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k-FORESTED CHOOSABILITY OF GRAPHS WITH BOUNDED MAXIMUM AVERAGE DEGREE

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ABSTRACT. A proper vertex coloring of a simple graph is k-forested if the graph induced by the vertices of any two color classes is a forest with maximum degree less than k. A graph is k-forested q-choosable if for a given list of q colors associated with each vertex v, there exists a k-forested coloring of G such that each vertex receives a color from its own list. In this paper, we prove that the k-forested choosability of a graph with maximum degree $\Delta \ge k \ge 4$ is at most $\left\lceil \frac{\Delta}{k-1} \right\rceil + 1, \left\lceil \frac{\Delta}{k-1} \right\rceil + 2$ or $\left\lceil \frac{\Delta}{k-1} \right\rceil + 3$ if its maximum average degree is less than $\frac{12}{5}, \frac{8}{3}$ or 3, respectively.

1. Introduction

In this paper, all graphs considered are finite, simple and undirected. We use V(G), E(G), $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. The maximum average degree of G is defined by $mad(G) = max\{2|E(H)|/|V(H)|, H \subseteq G\}$. Any undefined notation follows that of Bondy and Murty [1].

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A proper vertex coloring of G is called an *acyclic coloring* of G if there are no bichromatic cycles in G under this coloring. The smallest number of colors such that G has an acyclic coloring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$. This concept was introduced by Grünbaum [3], and has been extensively studied in many papers. A coloring such that for every vertex $v \in V(G)$ no color appears more than k - 1 times in the neighborhood of v is called a k-frugal coloring. The notation of k-frugality was introduced by Hind et al. in [4].

Yuster mixed these two notions (setting k = 3) in [6] and first introduced the concept of *linear coloring*, which is a proper coloring of G such that the graph induced by the vertices of any two color classes is the union of vertex-disjoint paths. The linear chromatic number lc(G) of the graph G is the smallest number t such that G has a linear t-coloring. Linear coloring was also investigated by Esperet, Montassier and Raspaud in [2], and by Raspaud and Wang in [5]. In [2], the authors introduced a concept of k-forested coloring of a graph G, which is defined to be a proper vertex coloring of G such that the union of any two color classes is a forest of maximum degree less than k. So a linear coloring is equivalent to a 3-forested coloring. The k-forested chromatic number of a graph G, denoted by $\Lambda_k(G)$, is the smallest number of colors appearing in a k-forested coloring of G. Note that $\Lambda_k(G) = \chi_a(G)$ for $k > \Delta(G)$. If L is an assignment of a list L(v)of colors to each vertex $v \in V(G)$, then G is said to be k-forested L*colorable* if it has a k-forested coloring where each vertex is colored with a color from its own list. We say G is k-forested q-choosable if G is k-forested L-colorable whenever |L(v)| = q for every vertex $v \in V(G)$. The k-forested choice number $\Lambda_k^l(G)$ is the smallest integer q such that G is k-forested q-choosable. When k = 3, this is just equivalent to the *linear choice number*, which has been investigated by Esperet et al. for the graphs with bounded maximum average degree [2]. Their result is as follows.

Theorem 1.1. [2] Let G be a graph with maximum degree Δ . (1) If $\Delta \geq 3$ and $\operatorname{mad}(G) < \frac{16}{7}$, then $\Lambda_3^l(G) = \left\lceil \frac{\Delta}{2} \right\rceil + 1$. (2) If $\operatorname{mad}(G) < \frac{5}{2}$, then $\Lambda_3^l(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2$. (3) If $\operatorname{mad}(G) < \frac{8}{3}$, then $\Lambda_3^l(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 3$.

This paper is devoted to the following extensions of Theorem 1.1.

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Theorem 1.2. Given a positive integer $M \ge k \ge 4$, let G be a graph with maximum degree $\Delta \le M$.

(1) If mad(G) <
$$\frac{12}{5}$$
, then $\Lambda_k^l(G) \le \left\lfloor \frac{M}{k-1} \right\rfloor + 1$
(2) If mad(G) < $\frac{8}{3}$, then $\Lambda_k^l(G) \le \left\lceil \frac{M}{k-1} \right\rceil + 2$.

(3) If mad(G) < 3, then $\Lambda_k^l(G) \le \left\lfloor \frac{M}{k-1} \right\rfloor + 3.$

By the definition of the k-forested choice number and k-forested chromatic number, one can easily say that $\Lambda_k^l(G) \ge \Lambda_k(G) \ge \left\lceil \frac{\Delta}{k-1} \right\rceil + 1$ for every graph G with maximum degree Δ . Now setting $M = \Delta$ in Theorem 1.2, we have the following theorem as a corollary.

Theorem 1.3. Let G be a graph with maximum degree $\Delta \ge k \ge 4$. (1) If $\operatorname{mad}(G) < \frac{12}{5}$, then $\Lambda_k^l(G) = \left\lceil \frac{\Delta}{k-1} \right\rceil + 1$. (2) If $\operatorname{mad}(G) < \frac{8}{3}$, then $\Lambda_k^l(G) \le \left\lceil \frac{\Delta}{k-1} \right\rceil + 2$. (3) If $\operatorname{mad}(G) < 3$, then $\Lambda_k^l(G) \le \left\lceil \frac{\Delta}{k-1} \right\rceil + 3$.

Since every planar or projective-planar graph G with girth g(G) satisfies $\operatorname{mad}(G) < \frac{2g(G)}{g(G)-2}$, we obtain the direct corollary from Theorem 1.3.

Corollary 1.4. Let G be a planar or projective-planar graph with maximum degree $\Delta \ge k \ge 4$.

(1) If $g(G) \ge 12$, then $\Lambda_k^l(G) = \left\lceil \frac{\Delta}{k-1} \right\rceil + 1$. (2) If $g(G) \ge 8$, then $\Lambda_k^l(G) \le \left\lceil \frac{\Delta}{k-1} \right\rceil + 2$. (3) If $g(G) \ge 6$, then $\Lambda_k^l(G) \le \left\lceil \frac{\Delta}{k-1} \right\rceil + 3$.

Remark 1.1. In Theorems 1.2 and 1.3, we always respectively assume $M \ge k$ or $\Delta \ge k$. That is because if we assume M < k or $\Delta < k$, then $\Lambda_k^l(G) = \chi_a^l(G)$ holds for any graph G, where $\chi_a^l(G)$ denotes the acyclic choice number of G.

2. Proof of Theorem 1.2

In Claim 2.1 below, we will use (p) to denote the relevant part of Theorem 1.2 (p = 1, 2, 3). For brevity we will write $Q = \left\lceil \frac{M}{k-1} \right\rceil$ and

q = Q + p, so that in part (p) we wish to prove that $\Lambda_k^l(G) \leq q$. Note that, since $M \geq k$,

(2.1)
$$Q \ge 2 \text{ and } q = Q + p \ge p + 2.$$

Suppose that part (p) of Theorem 1.2 is false. Let G be a minimal counterexample to it; that is, every proper subgraph H of G is k-forested q-choosable but G itself is not. (Here note that $mad(H) \leq mad(G)$ if H is a subgraph of G.) Let L be a list assignment of a list L(v) of q colors to each vertex $v \in V(G)$, such that G has no k-forested L-coloring.

By the minimality of G, every proper subgraph H of G has a k-forested L-coloring. If c is a k-forested L-coloring of a proper induced subgraph H of G, and $v \in V(G)$, we use $c(N_G(v))$ to denote the set of colors used by c on neighbors of v, and $C_{k-1}(v)$ to denote the set of colors that are each used by c on exactly k-1 neighbors of v. Note that if v has at least one neighbor that is uncolored, then (2.2)

$$|C_{k-1}(v)| \le \left\lfloor \frac{d_G(v) - 1}{k - 1} \right\rfloor \le \left\lfloor \frac{\Delta - 1}{k - 1} \right\rfloor \le \left\lfloor \frac{M - 1}{k - 1} \right\rfloor = \left\lceil \frac{M}{k - 1} \right\rceil - 1 = Q - 1.$$

Claim 2.1. *G* does not contain any of the following configurations: (C1) a 1-vertex;

(C2) a 2-vertex adjacent to a ($\leq p$)-vertex;

(C3) if $p \leq k-2$, a 2-vertex adjacent to a $(\leq p+1)$ -vertex and a $(\leq 2p+1)$ -vertex;

(C4) if p = 3, a 4-vertex adjacent to three or more 2-vertices;

(C5) if p = 3, a 5-vertex adjacent to five 2-vertices.

Remark 2.1. During proving Claim 2.1, we assume only that $k \ge 2$ in (C1), $k \ge p+1$ in (C2), $k \ge \max\{p+2,4\}$ in (C3), $k, p \ge 3$ in (C4), and $k \ge 4, p \ge 3$ in (C5). These conditions certainly hold if the conditions given in (C3)–(C5) hold and also $p \le 3$ and $k \ge 4$, as stated in Theorem 1.2.

Remark 2.2. In each part of the following proof, we first delete a set of vertices $\{x_1, \ldots, x_n\}$ from G to obtain an induced subgraph H that satisfies Theorem 1.2, and then extend the coloring c of H to each of x_1, \ldots, x_n one by one. One should be careful here to update the color set $C_{k-1}(\cdot)$ each time when c has been extended. For example, the color set $C_{k-1}(\cdot)$ in terms of the coloring c of H may be different from the one in terms of the coloring c of $H + v_1$ after extending c to v_1 , but we still use the same notation for simplicity.

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Proof. (C1) Suppose G contains a 1-vertex v. Let c be a k-forested L-coloring of G - v, which exists by the minimality of G. Denote the neighbor of v by u, and define

$$F(v) := \{c(u)\} \cup C_{k-1}(u).$$

Then $|F(v)| \leq Q$ by (2.2), and so $L(v) \setminus F(v) \neq \emptyset$ since |L(v)| = q > Q by (2.1). So we can color v with a color in $L(v) \setminus F(v)$, and the coloring obtained is a k-forested L-coloring of G, which is a contradiction.

(C2) Suppose G contains a 2-vertex v which is adjacent to a $(\leq p)$ -vertex u. Let the other neighbor of v be w. In view of (C1) we may assume that $p \geq 2$. Let c be a k-forested L-coloring of G - v. Note that $C_{k-1}(u) = \emptyset$, since p - 1 < k - 1. Define

$$F(v) := \begin{cases} \{c(u)\} \cup c(N_G(u)) \cup C_{k-1}(w), & \text{if } c(u) = c(w); \\ \{c(u), c(w)\} \cup C_{k-1}(w), & \text{if } c(u) \neq c(w). \end{cases}$$

Then, by (2.2), $|F(v)| \leq 1 + (p-1) + (Q-1) < Q + p = q = |L(v)|$, and so we can color v with a color in $L(v) \setminus F(v)$. This gives a k-forested L-coloring of G, which is a contradiction.

(C3) Suppose G contains a 2-vertex v which is adjacent to a $(\leq p+1)$ -vertex u and a $(\leq 2p+1)$ -vertex w. Let c be a k-forested L-coloring of G-v. Note that $C_{k-1}(u) = \emptyset$, since p < k-1 by hypothesis. Define

$$F(v) := \begin{cases} \{c(u)\} \cup [c(N_G(u)) \cap c(N_G(w))] \cup C_{k-1}(w), & \text{if } c(u) = c(w); \\ \{c(u), c(w)\} \cup C_{k-1}(w), & \text{if } c(u) \neq c(w). \end{cases}$$

Let $i = |c(N_G(u)) \cap c(N_G(w))| \le |c(N_G(u))| \le p$. If c(u) = c(w) then

$$|F(v)| \le 1 + i + \left\lfloor \frac{2p - i}{k - 1} \right\rfloor \le 1 + p + \left\lfloor \frac{p}{k - 1} \right\rfloor = 1 + p < q = |L(v)|$$

by (2.1), since p < k - 1. So suppose $c(u) \neq c(w)$. If p = 1 then $|c(N_G(w))| \leq 2$ and so $C_{k-1}(w) = \emptyset$, since k - 1 > 2; thus $|F(v)| \leq 2 < 3 \leq |L(v)|$ by (2.1). If $p \geq 2$, then |F(v)| < |L(v)| by the same argument as in (C2). In every case we can color v with a color from $L(v) \setminus F(v)$ to get a k-forested L-coloring of G, which is a contradiction.

(C4) Suppose p = 3 and G contains a 4-vertex v which is adjacent to three 3-vertices x, y, z. Denote the other neighbors of v, x, y, zby w, x', y', z' respectively. Let c be a k-forested L-coloring of $G - \{v, x, y, z\}$. Clearly $C_{k-1}(v) = \emptyset$. Give z a color $c(z) \in L(z) \setminus F(z)$ where

$$F(z) := \{c(w), c(z')\} \cup C_{k-1}(z');$$

this is possible since $|L(z)| \ge Q+3$ by (2.1), while $|C_{k-1}(z')| \le Q-1$ by (2.2). Next, noting that v has colored neighbors z, w where $c(z) \ne c(w)$, and $C_{k-1}(u) = \emptyset$ for all $u \in N_G(v) \setminus \{w\}$, give v a color $c(v) \in L(v) \setminus F(v)$ where

$$F(v) := \{c(w), c(z), c(x')\} \cup C_{k-1}(w).$$

Then, noting that $|C_{k-1}(v)| = \left\lfloor \frac{1}{k-1} \right\rfloor = 0$ since $c(z) \neq c(w)$, give y a color from $L(y) \setminus F(y)$ where

$$F(y) := \begin{cases} \{c(v), c(w), c(z)\} \cup C_{k-1}(y'), & \text{if } c(v) = c(y'); \\ \{c(v), c(y')\} \cup C_{k-1}(y'), & \text{if } c(v) \neq c(y'). \end{cases}$$

Finally, noting that $c(v) \neq c(x')$, give x a color from $L(x) \setminus F(x)$ where

$$F(x) := \{c(v), c(x')\} \cup C_{k-1}(v) \cup C_{k-1}(x'),$$

which is possible since now $C_{k-1}(v) \leq \left\lfloor \frac{2}{k-1} \right\rfloor \leq 1$. This result is a *k*-forested coloring of *G*, a contradiction.

(C5) Suppose p = 3 and G contains a 5-vertex v which is adjacent to five 2-vertices x_1, \dots, x_5 . Denote the other neighbor of x_i by x'_i $(i = 1, \dots, 5)$. Let c be a k-forested L-coloring of $G - \{v, x_1, x_2, x_3, x_4\}$. (In fact we do not need $d(x_5) = 2$, only assuming $d(x_5) < k$ is enough so that when we prepare to color v, $C_{k-1}(x_5) = \emptyset$.) Give x_1 a color $c(x_1) \in L(x_1) \setminus F(x_1)$ where

$$F(x_1) := \{c(x_1'), c(x_5)\} \cup C_{k-1}(x_1'),$$

then give v a color $c(v) \in L(v) \setminus F(v)$ where

$$F(v) := \{c(x_1), c(x_5), c(x'_2), c(x'_3)\},\$$

which is possible since $|L(v)| \ge p + 2 = 5$ by (2.1). Now, noting that $c(x_1) \ne c(x_5)$ so that (even after x_2 is colored) $|C_{k-1}(v)| \le \left\lfloor \frac{2}{k-1} \right\rfloor = 0$, and $c(v) \notin \{c(x'_2), c(x'_3)\}$, give x_i a color from $L(x_i) \setminus F(x_i)$ where

$$F(x_i) := \{c(v), c(x'_i)\} \cup C_{k-1}(x'_i) \ (i = 2, 3).$$

Finally, give x_4 a color from $L(x_4) \setminus F(x_4)$ where

$$F(x_4) := \begin{cases} \{c(v), c(x_1), c(x_5)\} \cup C_{k-1}(x'_4), & \text{if } c(v) = c(x'_4); \\ \{c(v), c(x'_4)\} \cup C_{k-1}(v) \cup C_{k-1}(x'_4), & \text{if } c(v) \neq c(x'_4), \end{cases}$$

which is possible since now $|C_{k-1}(v)| \leq \left\lfloor \frac{3}{k-1} \right\rfloor \leq 1$. This result is a *k*-forested *L*-coloring of *G*, a contradiction.

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In the next, we will complete the proof of each part of Theorem 1.2 by a discharging procedure applying to the minimal counterexample G to the theorem. We involve the same idea during each of the three proofs (assign each vertex $v \in V(G)$ an initial charge w(v) = d(v)) and the only differences are the definition of the discharging rules and the estimation on the final charge $w^*(v)$ of each vertex v in G.

Proof of Theorem 1.2(1). We define discharging rules as follows.

R1.1. Each 3-vertex gives $\frac{1}{5}$ to each adjacent 2-vertex;

R1.2. Each \geq 4-vertex gives $\frac{2}{5}$ to each adjacent 2-vertex.

Since the configuration (C1) in Claim 2.1 is forbidden in G, we assume that $d(v) \geq 2$ for any vertex $v \in V(G)$. Suppose d(v) = 2. If v is adjacent to a 2-vertex, then by the forbiddance of configuration (C3) in G, v receives $\frac{2}{5}$ from its another neighbor; if v is not adjacent to any 2-vertex, then v also receives at least $\frac{2}{5}$ from its neighbors. So $w^*(v) \geq w(v) + \frac{2}{5} = \frac{12}{5}$, since v gives nothing. Assume that d(v) = 3. By R1.1, it gives out at most $\frac{3}{5}$. So $w^*(v) \geq w(v) - \frac{3}{5} = 3 - \frac{3}{5} = \frac{12}{5}$. Assume that $d(v) = d \geq 4$. By R1.2, it gives out at most $\frac{2d}{5}$. So $w^*(v) \geq w(v) - \frac{2d}{5} = d - \frac{2d}{5} = \frac{3d}{5} \geq \frac{12}{5}$. Thus $w^*(v) \geq \frac{12}{5}$ for each vertex $v \in V(G)$, proving that

$$\operatorname{mad}(G) \ge \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{\sum_{v \in V(G)} w(v)}{|V(G)|} \\ = \frac{\sum_{v \in V(G)} w^*(v)}{|V(G)|} \ge \frac{12|V(G)|/5}{|V(G)|} = \frac{12}{5}.$$

This contradiction proves Theorem 1.2(1).

Proof of Theorem 1.2(2). We define discharging rules as follows.

- **R2.1**. Each 3-vertex gives $\frac{1}{9}$ to each adjacent 2-vertex;
- **R2.2**. Each *d*-vertex $(4 \le d \le 5)$ gives $\frac{1}{3}$ to each adjacent 2-vertex;
- **R2.3**. Each \geq 6-vertex gives $\frac{5}{9}$ to each adjacent 2-vertex.

Similarly as above, we assume that $d(v) \ge 2$ for any vertex $v \in V(G)$. Suppose d(v) = 2. Then v cannot be adjacent to any 2-vertex since (C2) can not appear in G by Claim 2.1. If v is adjacent to a 3-vertex, then by the forbiddance of configuration (C3) in G, another neighbor of v must be a (≥ 6)-vertex, so v receives totally $\frac{1}{9} + \frac{5}{9} = \frac{2}{3}$ by R2.1 and R2.3. If v is not adjacent to any 3-vertex, then by R2.2 and R2.3, v receives at least $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. So $w^*(v) \ge w(v) + \frac{2}{3} = \frac{8}{3}$, since v gives nothing. Suppose d(v) = 3. Then v gives out at most $\frac{1}{3}$ by R2.1, so $w^*(v) \ge w(v) - \frac{1}{3} = \frac{8}{3}$. Similarly, we can prove that $w^*(v) \ge \frac{8}{3}$ for any (≥ 4) -vertex. Thus $w^*(v) \ge \frac{8}{3}$ for each vertex $v \in V(G)$, proving that mad $(G) \ge \frac{8}{3}$. This contradiction completes the proof of Theorem 1.2(2).

Proof of Theorem 1.2(3). We define discharging rules as follows.

R3. Each \geq 4-vertex gives $\frac{1}{2}$ to each adjacent 2-vertex.

Similarly we first assume $d(v) \ge 2$ for any $v \in V(G)$. Suppose d(v) = 2. Then the two neighbors of v must be (≥ 4) -vertices since the configuration (C2) in Claim 2.1 is forbidden in G. Thus, v receives together 1 from its neighbors but gives nothing by R3, which implies that $w^*(v) \ge w(v) + 1 \ge 3$. Suppose d(v) = 3. Note that v receives and gives nothing by R3, so $w^*(v) = w(v) = 3$. Suppose d(v) = 4. By the forbiddance of configuration (C4) in G, v can be adjacent to at most two 2-vertices, so it gives out at most $2 \times \frac{1}{2} = 1$ by R3. This implies $w^*(v) \ge w(v) - 1 = 3$. Suppose d(v) = 5. Noting that the configuration (C5) can not occur in G, v can be adjacent to at most four 2-vertices, so it gives out at most $4 \times \frac{1}{2} = 2$ by R3. This implies $w^*(v) \ge w(v) - 2 = 3$. Suppose $d(v) = t \ge 6$. We have $w^*(v) \ge w(v) - \frac{1}{2}t = \frac{1}{2}t \ge 3$ by R3. Thus $w^*(v) \ge 3$ for each vertex $v \in V(G)$, proving that mad $(G) \ge 3$. This contradiction completes the proof of Theorem 1.2.

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