edge is crossed by at most one other edge;

(2) the number of crossings is as small as possible.

The *associated plane graph* $G^\times$ of G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. We call the new vertices in

$G^\times$ *crossing vertices*. The original edge of G that contains a crossing vertex is called *crossing edge*. One can easily observe that if v is not a crossing vertex, then

$$d_{G^\times}(v) = d_G(v).$$

Therefore, in the following, we do not distinguish the two notations $d_{G^\times}(v)$ and $d_G(v)$ when v is not a crossing vertex, in which case we only use the brief notation $d(v)$ to represent both $d_{G^\times}(v)$ and $d_G(v)$.

To begin with, we display some basic properties of 1-planar graph G and its associated plane graph $G^\times$, which were proved by Zhang and Wu [27]. For a vertex $v \in V(G^\times)$, we use $f_k(v)$ to denote the number of k-faces that are incident with it and use $n_c(v)$ to denote the number of crossing vertices that are adjacent to v in $G^\times$.

Lemma 1 [27] *Let G be a 1-planar graph. Then the following results hold.*

(a) *For any two crossing vertices u and v in $G^\times$, $uv \notin E(G^\times)$.*

(b) *If there is a 3-face $uvwu$ in $G^\times$ such that $d_{G^\times}(v) = 2$, then u and w are not crossing vertices.*

(c) *If a 3-vertex v is incident with two 3-faces and adjacent to two crossing vertices in $G^\times$, then v is also incident with a 5^+-face in $G^\times$.*

(d) *There exists no edge uv in $G^\times$ such that $d_{G^\times}(u) = 3$, v is a crossing vertex, and uv is incident with two 3-faces in $G^\times$.*

Lemma 2 [27] *Let G be a 1-plane graph. Then for every vertex $v \in V(G)$, we have*

$$f_3(v) + n_c(v) \leqslant \begin{cases} 3, & d_G(v) = 3,\ f_3(v) \neq 2; \\ 4, & d_G(v) = 3,\ f_3(v) = 2; \\ 5, & d_G(v) = 4; \\ \left\lfloor \dfrac{3d_G(v)}{2} \right\rfloor, & d_G(v) \geqslant 5. \end{cases}$$

Theorem 1 *Let G be a 1-planar graph with maximum degree $\Delta \geqslant 16$. Then*

$$\chi'_l(G) \leqslant \Delta + 1, \quad \chi''_l(G) \leqslant \Delta + 2.$$

Proof Let G be a minimal counterexample to the theorem. Then G has the following properties.

(a) G is connected.

(b) G contains no edge uv such that

$$\min\{d(u), d(v)\} \leqslant \left\lfloor \frac{\Delta+1}{2} \right\rfloor, \quad d(u) + d(v) \leqslant \Delta + 2.$$

(c) For each integer $3 \leqslant k \leqslant 5$, let

$$X_k = \{x \in V(G) \mid d_G(x) \leqslant k\}, \quad Y_k = \bigcup_{x \in X_k} N_G(x).$$

If $X_k \neq \emptyset$, then there exists a bipartite subgraph $M_k = (X_k, Y_k)$ of G such that $d_{M_k}(x) = 1$ for any $x \in X_k$ and $d_{M_k}(y) \leqslant k-2$ for any $y \in Y_k$. We call y the *k-master* of x if $xy \in M_k$ and $x \in X_k$.

(a) and (b) are easy to be proved. The proof of (c) is just similar to the one in [20], with only quite a little minor changes. Therefore, we omit it here and refer the reader to [20, Lemma 2.4].

From (b), we deduce that $\delta(G) \geqslant 3$. By (c), each i-vertex $(3 \leqslant i \leqslant 5)$ has a j-master $(i \leqslant j \leqslant 5)$. The concept of j-master will be used to define discharging rules in the following proofs.

Now, we apply the discharging method to the associated plane graph $G^{\times}$ of G and complete the proof by a contradiction.

Since $G^{\times}$ is a plane graph, Euler's formula on $G^{\times}$ can be rewritten as

$$\sum_{v \in V(G^{\times})} (d(v)-4) + \sum_{f \in F(G^{\times})} (d(f)-4) = -8.$$

Define $ch(x)$ to be the initial charge of $x \in V(G^{\times}) \cup F(G^{\times})$. Let

$$ch(v) = d(v) - 4, \quad \forall\ v \in V(G^{\times})$$

and let

$$ch(f) = d(f) - 4, \quad \forall\ f \in F(G^{\times}).$$

It follows that

$$\sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch(x) < 0.$$

In the next, we will reassign a new charge, denoted by $ch'(x)$, to each $x \in V(G^{\times}) \cup F(G^{\times})$ according to the following discharging rules. Since our rules only move charge around, and do not affect the sum, we have

$$\sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch'(x) = \sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch(x) < 0.$$

However, we will prove the final charge $ch'(x)$ of every element $x \in V(G^{\times}) \cup F(G^{\times})$ is nonnegative in what follows. This leads to

$$\sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch'(x) \geqslant 0,$$

a contradiction.

A vertex v in G is *small* if $d(v) \leqslant 7$ and is *big* if $d(v) \geqslant 8$. A 3-face in $G^{\times}$ is of *type one* if it is incident with one crossing vertex, one small vertex, and one big vertex, and is of *type two* otherwise. Note that if f is a type two 3-face, then f shall be incident with at least two big vertices because no two small vertices are adjacent in G by Lemma 1 (b) and no two crossing vertices are adjacent in $G^{\times}$ by Lemma 1 (a). Our discharging rules are defined as follows.

Rule 1.1 Each 3-vertex in G receives $1/3$ from its 3-master, $2/3$ from its 4-master, $1/3$ from its 5-master, and sends $1/3$ to each type one 3-face incident with it in $G^\times$.

Rule 1.2 Each 4-vertex in G receives $2/3$ from its 4-master, $1/3$ from its 5-master, and sends $1/3$ to each type one 3-face incident with it in $G^\times$.

Rule 1.3 Each 5-vertex in G receives $1/3$ from its 5-master and sends $1/3$ to each type one 3-face incident with it in $G^\times$.

Rule 1.4 Each 6-vertex in G sends $1/3$ to each type one 3-face incident with it in $G^\times$.

Rule 1.5 Each 7-vertex in G sends $1/2$ to each type one 3-face incident with it in $G^\times$.

Rule 1.6 Each d-vertex in G ($8 \leqslant d \leqslant \Delta - 4$) sends $1/2$ to each 3-face incident with it in $G^\times$.

Rule 1.7 Each d-vertex in G ($\Delta - 3 \leqslant d \leqslant \Delta$) sends $2/3$ to each type one 3-face and $1/2$ to each type two 3-face incident with it in $G^\times$.

Rule 1.8 Each k-face ($k \geqslant 5$) in $G^\times$ sends $(k-4)/t(f)$ to each 3-vertex incident with it, where $t(f)$ is the number of 3-vertices incident with face f.

Clearly, if $d(f) = 4$, then

$$ch'(f) = ch(f) = d(f) - 4 = 0,$$

and if $d(f) \geqslant 5$, then

$$ch'(f) \geqslant ch(f) - t(f)\frac{d(f) - 4}{t(f)} = 0$$

by Rule 1.8. Now, we shall consider the initial charges of 3-faces. First, suppose that f is of type one. Then by (b), if the small vertex v incident with it is of degree i ($3 \leqslant i \leqslant 7$), then f is also incident with one crossing vertex and one big vertex of degree at least $\Delta + 3 - i$. Therefore, by Rules 1.1–1.7, f receives at least 1 from the vertices incident with it and sends out none. Thus,

$$ch'(f) \geqslant ch(f) + 1 = 0.$$

Suppose that f is of type two, i.e., f is incident with at least two big vertices. By Rules 1.6 and 1.7, each big vertex sends $1/2$ to each type two 3-face incident with it, and thus, f receives at least $2 \times 1/2 = 1$. Note that f sends out none. Therefore,

$$ch'(f) \geqslant ch(f) + 1 = 0.$$

Now, we check the initial charge of the vertex $v \in V(G^\times)$ with $d_{G^\times}(v) = d$. Recall that the vertices of $G^\times$ except the crossing ones are just the vertices of G. Suppose $d = 3$. Then v has one 3-master, one 4-master, and one 5-master. Therefore, v receives totally $4/3$ from its masters by Rule 1.1. If $n_c(v) = 3$, then $f_3(v) = 0$ by Lemma 2, and then v sends out none. Therefore,

$$ch'(v) = ch(v) + \frac{4}{3} > 0.$$

If $n_c(v) = 2$, then by Lemma 1 (a), $f_3(v) \leqslant 2$. If $f_3(v) \leqslant 1$, then

$$ch'(v) = ch(v) + \frac{4}{3} - \frac{1}{3} = 0.$$

If $f_3(v) = 2$, then by Lemma 1 (c), v must be incident with a 5^+-face f. It follows that v receives at least $1/2$ from f by Rule 1.8. Therefore,

$$ch'(v) = ch(v) + \frac{4}{3} - \frac{2}{3} + \frac{1}{2} > 0.$$

If $n_c(v) \leqslant 1$, then by Lemma 1 (d), v is incident with at most one type one 3-face. Thus, by R1.1, we have

$$ch'(v) \geqslant ch(v) + \frac{4}{3} - \frac{1}{3} = 0.$$

Suppose $d = 4$. If v is a crossing vertex, then

$$ch'(v) = ch(v) = 0.$$

If not, then by Lemma 2, v is incident with at most three type one 3-faces. Note that v have a 4-master and a 5-master. Therefore, by Rule 1.2, we have

$$ch'(v) \geqslant ch(v) - 3 \times \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = 0.$$

Suppose $d = 5$. Then v is incident with at most four type one 3-faces by Lemma 2. Note that v has a 5-master. Therefore, we have

$$ch'(v) \geqslant ch(v) - 4 \times \frac{1}{3} + \frac{1}{3} = 0$$

by Rule 1.3. Suppose $d = 6$. Then we have

$$ch'(v) \geqslant ch(v) - 6 \times \frac{1}{3} = 0$$

by Rule 1.4. Suppose $d = 7$. Then v is incident with at most six type one 3-faces by Lemma 2. Therefore, we have

$$ch'(v) \geqslant ch(v) - 6 \times \frac{1}{2} = 0$$

by Rule 1.5. Suppose

$$8 \leqslant d \leqslant \Delta - 4.$$

Then by Rule 1.6, we have

$$ch'(v) \geqslant ch(v) - \frac{d(v)}{2} = \frac{d(v) - 8}{2} \geqslant 0.$$

Suppose $d = \Delta - 3$. Then by Rule 1.7, we have

$$ch'(v) \geqslant ch(v) - \frac{2d(v)}{3} = \frac{\Delta - 15}{3} \geqslant 0.$$

In the following, we will deal with the last three cases when $\Delta - 2 \leqslant d \leqslant \Delta$ much more carefully since the vertices of such a degree may give out charges as masters of some vertices. A fan $F = [u; v_1, v_2, \dots, v_n]$ is a graph with

$$V(F) = \{u, v_1, v_2, \dots, v_n\}, \quad E(F) = \{uv_1, uv_2, \dots, uv_n, v_1v_2, \dots, v_{n-1}v_n\}.$$

We first show two claims.

Claim 1 *There is no fan $F = [v; x, y, z]$ in $G^\times$ such that x, z are small vertices and y is a crossing vertex.*

Proof Suppose, to be contrary, that such a fan F exists. Since the ordinary graph G is simple and y is a crossing vertex in $G^\times$, xz must be a crossing edge in G. But both x and z are small vertices, this contradicts the fact that no two small vertices are adjacent in G.

Claim 2 *Each big vertex v in $G^\times$ is incident with at most $\lceil f_3(v)/2 \rceil + 1$ type one 3-faces if $f_3(v) = d(v) - 2$, at most $\lceil f_3(v)/2 \rceil$ type one 3-faces if $f_3(v) = d(v) - 1$, and at most $\lfloor f_3(v)/2 \rfloor$ type one 3-faces if $f_3(v) = d(v)$.*

Proof Otherwise, there must exist three consecutive type one 3-faces that are incident with v. Denote those three 3-faces by vv_1v_2, vv_2v_3, and vv_3v_4, respectively. Then $F_1 = [v; v_1, v_2, v_3]$ and $F_2 = [v; v_2, v_3, v_4]$ are two fans. By the definition of type one 3-face and Lemma 1 (a), one of F_1 and F_2 must be the fan described in Claim 1, a contradiction.

Suppose $d = \Delta - 2$. Then by (b), for any small vertex u such that $uv \in E(G^\times)$, we have $d(u) \geqslant 5$. Therefore, v can be 5-master of at most three vertices by (c). Thus, v sends out at most $3 \times 1/3 = 1$ as masters of some vertices by Rule 1.3. If $f_3(v) \leqslant d(v) - 2$, then we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - 1 \geqslant \frac{\Delta - 13}{3} > 0.$$

If $f_3(v) \geqslant d(v) - 1$, then by Claim 2 and Rule 1.7, we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\left\lceil \frac{f_3(v)}{2} \right\rceil - \frac{1}{2}\left(f_3(v) - \left\lceil \frac{f_3(v)}{2} \right\rceil\right) - 1 \geqslant \frac{5\Delta - 71}{12} > 0.$$

Suppose $d = \Delta - 1$. Then by (b), for any small vertex u such that $uv \in E(G^\times)$, we have $d(u) \geqslant 4$. Therefore, v can be 4-master of at most two vertices and 5-master of at most three vertices by (c). Thus, v sends out at most

$$2 \times \frac{2}{3} + 3 \times \frac{1}{3} = \frac{7}{3}$$

as masters of some vertices by Rules 1.2 and 1.3. By Claim 2 and Rule 1.7, if $f_3(v) \leqslant d(v) - 2$, then

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - \frac{7}{3} \geqslant \frac{\Delta - 16}{3} \geqslant 0;$$

if $f_3(v) = d(v) - 1$, then

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\left\lceil \frac{f_3(v)}{2} \right\rceil - \frac{1}{2}\Big(f_3(v) - \left\lceil \frac{f_3(v)}{2} \right\rceil\Big) - \frac{7}{3} \geqslant \frac{5\Delta - 75}{12} > 0;$$

and if $f_3(v) = d(v)$, we still have

$$\begin{aligned} ch'(v) &\geqslant ch(v) - \frac{2}{3}\left\lfloor \frac{f_3(v)}{2} \right\rfloor - \frac{1}{2}\Big(f_3(v) - \left\lfloor \frac{f_3(v)}{2} \right\rfloor\Big) - \frac{7}{3} \\ &\geqslant \frac{3\Delta - \lfloor (\Delta - 1)/2 \rfloor - 41}{6} \\ &\geqslant 0 \end{aligned}$$

since $\Delta \geqslant 16$. Suppose $d = \Delta$. Then by (b), for any small vertex u such that $uv \in E(G^{\times})$, we have $d(u) \geqslant 3$. Therefore, v can be 3-master of at most one vertex, 4-master of at most two vertices, and 5-master of at most three vertices by (c). Thus, v sends out at most

$$\frac{1}{3} + 2 \times \frac{2}{3} + 3 \times \frac{1}{3} = \frac{8}{3}$$

as masters of some vertices by Rules 1.2 and 1.3. If $f_3(v) \leqslant d(v) - 2$, then we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - \frac{8}{3} \geqslant \frac{\Delta - 16}{3} \geqslant 0;$$

if $f_3(v) = d(v) - 1$, then by Claim 2, we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\left\lceil \frac{f_3(v)}{2} \right\rceil - \frac{1}{2}\Big(f_3(v) - \left\lceil \frac{f_3(v)}{2} \right\rceil\Big) - \frac{8}{3} \geqslant \frac{5\Delta - 74}{12} > 0$$

by Rule 1.7; if $f_3(v) = d(v)$, then we still have

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\left\lfloor \frac{f_3(v)}{2} \right\rfloor - \frac{1}{2}\Big(f_3(v) - \left\lfloor \frac{f_3(v)}{2} \right\rfloor\Big) - \frac{8}{3} \geqslant \frac{5\Delta - 80}{12} \geqslant 0$$

by Claim 2 and R1.7, since $\Delta \geqslant 16$. □

Theorem 2 *Let G be a 1-planar graph with maximum degree $\Delta \geqslant 21$. Then*

$$\chi'_l(G) = \Delta, \quad \chi''_l(G) = \Delta + 1.$$

Proof Let G be a minimal counterexample to the theorem. Then G has the following properties.

(a) G is connected.

(b) G contains no edge uv such that

$$\min\{d(u), d(v)\} \leqslant \left\lfloor \frac{\Delta}{2} \right\rfloor, \quad d(u) + d(v) \leqslant \Delta + 1.$$

(c) For each integer $2 \leqslant k \leqslant 5$, let

$$X_k = \{x \in V(G) \mid d_G(x) \leqslant k\}, \quad Y_k = \bigcup_{x \in X_k} N_G(x).$$

If $X_k \neq \emptyset$, then there exists a bipartite subgraph $M_k = (X_k, Y_k)$ of G such that $d_{M_k}(x) = 1$ for any $x \in X_k$ and $d_{M_k}(y) \leqslant k - 1$ for any $y \in Y_k$. We also call y the *k-master* of x if $xy \in M_k$ and $x \in X_k$.

(a) and (b) are easy to be proved. The proof of (c) can be found in [20].

It is also easy to see from (b) that $\delta(G) \geqslant 2$. By (c), each i-vertex $(2 \leqslant i \leqslant 5)$ has one j-master $(i \leqslant j \leqslant 5)$.

Now, we also apply the discharging method to the associated plane graph $G^\times$ of G and complete the proof by a contradiction.

Since $G^\times$ is a plane graph, by Euler's formula, we have

$$\sum_{v \in V(G^\times)} (d(v) - 4) + \sum_{f \in F(G^\times)} (d(f) - 4) = -8$$

Here, we define the initial charge of $x \in V(G^\times) \cup F(G^\times)$, small vertex, big vertex, type one 3-face, and type two 3-face just as the same as we defined in the proof of Theorem 1, respectively. Our discharging rules are redefined as follows.

Rule 2.1 Each 2-vertex in G receives 2/3 from its 2-master, 1/3 from its 3-master, 2/3 from its 4-master, and 1/3 from its 5-master.

Rule 2.2 Each 3-vertex in G receives 1/3 from its 3-master, 2/3 from its 4-master, 1/3 from its 5-master, and sends 1/3 to each type one 3-face incident with it in $G^\times$.

Rule 2.3 Each 4-vertex in G receives 2/3 from its 4-master, 1/3 from its 5-master, and sends 1/3 to each type one 3-face incident with it in $G^\times$.

Rule 2.4 Each 5-vertex in G receives 1/3 from its 5-master and sends 1/3 to each type one 3-face incident with it in $G^\times$.

Rule 2.5 Each 6-vertex in G sends 1/3 to each type one 3-face incident with it in $G^\times$.

Rule 2.6 Each 7-vertex in G sends 1/2 to each type one 3-face incident with it in $G^\times$.

Rule 2.7 Each d-vertex in G $(8 \leqslant d \leqslant \Delta - 5)$ sends 1/2 to each 3-face incident with it in $G^\times$.

Rule 2.8 Each d-vertex in G $(\Delta - 4 \leqslant d \leqslant \Delta)$ sends 2/3 to each type one 3-face and 1/2 to each type two 3-face incident with it in $G^\times$.

Rule 2.9 Each k-face in $G^{\times}$ ($k \geqslant 5$) sends $(k-4)/t(f)$ to each 3-vertex incident with it, where $t(f)$ is the number of 3-vertices incident with face f.

By the similar means as we applied in the proof of Theorem 1, we can check that $ch'(f) \geqslant 0$ for every face $f \in F(G^{\times})$ (here we shall in particular note that each 2-vertex is not incident with any type one face in $G^{\times}$ by Lemma 1 (b)) and $ch'(v) \geqslant 0$ for every vertex $v \in V(G^{\times})$ with $3 \leqslant d(v) \leqslant \Delta - 5$. Note that Claims 1 and 2 in the proof of Theorem 1 are also available here. In the following, we just need to check that $ch'(v) \geqslant 0$ for each vertex v with

$$d(v) \in \{2, \Delta-4, \Delta-3, \Delta-2, \Delta-1, \Delta\}.$$

Let v be a 2-vertex. By (c), v has a j-master, where $2 \leqslant j \leqslant 5$. Then we have

$$ch'(v) = ch(v) + \frac{2}{3} + \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = 0$$

by Rule 2.1. Let v be a $(\Delta-4)$-vertex. Then by (b), for any small vertex u such that $uv \in E(G^{\times})$, we have $d(u) \geqslant 6$. Therefore, v cannot send out charges as masters of some vertices by Rules 2.1–2.4. Therefore,

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) \geqslant \frac{\Delta - 16}{3} > 0$$

by Rule 2.8. Let v be a $(\Delta-3)$-vertex. Then by (b), for any small vertex u such that $uv \in E(G^{\times})$, we have $d(u) \geqslant 5$. Therefore, v can be 5-master of at most four vertices by (c). Thus, v sends out at most $4 \times 1/3 = 4/3$ as masters of some vertices by Rules 2.1–2.4. Therefore,

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - \frac{4}{3} \geqslant \frac{\Delta - 19}{3} > 0$$

by Rule 2.8. Let v be a $(\Delta-2)$-vertex. Then by (b), for any small vertex u such that $uv \in E(G^{\times})$, we have $d(u) \geqslant 4$. Therefore, v can be 4-master of at most three vertices and 5-master of at most four vertices by (c). Thus, v sends out at most

$$3 \times \frac{2}{3} + 4 \times \frac{1}{3} = \frac{10}{3}$$

as masters of some vertices by Rules 2.1–2.4. If $f_3(v) \leqslant d(v) - 2$, then we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - \frac{10}{3} \geqslant \frac{\Delta - 20}{3} > 0;$$

if $f_3(v) \geqslant d(v) - 1$, then by Claim 2 in the proof of Theorem 1, we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\left\lceil \frac{f_3(v)}{2} \right\rceil - \frac{1}{2}\left(f_3(v) - \left\lceil \frac{f_3(v)}{2} \right\rceil\right) - \frac{10}{3} \geqslant \frac{5\Delta - 89}{12} > 0$$

by Rule 2.8. Let v be a $(\Delta-1)$-vertex. Then by (b), for any small vertex u such that $uv \in E(G^{\times})$, we have $d(u) \geqslant 3$. Therefore, v can be 3-master of at most two vertices, 4-master of at most three vertices, and 5-master of at most four vertices by (c). Thus, v sends out at most

$$2 \times \frac{1}{3} + 3 \times \frac{2}{3} + 4 \times \frac{1}{3} = 4$$

as masters of some vertices by Rules 2.1–2.4. If $f_3(v) \leqslant d(v) - 2$, then we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - 4 \geqslant \frac{\Delta - 21}{3} \geqslant 0;$$

if $f_3(v) \geqslant d(v) - 1$, then by Claim 2 in the proof of Theorem 1, we also have

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\Big\lceil \frac{f_3(v)}{2} \Big\rceil - \frac{1}{2}\Big(f_3(v) - \Big\lceil \frac{f_3(v)}{2} \Big\rceil\Big) - 4 \geqslant \frac{5\Delta - 97}{12} > 0$$

by Rule 2.8. Let v be a Δ-vertex. Then by (b), for any small vertex u such that $uv \in E(G^{\times})$, we have $d(u) \geqslant 2$. Therefore, v can be 2-master of at most one vertex, 3-master of at most two vertices, 4-master of at most three vertices, and 5-master of at most four vertices by (c). Thus, v sends out at most

$$\frac{2}{3} + 2 \times \frac{1}{3} + 3 \times \frac{2}{3} + 4 \times \frac{1}{3} = \frac{14}{3}$$

as masters of some vertices by Rules 2.1–2.4. If $f_3(v) \leqslant d(v) - 3$, then we have

$$ch'(v) \geqslant ch(v) - \frac{2}{3} f_3(v) - \frac{14}{3} \geqslant \frac{\Delta - 20}{3} > 0;$$

if

$$d(v) - 2 \leqslant f_3(v) \leqslant d(v) - 1,$$

then by Claim 2 in the proof of Theorem 1, we have

$$\begin{aligned} ch'(v) &\geqslant ch(v) - \frac{2}{3}\Big(\Big\lceil \frac{f_3(v)}{2} \Big\rceil + 1\Big) - \frac{1}{2}\Big(f_3(v) - \Big\lceil \frac{f_3(v)}{2} \Big\rceil - 1\Big) - \frac{14}{3} \\ &\geqslant \frac{5\Delta - 100}{12} \\ &> 0 \end{aligned}$$

by Rule 2.8; if $f_3(v) = d(v)$, then we still have

$$ch'(v) \geqslant ch(v) - \frac{2}{3}\Big\lfloor \frac{f_3(v)}{2} \Big\rfloor - \frac{1}{2}\Big(f_3(v) - \Big\lfloor \frac{f_3(v)}{2} \Big\rfloor\Big) - \frac{14}{3} \geqslant \frac{5\Delta - 104}{12} > 0$$

by Rule 2.8. Therefore,

$$\sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch(x) = \sum_{x \in V(G^{\times}) \cup F(G^{\times})} ch'(x) > 0.$$

This contradiction completes the proof of the theorem. $\square$

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