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On $(p, 1)$-total labelling of plane graphs with independent crossings

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Abstract. Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing in the plane so that every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short. In this paper, it is proved that the $(p, 1)$-total labelling number of every IC-planar graph G is at most $\Delta(G) + 2p - 2$ provided that $\Delta(G) \geq \Delta$ and $g(G) \geq g$, where $(\Delta, g) \in \{(6p+2, 3), (4p+2, 4), (2p+5, 5)\}$. As a consequence, we generalize and improve some results obtained in [F. Bazzaro, M. Montassier, A. Raspaud, $(d, 1)$-Total labelling of planar graphs with large girth and high maximum degree, Discrete Math. 307 (2007) 2141–2151].

1. Introduction

In the channel assignment problems, we need to assign different channels to close transmitters so that they can avoid interference and communication link failure. Moreover, a sufficient separation of the channels assigned to two close transmitters is also necessary. An $L(p, q)$-labeling is a popular graph theoretic model for this problem. An $L(p, q)$-labelling of a graph G is a mapping f form the set of vertices $V(G)$ to the set of integers $\mathbb{Z}_k = \{0, 1, \cdots, k\}$ such that $|f(x) - f(y)| \geq p$ if x and y are adjacent and $|f(x) - f(y)| \geq q$ if x and y are at distance 2. This notion has been studied many times and gives many challenging problems. The interested readers can refer to the surveys by Calamoneri [3] and by Yeh [11].

The incidence graph $I(G)$ of a graph G is the graph obtained from G by replacing each edge with a path of length 2. Given a graph G, Whittlesey et al. [9] studied the $L(2, 1)$-labelling of $I(G)$ in 1995. Indeed, such a labelling of $I(G)$ is equivalent to an assignment of the integer set $\{0, 1, \cdots, k\}$ to each element of $V(G) \cup E(G)$ such that the restrained vertex coloring and edge coloring of G is proper and the difference between the integer assigned to a vertex and these assigned to its incident edges is at least 2. This assignment introduced by Havet and Yu [4, 5] is called a $(2, 1)$-total labelling of G and can be generalized to the notation of $(p, 1)$-total labelling.

A k-$(p, 1)$-total labelling of a graph G is a function f from $V(G) \cup E(G)$ to the color set $\{0, 1, \cdots, k\}$ such that $|f(u) - f(v)| \geq 1$ if $uv \in E(G)$, $|f(e_1) - f(e_2)| \geq 1$ if e_1 and e_2 are two adjacent edges and $|f(u) - f(e)| \geq p$ if the vertex u is incident to the edge e. The minimum k such that G has a k-$(p, 1)$-total labelling, denoted by $\lambda_p^T(G)$, is called the $(p, 1)$-total labelling number of G. One can easily see that the $(1, 1)$-total labelling and the total coloring are equivalent and thus the following $(p, 1)$-Total Labelling Conjecture can be seen as a generalization of the well-known Total Coloring Conjecture, which asserts that every graph is $(\Delta + 2)$-total colorable.

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Conjecture 1. [5, 6] *Let G be a graph. Then* $\lambda_p^T(G) \le \min\{\Delta(G) + 2p - 1, 2\Delta(G) + p - 1\}$.

This conjecture is now confirmed for some planar graphs with high girth and high maximum degree [2] and for graphs with a given maximum average degree [10]. In particular, Bazzaro et al. [2] proved the following theorem for planar graphs.

Theorem 2. [2] *Let G be a planar graph with maximum degree* Δ *and girth g. Then* $\lambda_p^T(G) \le \Delta + 2p - 2$ *with* $p \ge 2$ *in the following cases:*
(1) $\Delta \ge 2p + 1$ *and* $g \ge 11$;
(2) $\Delta \ge 2p + 2$ *and* $g \ge 6$;
(3) $\Delta \ge 2p + 3$ *and* $g \ge 5$;
(4) $\Delta \ge 8p + 2$.

In this paper, we focus on plane graphs with independent crossings. Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing in the plane in which every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short throughout this paper. This definition of IC-planar graph was introduced by Alberson [1] in 2008. Setting a conjecture of Alberson [1], Král and Stacho [7] showed that every IC-planar graph is 5-colorable.

Throughout this paper, we always assume that every IC-planar graph has already been drawn in the plane with all its crossings independent and with the number of crossings minimum. Such a drawing is called IC-plane graph. The associated plane graph $G^\times$ of an IC-plane graph G is the graph obtained from G by turning all crossings of G into new 4-valent vertices. A vertex in $G^\times$ is called false if it is a new added vertex and is called true otherwise. We call a face in $G^\times$ false or true according to whether it is incident with a false vertex or not. A crossed edge in G is an edge $e \in E(G) \setminus E(G^\times)$. By the definition of IC-plane graph, one can see that every vertex in $G^\times$ is adjacent to at most one false vertex and is incident with at most two false faces in $G^\times$. For other basic undefined concepts we refer the reader to [8].

2. Main results and their proofs

This section is dedicated to the proof the following main theorem. Note that IC-planar graphs is a larger class than planar graphs. So Theorem 3(3) improves and generalizes Theorem 2(4) in some sense. On the other hand, one can also see that the bound for Δ in Theorem 3(1) is very close to the corresponding one in Theorem 2(3), even though we consider a larger class.

Theorem 3. *Let G be an IC-planar graph with maximum degree* Δ *and girth g. Then* $\lambda_p^T(G) \le \Delta + 2p - 2$ *with* $p \ge 2$ *in the following cases:*
(1) $\Delta \ge 2p + 5$ *and* $g \ge 5$;
(2) $\Delta \ge 4p + 2$ *and* $g \ge 4$;
(3) $\Delta \ge 6p + 2$.

Instead of proving Theorem 3 directly, we would prove the following slightly stronger theorem. Indeed, this is only a technical strengthening of Theorems 3, without which we would get complications when considering a subgraph $G' \subset G$ such that $\Delta(G') < \Delta(G)$ (the readers can make themselves sure of that). Of course, the interesting case of it is when $M = \Delta$.

Theorem 4. *Let G be an IC-planar graph with maximum degree* $\Delta \le M$ *and girth g. Then* $\lambda_p^T(G) \le M + 2p - 2$ *with* $p \ge 2$ *in the following cases:*
(1) $M \ge 2p + 5$ *and* $g \ge 5$;
(2) $M \ge 4p + 2$ *and* $g \ge 4$;
(3) $M \ge 6p + 2$.

Before proving it, let us recall some useful lemmas on the minimum counterexample G to Theorem 4 in terms of $|V(G)| + |E(G)|$.

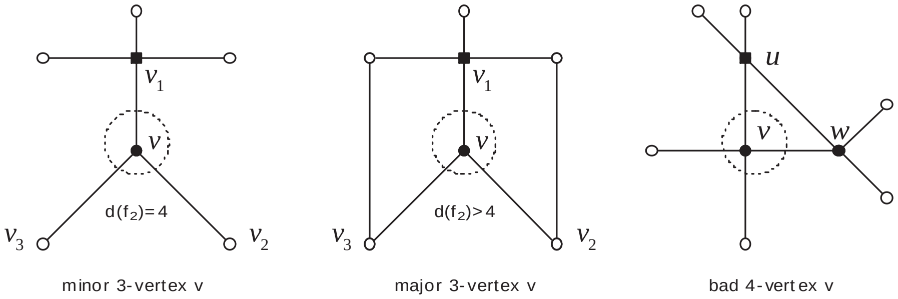

Figure 1: Depictions of some useful configurations

Lemma 5. [13] *For any edge $uv \in E(G)$, if* $\min\{d_G(u), d_G(v)\} \leq \lfloor \frac{M+2p-2}{2p} \rfloor$, *then* $d_G(u) + d_G(v) \geq M + 2$.

Lemma 6. [13] *For any edge $uv \in E(G)$, $d_G(u) + d_G(v) \geq M - 2p + 3$.*

Lemma 7. [13] *For any integer $2 \leq k \leq \lfloor \frac{M+2p-2}{2p} \rfloor$, let $X_k = \{x \in V(G) \mid d_G(x) \leq k\}$ and $Y_k = \bigcup_{x \in X_k} N_G(x)$. If $X_k \neq \emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x) = 1$ for every $x \in X_k$ and $d_{M_k}(y) \leq k - 1$ for every $y \in Y_k$.*

Let M_k and X_k be the bipartite graph and the vertex set stated in Lemma 7. If $xy \in M_k$ and $x \in X_k$, then we call y the k-master of x and x the k-dependent of y. By this definition, the following corollary of Lemma 7 is natural.

Corollary 8. *Every i-vertex in G has a j-master when* $2 \leq i \leq j \leq \lfloor \frac{M+2p-2}{2p} \rfloor$ *and every vertex in G has at most* $k - 1$ *k-dependents when* $2 \leq k \leq \lfloor \frac{M+2p-2}{2p} \rfloor$.

Let v be a 3-vertex in G with v_1, v_2, v_3 being its neighbors in $G^\times$ in a clockwise order. Let f_i $(1 \leq i \leq 3)$ be the face incident with the path $v_i v v_{i+1}$ in $G^\times$, where i is taken modular 3. If v_1 is false and $d_{G^\times}(f_2) = 4$, then we call v a minor 3-vertex; if v_1 is false, $d_{G^\times}(f_1) = d_{G^\times}(f_3) = 4$ and $d_{G^\times}(f_2) \geq 5$, then we call v a major 3-vertex (see the first two configurations of Figure 1). We call a 4-vertex v in G bad if v is incident with a false 3-face uvw in $G^\times$ so that u is a false vertex and w is a 4-vertex in G (see the third configuration of Figure 1). A 5^+-vertex in G is called good if it is incident with no false 3-faces. The following lemmas deal with the structural properties of G as an IC-plane graph.

Lemma 9. *There is no* 2*-vertices that is incident with a false* 3*-face in* $G^\times$.

Proof. The same result has already been proved for 1-planar graphs (i.e., graphs that can be draw in the plane so that each edge is crossed by at most one other edge) in [12]. So this lemma follows from the fact that every IC-planar graph is 1-planar. □

Lemma 10. *If $g(G) \geq 5$ and the neighbors of any 3-vertex in G are of degree at least 5, then every 3-vertex that is not major in G is either incident with at least two 5^+-faces in $G^\times$, or incident with one 5^+-face $G^\times$ and adjacent to two good 5^+-vertices in G, or adjacent to three good 5^+-vertices in G.*

Proof. We prove this lemma by contradiction. Let v be a 3-vertex in G with v_1, v_2, v_3 being its neighbors in $G^\times$ in a clockwise order. Let f_i $(1 \leq i \leq 3)$ be the face incident with the path $v_i v v_{i+1}$ in $G^\times$, where i is taken modular 3. First suppose that all of v_1, v_2 and v_3 are true. Since $g(G) \geq 5$, $v_1v_2, v_2v_3, v_3v_1 \notin E(G^\times)$ and thus f_1, f_2, f_3 are all 4^+-faces. If v is incident with at most one 5^+-faces in $G^\times$, then v would be incident with at least two 4-faces in $G^\times$. Without loss of generality, assume that $d_{G^\times}(f_1) = d_{G^\times}(f_2) = 4$. Since $g(G) \geq 5$ and $v_1v_3 \notin E(G^\times)$, there exist two different false vertices x and y such that $xv_1, xv_2, yv_2, yv_3 \in E(G^\times)$. This is impossible since v_2 is adjacent two false vertices x and y now. Thus v is incident with at least two 5^+-faces in $G^\times$.

Now we assume that only one of v_1, v_2 and v_3, say v_1, is false, and in addition assume that the 3-vertex v is incident with at most one 5^+-face in $G^\times$. If $d_{G^\times}(f_1) = 3$, then $y \neq v_3$ and $v_3y \notin E(G^\times)$, where is assumed that vx crosses v_2y in G at the point v_1, because otherwise vv_2v_3 or vv_2yv_3 would be a triangle or a quadrilateral in G, a contradiction to $g(G) \geq 5$. This implies that $d_{G^\times}(f_3) \geq 5$ and thus the degree of f_2 in $G^\times$ must be 4 by our assumption since $v_2v_3 \notin E(G^\times)$, which follows that there exists a false vertex $z \neq v_1$ such that $zv_2, zv_3 \in E(G^\times)$. However, this is impossible since v_2 is adjacent two false vertices z and v_1 now. Thus we shall assume that $\min\{d_{G^\times}(f_1), d_{G^\times}(f_3)\} \geq 4$.

If $d_{G^\times}(f_2) = 4$, then there exists a false vertex $x \neq v_1$ such that $xv_2, xv_3 \in E(G^\times)$. Suppose that v_2y crosses v_3z at x. Then $v_2z \notin E(G^\times)$, because otherwise vv_2zv_3 would be a quadrilateral in G, a contradiction to $g(G) \geq 5$. This follows that v_2 is incident with no false 3-faces in $G^\times$ and thus v_2 is a good 5^+-vertex in G. Similarly one can show that v_3 is also a good 5^+-vertex in G. Since v is now adjacent to two good 5^+-vertices in G, by our assumption, we have to assume that $d_{G^\times}(f_1) = d_{G^\times}(f_3) = 4$. Suppose that vx_1 crosses y_1z_1 at v_1. Then $y_1v_3, z_1v_2 \in E(G^\times)$. This implies that $x_1y_1, x_1z_1 \notin E(G^\times)$, because otherwise a quadrilateral would appear in G. So x_1 is incident with no false 3-faces in $G^\times$ and thus x_1 is the third good 5^+-vertex in G that is adjacent to v in G.

The last case is when $d_{G^\times}(f_2) \geq 5$. However, under this case we shall assume $d_{G^\times}(f_1) = d_{G^\times}(f_3) = 4$, which implies that v is a major 3-vertex in G, a contradiction. □

By the proof of Lemma 10, we also have the following useful lemma as a corollary.

Lemma 11. *If $g(G) \geq 5$ and v is a 3-vertex in G that is neither minor nor major, then v is incident with at least two 5^+-faces in $G^\times$.*

Lemma 12. *If $g(G) \geq 5$, then every 5^+-vertex is adjacent to at most three minor 3-vertices in G.*

Proof. Suppose, to the contrary, that v is a 5^+-vertex that is adjacent to four minor 3-vertices v_1, v_2, v_3 and v_4 in G, which are lying in a clockwise order. Since v is adjacent to at most one false vertex in $G^\times$, without loss of generality, assume that vv_1 and vv_2 are not crossed edges. Since $g(G) \geq 5$ and v_2 is a minor 3-vertex, vv_2 must be incident with an edge vv_0 such that vv_0 is crossed by another edge xy at a false vertex z and $xv_2 \in E(G^\times)$. Furthermore, the three neighbors v_1, v_0, v_3 of v should be lying in a clockwise order. First suppose that $v_0 = v_3$. Then consider the 3-vertex v_1. Since v_1 is minor and vv_0 is the unique crossed edge that is incident with v, there exists an edge x_1y_1 in G such that $x_1v_1 \in E(G^\times)$ and x_1y_1 crosses vv_0 in G. Note that vv_0 has already been crossed by xy at z, we should have $x_1y_1 = xy$ and $x_1z, y_1z \in E(G^\times)$. This implies that the four vertices v_1, x_1, z and v cannot form a 4-face in $G^\times$, a contradiction to the definition of minor 3-vertices. Thus we shall assume that $v_0 \neq v_3$. Under this case we consider the minor 3-vertex v_4. Let s be a vertex in G such that $sv_4 \in E(G^\times)$. By a similar argument as above one can also show that $s \in \{x, y\}$ and thus the face incident with the path vv_4s in $G^\times$ cannot be of degree 4 by the drawing of G. This contradiction completes the proof. □

Lemma 13. *Let uv be a crossed edge in G such that u is a 5^+-vertex and v is a major 3-vertex. If $g(G) \geq 5$, then u is a good 5^+-vertex that is adjacent to at most two minor 3-vertices in G.*

Proof. Let v_1, v_2, v_3 be the neighbors of v in $G^\times$ in a clockwise order. Without loss of generality, assume that xy crosses uv in G at v_1. Since v is a major 3-vertex, we can also assume that $xv_2, yv_3 \in E(G)$. This follows that $ux, uy \notin E(G)$, because otherwise there would be a quadrilateral in G. Thus u is a good 5^+-vertex. Let z be a minor 3-vertex that is adjacent to u in G. Then $uz \in E(G^\times)$ since uv is a crossed edge in G and $v \neq z$. This implies that $zx \in E(G) \cap E(G^\times)$ or $zy \in E(G) \cap E(G^\times)$ by the definition of z (recall that $g(G) \geq 5$ here). Suppose that u is adjacent to three minor 3-vertices z_1, z_2 and z_3 in G. Then by the above argument, there are at least two vertices among them, say z_1 and z_2, such that $z_1x, z_2x \in E(G) \cap E(G^\times)$. Since z_1 and z_2 are both minor and $z_1x, z_1u, z_2x, z_2u \in E(G) \cap E(G^\times)$, by the definition of minor 3-vertices, there must exist a 4-face h_1 that is incident with the four vertices u, z_1, x, v_1 and another 4-face h_2 that is incident with the four vertices

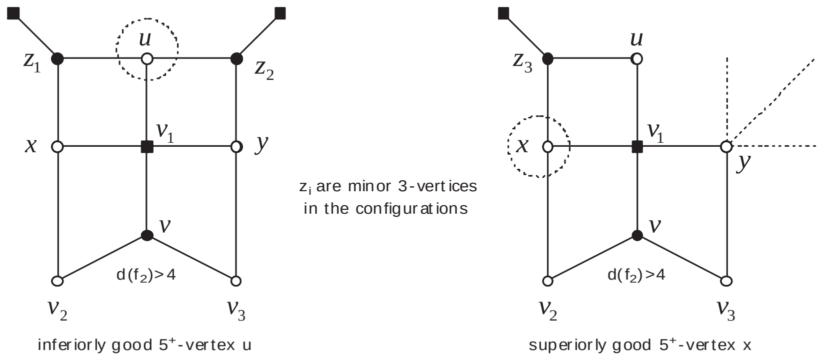

Figure 2: Definitions of two kinds of good 5^+-vertices

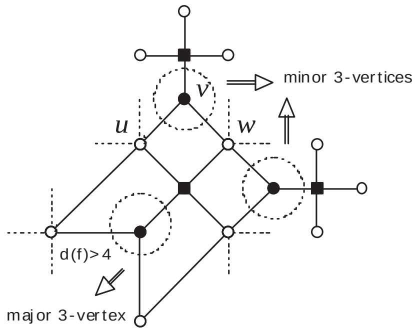

Figure 3: w is inferiorly good only if u is superiorly good

u, z_2, x, v_1. However, h_1 and h_2 cannot simultaneously appear in $G^\times$ by the drawing of G unless $z_1 = z_2$. This is a contradiction. □

Lemma 14. *Let uv be an edge that is crossed by xy in G such that u, x, y are 5^+-vertices and v is a major 3-vertex. If $g(G) \geq 5$, then x is a good 5^+-vertex that is adjacent to at most one minor 3-vertex in G.*

Proof. Let v_1, v_2, v_3 be the neighbors of v in $G^\times$ in a clockwise order. Without loss of generality, assume that v_1 is the false vertex such that $uv_1, vv_1, xv_1, yv_1 \in E(G^\times)$. By a similar argument as in Lemma 13, we have $ux \notin E(G)$. Since v is a major 3-vertex, the face incident with the path xv_1v in $G^\times$ is of degree 4. These two facts implies that x is a good 5^+-vertex. Let $z \neq y$ be a minor 3-vertex that is adjacent to x in G. Then $xz \in E(G^\times)$ since xy is a crossed edge in G and $y \neq z$. This implies that $zu \in E(G)$ and there is a 4-face in $G^\times$ that is incident with x, z, u and v_1 by the definition of minor 3-vertices (here, also remind that $g(G) \geq 5$). Suppose that x is adjacent to two minor 3-vertices z_1 and z_2 in G. Then by a similar argument as in Lemma 13, one can claim that $z_1 = z_2$. Thus this lemma follows. □

Let u (resp. x) be the vertex stated in Lemma 13 (resp. Lemma 14). If u (resp. x) is adjacent to exactly two (resp. one) minor 3-vertices in G, then u (resp. x) is called inferiorly (resp. superiorly) good 5^+-vertex (see Figure 2). Other good 5^+-vertices (neither superior nor inferior) contained in G is called to be generally good 5^+-vertices from now on. By Lemmas 13 and 14 along with the proofs of them, one can deduce the following lemma as a corollary (see Figure 3).

Lemma 15. *Let v be a minor 3-vertex in G. If $g(G) \geq 5$ and the neighbors of any 3-vertex in G are of degree at least 5, then v is adjacent to an inferiorly good 5^+-vertex in G only if v is also adjacent to a superiorly good 5^+-vertex in G.*

Lemma 16. *Let v be a bad 4-vertex in G with v_1, v_2, v_3, v_4 being its neighbors in $G^\times$ in a clockwise order, where v_1 is false. Let f_i $(1 \leq i \leq 4)$ be the face incident with the path v_ivv_{i+1} in $G^\times$, where i is taken modular 4. If $g(G) \geq 5$ and $d_{G^\times}(f_4) = 3$, then $\min\{d_{G^\times}(f_1), d_{G^\times}(f_3)\} \geq 5$. Furthermore, if $d_{G^\times}(f_2) = 4$, then v_3 cannot be a bad 4-vertex.*

Proof. Let v_4u and vw be two mutually crossed edges in G that intersect at v_1. Then $u \neq v_2$, because otherwise vv_4v_2 would be a triangle in G, a contradiction. Meanwhile, $uv_2 \notin E(G)$ because otherwise vv_4uv_2 would be a quadrilateral in G, again a contradiction. These two facts imply that $d_{G^\times}(f_1) \geq 5$. Since v_1 is false and $g(G) \geq 5$, v_2, v_3 and v_4 are true and thus $v_3v_4 \notin E(G)$. If $d_{G^\times}(f_3) = 4$, then there exists a false vertex $x \neq v_1$ such that $xv_3, xv_4 \in E(G^\times)$, which implies that v_4 is adjacent to two false vertices v_1 and x in $G^\times$, a contradiction to the definition of G. Thus $d_{G^\times}(f_3) \geq 5$. If $d_{G^\times}(f_2) = 4$, then there exists a false vertex y such that $yv_2, yv_3 \in E(G^\times)$. Suppose v_2z_2 crosses v_3z_1 at the false vertex y. Then $v_3z_2 \notin E(G)$, because otherwise $vv_2z_2v_3$ would be a quadrilateral in G, a contradiction. So v_3 is incident with no false 3-faces and thus v_3 cannot be a bad 4-vertex. □

In the following, we prove each part of Theorem 4 by discharging method. First of all, we assign an initial charge $c(v) = d_G(v) - 4$ to every vertex v in G and $c(f) = d_{G^\times}(f) - 4$ to every face f in $G^\times$. Then by Euler's formula on the plane graph $G^\times$ and by the fact that $d_{G^\times}(v) = 4$ for every $v \in V(G^\times) \setminus V(G)$, we have

$$\begin{aligned}\sum_{x \in V(G)\cup F(G^\times)} c(x) &= \sum_{v\in V(G)} (d_G(v) - 4) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 4)\\ &= \sum_{v \in V(G^\times)} (d_{G^\times}(v) - 4) + \sum_{f\in F(G^\times)} (d_{G^\times}(f) - 4)\\ &= -4(|V(G^\times)| + |F(G^\times)| - |E(G^\times)|) = -8.\end{aligned}$$

Whereafter, we redistribute the initial charge by discharging rules and obtain a final charge $c'(x)$ for every $x \in V(G) \cup F(G^\times)$. We then check that the final charge on each vertex and face is nonnegative. However, our rules only move charge around and do not affect the sum; this implies that $\sum_{x\in V(G)\cup F(G^\times)} c'(x) = -8$, a contradiction.

Part I. Proof of Theorem 4(1)

Let f be a face in $G^\times$. Denote by $n_i(f)$ the number of true i-vertices that are incident with f in $G^\times$ and by $n'_4(f)$ the number of bad 4-vertices that are incident with f in $G^\times$. By Lemma 5, Lemma 6 and Corollary 8, G has the following basic properties.

(P1) $\delta(G) \geq 2$.

(P2) Every 2-vertex is adjacent to two M-vertices, one of which is the 2-master of it.

(P3) For a 3-vertex $v \in V(G)$ and an edge $uv \in E(G)$, $d_G(u) \geq M - 2p \geq 5$.

(P4) For a 4-vertex $v \in V(G)$ and an edge $uv \in E(G)$, $d_G(u) \geq M - 2p - 1 \geq 4$.

(P5) Every M-vertex has at most one 2-dependent.

Now let us discharging along the following rules.

R1. Let $f = uvw$ be a false 3-face in $G^\times$ with u being false. If $d_G(v) \geq 5$, then f receives 1 from v; if $d_G(v) = d_G(w) = 4$, then f receives $\frac{1}{2}$ from each of v and w.

R2. Let f be a 5^+-face. Then f sends $\frac{1}{2}$ to each of 3-vertices incident with it and $\frac{2d_{G^\times}(f)-n_3(f)-8}{2n'_4(f)}$ to each of bad 4-vertices incident with it.

R3. Every 2-vertex receives 2 from its 2-master.

R4. Let uv be an edge of G such that u is a good 5^+-vertex and v is a minor 3-vertex. Then v receives $\frac{1}{2}$, $\frac{1}{3}$ or $\frac{1}{4}$ from u if u is superiorly good, generally good or inferiorly good, respectively.

R5. Let uv be a crossed edge of G such that u is a 5^+-vertex and v is a major 3-vertex. Then v receives $\frac{1}{2}$ from u.

Claim 1. *Let f be a face in $G^\times$ and let v be a bad 4-vertex that is incident with f. If $d_{G^\times}(f) \geq 6$, then f sends at least $\frac{1}{3}$ to v, and if $d_{G^\times}(f) = 5$, then f sends at least $\frac{1}{5}$ to v.*

Proof. Since every neighbor of a 3-vertex in G is of degree at least 5 by (P3), one can easily deduce that $2n_3(f)+n'_4(f) \le 2n_3(f)+n_4(f) \le d_{G^\times}(f)$. So by R2, f sends to v at least $\frac{2d_{G^\times}(f)-n_3(f)-8}{2n'_4(f)} \ge \frac{2d_{G^\times}(f)-n_3(f)-8}{2d_{G^\times}(f)-4n_3(f)} \ge \frac{d_{G^\times}(f)-4}{d_{G^\times}(f)}$ for $d_{G^\times}(f) \ge 5$. Hence this claim follows. □

Claim 2. *Let f be a 5-face that is incident with at most four bad 4-vertices. Then f sends at least $\frac{1}{4}$ to each of its incident bad 4-vertices.*

Proof. If $n_3(f) = 0$, then f sends at least $\frac{10-8}{8} = \frac{1}{4}$ to each of its incident bad 4-vertices by R2. If $n_3(f) \ge 1$ and $n'_4(f) \ge 1$, then $n_3(f) = 1$ and $n'_4(f) \le 2$ by (P3), which follows that f sends at least $\frac{10-1-8}{4} = \frac{1}{4}$ to each of its incident bad 4-vertices by R2. □

Claim 3. *Every false 5-face sends at least $\frac{1}{4}$ to each of its incident bad 4-vertices.*

Proof. Since every false 5-face is incident with at most four bad 4-vertices, this is a direct corollary of Claim 2. □

Now we check the nonnegativity of the final charges of the vertices and faces. By Lemma 9 and (P3), every false 3-face in $G^\times$ is either incident with two 4-vertices or incident with at least one 5^+-vertex. So by R1 $c'(f) \ge -1+\min\{2\times\frac{1}{2},1\} = 0$ for any false 3-face f. Note that there is no true 3-faces (since $g(G) \ge 5$) and every 4-face (whose initial charge is 0) has not involved in the above rules. So we only need to consider 5^+-faces. By R2, for any 5^+-face $f \in F(G^\times)$, $c'(f) \ge d_{G^\times}(f) - 4 - \frac{1}{2}n_3(f) - \frac{2d_{G^\times}(f)-n_3(f)-8}{2n'_4(f)}n'_4(f) = 0$.

Let v be a vertex in G. If $d_G(v) = 2$, then by (P2) and R3, $c'(v) \ge -2+2 = 0$.

If $d_G(v) = 3$, then we consider three cases. First, suppose that v is neither minor nor major. Then by Lemma 11, v is incident with at least two 5^+-faces in $G^\times$, which implies that $c'(v) \ge -1+2\times\frac{1}{2} = 0$ by R2. Second, suppose that v is minor (also assume that v is incident with at most one 5^+-face in $G^\times$). Then by Lemma 10, we have two subcases. If v is incident with exactly one 5^+-face and adjacent to two good 5^+-vertices, then $c'(v) \ge -1+\frac{1}{2}+2\times\frac{1}{4} = 0$ by R2 and R4. If v is adjacent to three good 5^+-vertices x, y and z, then by R2, R4 and Lemma 15, $c'(v) \ge -1+3\times\frac{1}{3} = 0$ when none of x, y and z is inferiorly good and $c'(v) \ge -1+\frac{1}{2}+2\times\frac{1}{4} = 0$ when at least one of x, y and z is inferiorly good. Third, suppose that v is a major 3-vertex. Then by its definition, v is incident with a 5^+-face in $G^\times$ and is incident with a crossed edge uv in G. So by R2, R5 and (P3), $c'(v) \ge -1+\frac{1}{2}+\frac{1}{2} = 0$.

If $d_G(v) = 4$, then by R1, (P3) and (P4), $c'(v) = c(v) = 0$ unless v is incident with a false 3-face vv_1v_4 such that v_1 is false and v_4 is a true 4-vertex in G (i.e., v is a bad 4-vertex). Let v_2 and v_3 be another two neighbors of v in $G^\times$ such that v_1, v_2, v_3, v_4 are lying in a clockwise order. Let f_i ($1 \le i \le 4$) be the face incident with the path v_ivv_{i+1} in $G^\times$, where i is taken modular 4. By Lemma 16, $\min\{d_{G^\times}(f_1), d_{G^\times}(f_3)\} \ge 5$. If $d_{G^\times}(f_2) \ge 5$, then by R1 and Claim 1, $c'(v) \ge 0-\frac{1}{2}+3\times\frac{1}{5} > 0$. So we assume that $d_{G^\times}(f_2) = 4$. Under this case, f_1 is a false 5^+-face and f_3 is a 5^+-face that is incident with at most $d_{G^\times}(f_3)-1$ bad 4-vertices by Lemma 16. Thus by R1, Claim 1, Claim 2 and Claim 3, $c'(v) \ge 0-\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 0$.

If $d_G(v) \ge 5$ and v is not good, then $c'(v) \ge d_G(v)-4-1-\beta(v) > 0$ by R1, R3 and (P5), where $\beta(v) = 2$ if $d_G(v) = M$ and $\beta(v) = 0$ otherwise. Recall that $M \ge 2p+5 \ge 9$ here. If $d_G(v) \ge 5$ and v is good, then we divide our discussions into two cases.

First, assume that v is adjacent to a major 3-vertex u such that uv is a crossed edge in G. By Lemma 13, v is now adjacent to at most two minor 3-vertices. If v is adjacent to exactly two minor 3-vertices (i.e., v is inferiorly good), then by R3, R4, R5 and the IC-planarity of G , $c'(v) \ge d_G(v)-4-\frac{1}{2}-2\times\frac{1}{4}-\beta(v) > 0$. If v is adjacent to at most one minor 3-vertex, then by the same rules, $c'(v) \ge d_G(v)-4-\frac{1}{2}-\frac{1}{2}-\beta(v) > 0$.

Second, assume that v is adjacent to no major 3-vertices u such that uv is a crossed edge in G. Then v is not superiorly good and v sends no charges to major 3-vertices by R5. One the other hand, v is adjacent to at most three minor 3-vertices by Lemma 12, to which v sends at most $3\times\frac{1}{3} = 1$ by R4. Therefore, $c'(v) \ge d_G(v)-4-1-\beta(v) > 0$ by R3 in final. □

Part II. Proof of Theorem 4(2)

Note that $M \ge 4p+2 \ge 10$. So by Lemma 5, Lemma 6 and Corollary 8, G has the following basic properties.

(P1) $\delta(G) \geq 2$.
(P2) Every 2-vertex has one 2-master and one 3-master.
(P3) Every 3-vertex has one 3-master.
(P4) For a 4^--vertex $v \in V(G)$ and an edge $uv \in E(G)$, $d_G(u) \geq M - 2p - 1 \geq 5$.
(P5) Every M-vertex has at most one 2-dependent and at most two 3-dependents.
(P6) Every $(M-1)$-vertex has no 2-dependents and has at most two 3-dependents.

Let v be a vertex in G. It is easy to see that v is incident with at most two false faces in $G^\times$. If v is incident with exactly two false 3-faces uvx and uvy in $G^\times$, where u is a false vertex, then xy must be a crossed edge in G and thus vxy is a triangle in G, which contradicts the fact that $g(G) \geq 4$. Hence we have the following property (P7).

(P7) Every vertex in G is incident with at most one false 3-face in $G^\times$.

In the following, we prove this theorem by discharging along the following rules.

R1. Every false 3-face receives 1 from each of its incident 5^+-vertices.
R2. Every 2-vertex receives 1 from its 2-master and 1 from its 3-master.
R3. Every 3-vertex receives 1 from its 3-master.

Now we check the nonnegativity of the final charges of the vertices and faces. Since every false 3-face f in $G^\times$ is incident with at least one 5^+-vertex by (P4), $c'(f) \geq -1 + 1 = 0$ by R1. Thus we can easily claim that $c'(f) \geq 0$ for every $f \in F(G^\times)$ since there is no true 3-face and every 4^+-face has not been involved in the rules. Let v be a vertex in G. If $d_G(v) = 2$, then by (P2) and R2, $c'(v) \geq -2 + 1 + 1 = 0$. If $d_G(v) = 3$, then by (P3) and R3, $c'(v) \geq -1 + 1 = 0$. If $d_G(v) = 4$, then it is easy to see $c'(v) = c(v) = 0$. If $5 \leq d_G(v) \leq M - 2$, then by (P7) and R1, $c'(v) \geq d_G(v) - 4 - 1 \geq 0$. If $d_G(v) \geq M - 1$, then by (P5), (P6), (P7), R1, R2 and R3, $c'(v) \geq d_G(v) - 4 - 1 - 1 - 2 \times 1 > 0$. Thus the final charge of every vertex in G is also nonnegative. This completes the proof of Theorem 4(2). □

Part III. Proof of Theorem 4(3)

Note that $M \geq 6p + 2 \geq 14$. So by Lemma 5, Lemma 6 and Corollary 8, G has the following basic properties.

(P1) $\delta(G) \geq 2$.
(P2) Every 2-vertex has one 2-master and one 3-master.
(P3) Every 3-vertex has one 3-master.
(P4) For an edge $uv \in E(G)$, if $d_G(v) = 2, 3, 4, 5, 6$, then $d_G(u) \geq 14, 13, 12, 8, 7$, respectively.
(P5) Every M-vertex has at most one 2-dependent and at most two 3-dependents.
(P6) Every $(M-1)$-vertex has no 2-dependents and has at most two 3-dependents.

We call a false 3-face special if it is incident with a true 6^--vertex. Let v be a 7^+-vertex in G. If v is incident with two special false 3-faces uvx and uvy in $G^\times$, where u is a false vertex, then xy must be a crossed edge in G such that $\max\{d_G(x), d_G(y)\} \leq 6$. However, this is impossible since no two 6^--vertices are adjacent in G by (P4). Hence the following property (P7) holds.

(P7) Every 7^+-vertex in G is incident with at most one special false 3-face in $G^\times$.

Let u be a 2-vertex with neighbors v and w in G. If uv is crossed by another edge xy with $xv, yv \in E(G)$ and $w \neq x, y$, then we say that w is an assister of v and v needs assistance from w (see Figure 4).

Now let us discharge along the following rules.

R1. Let $f = uvw$ be a false 3-face in $G^\times$ with u being false and $d_G(v) \leq d_G(w)$. If $d_G(v) \leq 4$, then f receives 1 from w; if $d_G(v) = 5$, then f receives $\frac{1}{5}$ from v and $\frac{4}{5}$ from w; if $d_G(v) = 6$, then f receives $\frac{1}{3}$ from v and $\frac{2}{3}$ from w; if $d_G(v) \geq 7$, then f receives $\frac{1}{2}$ from each of v and w.
R2. Let $f = uvw$ be a true 3-face in $G^\times$ with $d_G(u) \leq d_G(v) \leq d_G(w)$. If $d_G(u) \leq 4$, then f receives $\frac{1}{2}$ from each of v and w; if $d_G(u) = 5$, then f receives $\frac{1}{5}$ from u and $\frac{2}{5}$ from each of v and w; if $d_G(u) \geq 6$, then f receives $\frac{1}{3}$ from each of u, v and w.
R3. Every 2-vertex receives 1 from its 2-master and 1 from its 3-master.
R4. Every 3-vertex receives 1 from its 3-master.
R5. If v has an assister w, then v receives $\frac{1}{2}$ from w

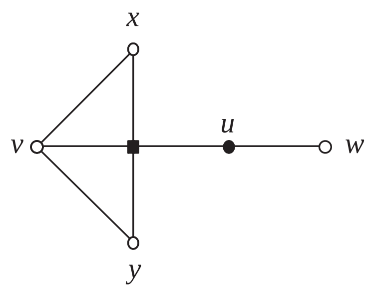

Figure 4: w is an assister of v

In what follows, we are to check the nonnegativity of the final charges of the vertices and faces. First of all, it is easy to check by R1 and R2 that every 3-face in $G^{\times}$ would totally receive exactly 1 from its incident true vertices. Meanwhile, the charge of any 4^{+}-face would not be updated after discharging. Thus one can claim that the final charge of every face in $G^{\times}$ is nonnegative.

Let v be a vertex in G. If $d_G(v) \leq 4$, then v would not send charges to its incident faces by R1 and R2. So by (P2), (P3), R3 and R4, $c'(v) \geq -2+1+1 = 0$ if $d_G(v) = 2$, $c'(v) \geq -1+1 = 0$ if $d_G(v) = 3$ and $c'(v) = c(v) = 0$ if $d_G(v) = 4$. If $d_G(v) = 5$, then by (P4), R1 and R2, v sends $\frac{1}{5}$ to each of its incident 3-faces, which implies that $c'(v) \geq 1 - 5 \times \frac{1}{5} = 0$. If $d_G(v) = 6$, then by (P4), R1 and R2, v sends $\frac{1}{3}$ to each of its incident 3-faces, which follows that $c'(v) \geq 2 - 6 \times \frac{1}{3} = 0$. If $d_G(v) = 7$, then by (P4), R1 and R2, v sends $\frac{1}{2}$ to each of its incident non-special false 3-faces, $\frac{2}{3}$ to each of its incident special false 3-faces and $\frac{1}{3}$ to each of its incident true 3-faces. Thus by (P7), $c'(v) \geq 3 - \frac{1}{2} - \frac{2}{3} - 5 \times \frac{1}{3} > 0$. If $8 \leq d_G(v) \leq 11$, then by (P4), R1 and R2, v sends $\frac{1}{2}$ to each of its incident non-special false 3-faces, at most $\frac{4}{5}$ to each of its incident special false 3-faces and at most $\frac{2}{5}$ to each of its incident true 3-faces. Thus by (P7), $c'(v) \geq 4 - \frac{1}{2} - \frac{4}{5} - 6 \times \frac{2}{5} > 0$. For a 12^{+}-vertex v, by (P4), R1 and R2, v would sends $\frac{1}{2}$ to each of its incident non-special false 3-faces, at most 1 to each of its incident special false 3-faces and at most $\frac{1}{2}$ to each of its incident true 3-faces. Thus if $d_G(v) = 12$, then $c'(v) \geq 8 - 1 - 11 \times \frac{1}{2} > 0$ by (P7). If $d_G(v) = 13$, then by (P4), the neighbors of v in G are of degree at least 3 and thus by (P6) (note that v may be a $(M-1)$-vertex since $M \geq 14$), v has no 2-dependents but may has at most two 3-dependents. So by R4 and (P7), $c'(v) \geq 9 - 1 - 12 \times \frac{1}{2} - 2 \times 1 = 0$.

At last, if $d_G(v) \geq 14$, then v is possible to be a M-vertex that has one 2-dependents and two 3-dependents and moreover, v may be assisters of some other vertices. Let $a(v)$ be the number of vertices that need assistance from v. It is easy to verify that v is incident with at most $d_G(v) - 2a(v)$ faces of degree 3 in $G^{\times}$. First of all, if $a(v) \geq 1$, then by R1–R5 and (P7), $c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 2a(v) - 1) - \frac{1}{2}a(v) - 1 - 2 \times 1$. So we assume that $a(v) = 0$. If v is incident with at least one 4^{+}-face, then by R1–R5 and (P7), $c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 2) - 1 - 2 \times 1 = \frac{1}{2}(d_G(v) - 14) \geq 0$. If v is adjacent to no 2-vertices in G, then by the same reason we also have $c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 1) - 2 \times 1 = \frac{1}{2}(d_G(v) - 13) > 0$. Thus in the end we assume that v is incident only with 3-faces in $G^{\times}$ and v is adjacent to at least one 2-vertex in G. Actually, in this case v can be adjacent to only one 2-vertex u and moreover, vu is a crossed edge in G. Let w be the other neighbor of u in G and let xy be the edge that crosses uv. It is easy to check that $w \neq x, y$ because otherwise we can redraw the figure of G so that the number of crossings is reduced by 1. Thus, w is an assistant of v, to which w sends $\frac{1}{2}$ by R5. Therefore, $c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 1) - 1 - 2 \times 1 + \frac{1}{2} = \frac{1}{2}(d_G(v) - 14) \geq 0$ by R1–R4 and (P7), and then the proof of Theorem 4 is complete. □

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