List (*d*,1)-Total Labelling of Graphs Embedded in Surfaces*

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Abstract The (d,1)-total labelling of graphs was introduced by Havet and Yu. In this paper, we consider the list version of (d,1)-total labelling of graphs. Let G be a graph embedded in a surface with Euler characteristic ε whose maximum degree $\Delta(G)$ is sufficiently large. We prove that the list (d,1)-total labelling number $Ch^{\rm T}_{d,1}(G)$ of G is at most $\Delta(G)+2d$.

Keywords (d,1)-total labelling, list (d,1)-total labelling, list (d,1)-total labelling number, graphs

Chinese Library Classification O157.5

2010 Mathematics Subject Classification 05C15

关于可嵌入曲面图的列表 $(d,1)$- 全标号问题

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摘要 图的 $(d,1)$- 全标号问题最初是由 Havet 等人提出的. 在本文中，我们考虑了可嵌入曲面图的列表 $(d,1)$- 全标号问题，并证明了其列表 $(d,1)$- 全标号数不超过 $\Delta(G)+2d$.

关键词 $(d,1)$- 全标号，列表 (d,1)- 全标号，列表 (d,1)- 全标号数，图

中图分类号 O157.5

数学分类号 05C15

0 Introduction

In this paper, graph G is a simple connected graph with a finite vertex set $V(G)$ and a finite edge set $E(G)$. If X is a set, we usually denote the cardinality of X by $|X|$. Denote the set of vertices adjacent to v by $N(v)$. The degree of a vertex v in G, denoted by $d_G(v)$, is the number of edges incident with v. We sometimes write $V, E, d(v), \Delta, \delta$ instead of $V(G), E(G), d_G(v), \Delta(G), \delta(G)$, respectively. Let G be a plane graph. We always denote by $F(G)$ the face set of G. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where cut edge is counted twice. A k-, k^+- and k^--vertex (or face) in graph G is a vertex (or face) of degree k, at least k and at most k, respectively.

收稿日期： 2011 年 2 月 3 日.

* Supported by GIIFSDU(yzc11025), NNSF(61070230, 11026184, 10901097) and RFDP (200804220001, 20100131120017) and SRF for ROCS.

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The $(d,1)$-total labelling of graphs was introduced by Havet and Yu[1]. A *k-(d,1)-total labelling* of a graph G is a function c from $V(G) \cup E(G)$ to the color set $\{0, 1, \cdots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$, $c(e) \neq c(e')$ if e and e' are two adjacent edges, and $|c(u) - c(e)| \geqslant d$ if vertex u is incident to the edge e. The minimum k such that G has a k-$(d,1)$-total labelling is called the *(d,1)-total labelling number* and denoted by $\lambda_d^{\mathrm{T}}(G)$. Readers are referred to [2,4-7] for further research.

Suppose that $L(x)$ is a list of colors available to choose for each element $x \in V(G) \cup E(G)$. If G has a $(d,1)$-total labelling c such that $c(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that c is an L-$(d,1)$-total labelling of G, and G is L-$(d,1)$-total labelable (sometimes we also say G is list $(d,1)$-total labelable). Furthermore, if G is L-$(d,1)$-total labelable for any L with $|L(x)| = k$ for each $x \in V(G) \cup E(G)$, we say that G is k-$(d,1)$-total choosable. The list $(d,1)$-total labelling number, denoted by $Ch_{d,1}^{\mathrm{T}}(G)$, is the minimum k such that G is k-$(d,1)$-total choosable. Actually, when $d = 1$, the list (1,1)-total labelling is the well-known list total coloring of graphs. It is known that for list version of total colorings there is a list total coloring conjecture (LTCC). Therefore, it is natural to conjecture that $Ch_{d,1}^{\mathrm{T}}(G) = \lambda_d^{\mathrm{T}}(G) + 1$. Unfortunately, counterexamples that $Ch_{d,1}^{\mathrm{T}}(G)$ is strictly greater than $\lambda_d^{\mathrm{T}}(G) + 1$ can be found in [9]. Although we can not present a conjecture like LTCC, we conjecture that

$$Ch_{d,1}^{\mathrm{T}}(G) \leqslant \Delta + 2d$$

for any graph G. In [9], we studied the list $(d,1)$-total labelling of special graphs such as paths, trees, stars and outerplanar graphs which lend positive support to our conjecture.

In this paper, we prove that, for graphs embedded in a surface with Euler characteristic ε, the conjecture is still true when the maximum degree is sufficiently large. Our main results are the following:

Theorem 0.1 Let G be a graph embedded in a surface of Euler characteristic $\varepsilon \leqslant 0$ and

$$\Delta(G) \geqslant \frac{d}{2d-1}\left(10d - 8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon}\right) + 1,$$

where $d \geqslant 2$. Then

$$Ch_{d,1}^{\mathrm{T}}(G) \leqslant \Delta(G) + 2d.$$

Theorem 0.2 Let G be a graph embedded in a surface of Euler characteristic $\varepsilon > 0$. If $\Delta(G) \geqslant 5d + 2$ where $d \geqslant 2$, then

$$Ch_{d,1}^{\mathrm{T}}(G) \leqslant \Delta(G) + 2d.$$

We prove two conclusions which are slightly stronger than the theorems above as follows.

Theorem 0.3 Let G be a graph embedded in a surface of Euler characteristic $\varepsilon \leqslant 0$ and let positive integer

$$M \geqslant \frac{d}{2d-1}\left(10d - 8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon}\right) + 1,$$

where $d \geqslant 2$. If $\Delta(G) \leqslant M$, then

$$Ch_{d,1}^{\mathrm{T}}(G) \leqslant M + 2d.$$

In particular,

$$Ch^{\mathrm{T}}_{d,1}(G) \leqslant \Delta(G) + 2d \quad \text{if} \quad \Delta(G) = M.$$

Theorem 0.4 Let G be a graph embedded in a surface of Euler characteristic $\varepsilon > 0$ and let positive integer $M \geqslant 5d + 2$ where $d \geqslant 2$. If $\Delta(G) \leqslant M$, then

$$Ch^{\mathrm{T}}_{d,1}(G) \leqslant M + 2d.$$

In particular,

$$Ch^{\mathrm{T}}_{d,1}(G) \leqslant \Delta(G) + 2d \quad \text{if} \quad \Delta(G) = M.$$

The interesting cases of Theorem 0.3 and Theorem 0.4 are when $M = \Delta(G)$. Indeed, Theorem 0.3 and Theorem 0.4 are only technical strengthening of Theorem 0.1 and Theorem 0.2, respectively. But without them we would get complications when a subgraph $H \subset G$ such that $\Delta(H) < \Delta(G)$ is considered.

In Section 1, we prove some lemmas. In Section 2, we complete our main proof with discharging method.

1 Structural properties

From now on, we will use without distinction the terms *colors* and *labels*. Let c be a partial list (d,1)-total labelling of G. We denote by $A(x)$ the set of colors which are still available for coloring element x of G with the partial list (d,1)-total labelling c. Let G be a minimal counterexample in terms of $|V(G)| + |E(G)|$ to Theorem 0.3 or Theorem 0.4.

Lemma 1.1 G is connected.

Proof Suppose that G is not connected. Without loss of generality, let G_1 be one component of G and $G_2 = G \backslash G_1$. By the minimality of G, G_1 and G_2 are both $(M + 2d)$-(d,1)-total choosable which implies G is $(M + 2d)$-(d,1)-total choosable, a contradiction.

Lemma 1.2 For each $e = uv \in E(G)$,

$$d(u) + d(v) \geqslant M - 2d + 4.$$

Proof Suppose to the contrary that there exists some edge $e = uv \in E(G)$ such that

$$d(u) + d(v) \leqslant M - 2d + 3.$$

By the minimality of G, $G - e$ is $(M + 2d)$-(d,1)-total choosable. We denote this coloring by c. Since

$$\begin{aligned} |A(e)| &\geqslant M + 2d - (d(u) + d(v) - 2) - 2(2d - 1) \\ &\geqslant M + 2d - (M - 2d + 1) - 2(2d - 1) \\ &\geqslant 1 \end{aligned}$$

under the coloring c, we can extend c to G, a contradiction.

Lemma 1.3 For any edge $e = uv \in E(G)$ with

$$\min\{d(u), d(v)\} \leqslant \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor,$$

we have

$$d(u) + d(v) \geqslant M + 3.$$

Proof Suppose there is some $e = uv \in E(G)$ such that

$$d(u) \leqslant \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$$

and

$$d(u) + d(v) \leqslant M + 2.$$

By the minimality of G, $G - e$ is $(M + 2d)$-$(d$,1)-total choosable. Erase the color of vertex u, and let c be the partial list $(d$,1)-total labelling with $|L| = M + 2d$. Then

$$\begin{aligned} |A(e)| &\geqslant M + 2d - (d(u) + d(v) - 2) - (2d - 1) \\ &\geqslant M + 2d - M - (2d - 1) \\ &\geqslant 1, \end{aligned}$$

which implies that e can be properly colored. Next, for vertex u,

$$\begin{aligned} |A(u)| &\geqslant M + 2d - (d(u) + (2d - 1)d(u)) \\ &\geqslant M + 2d - (M + 2d - 1) \\ &\geqslant 1. \end{aligned}$$

Thus we extend the coloring c to G, a contradiction.

Lemma 1.4 ([2]) A bipartite graph G is edge f-choosable where $f(uv) = \max\{d(u), d(v)\}$ for any $uv \in E(G)$.

A k-*alternator* for some k ($3 \leqslant k \leqslant \lfloor \frac{M+2d-1}{2d} \rfloor$) is a bipartite subgraph $B(X, Y)$ of graph G such that $d_B(x) = d_G(x) \leqslant k$ for each $x \in V(G)$ and $d_B(y) \geqslant d_G(y) + k - M - 1$ for each $y \in Y$.

The concept of k-alternator was first introduced by Borodin, Kostochka and Woodall [3] and generalized by Wu and Wang [8].

Lemma 1.5 There is no k-alternator $B(X, Y)$ in G for any integer k with $3 \leqslant k \leqslant \lfloor \frac{M+2d-1}{2d} \rfloor$.

Proof Suppose that there exits a k-alternator $B(X, Y)$ in G. Obviously, X is an independent set of vertices in graph G by Lemma 2.3. By the minimality of G, we can color all elements of subgraph $G[V(G) \backslash X]$ from their lists of size $M + 2d$. We denote this partial list $(d$,1)-total labelling by c. Then for each edge $e = xy \in B(X, Y)$,

$$\begin{aligned} |A(e)| &\geqslant M + 2d - (d_G(y) - d_B(y) + (2d - 1)) \\ &\geqslant M + 2d - (M - d_B(y) + (2d - 1)) \\ &\geqslant d_B(y) \end{aligned}$$

and

$$|A(e)| \geqslant M + 2d - (d_G(y) - d_B(y) + (2d - 1))$$

$$\begin{aligned}&\geqslant M+2d-(M+2d-k)\\&\geqslant k\end{aligned}$$

because $B(X,Y)$ is a k-alternator. Therefore,

$$|A(e)|\geqslant \max\{d_B(y),d_B(x)\}.$$

By Lemma 1.4, it follows that $E(B(X,Y))$ can be colored properly from their new color lists. Next, for each vertex $x\in X$,

$$\begin{aligned}|A(x)|&\geqslant M+2d-(d(x)+(2d-1)d(x))\\&\geqslant M+2d-(M+2d-1)\\&\geqslant 1,\end{aligned}$$

because $d_G(x)\leqslant k\leqslant \lfloor\frac{M+2d-1}{2d}\rfloor$. Thus we extend the coloring c to G, a contradiction.

Lemma 1.6 Let

$$X_k=\{x\in V(G)\mid d_G(x)\leqslant k\}\quad\text{and}\quad Y_k=\bigcup_{x\in X_k}N(x)$$

for any integer k with $3\leqslant k\leqslant \lfloor\frac{M+2d-1}{2d}\rfloor$. If $X_k\neq\emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k, such that $d_{M_k}(x)=1$ for each $x\in X_k$ and $d_{M_k}(y)\leqslant k-2$ for each $y\in Y_k$.

Proof The proof is omitted here as it is similar with the proof of Lemma 2.4 in [8].

We call y the k-master of x if $xy\in M_k$ and $x\in X_k, y\in Y_k$. By Lemma 1.3, if $uv\in E(G)$ satisfies

$$d(v)\leqslant\left\lfloor\frac{M+2d-1}{2d}\right\rfloor\quad\text{and}\quad d(u)=M-i,$$

then

$$d(v)\geqslant M+3-d(u)\geqslant i+3.$$

Together with Lemma 1.6, it follows that each $(M-i)$-vertex can be a j-master of at most $j-2$ vertices, where $3\leqslant i+3\leqslant j\leqslant\lfloor\frac{M+2d-1}{2d}\rfloor$. Each i-vertex has a j-master by Lemma 1.6, where $3\leqslant i\leqslant j\leqslant\lfloor\frac{M+2d-1}{2d}\rfloor$.

2 Proof of main results

By our Lemmas above, G has structural properties in the following.

(C1) G is connected;

(C2) for each $e=uv\in E(G)$, $d(u)+d(v)\geqslant M-2d+4$;

(C3) if $e=uv\in E(G)$ and $\min\{d(u),d(v)\}\leqslant\lfloor\frac{M+2d-1}{2d}\rfloor$, then $d(u)+d(v)\geqslant M+3$;

(C4) each i-vertex (if exists) has one j-master, where $3\leqslant i\leqslant j\leqslant\lfloor\frac{M+2d-1}{2d}\rfloor$;

(C5) each $(M-i)$-vertex (if exists) can be a j-master of at most $j-2$ vertices, where $3\leqslant i+3\leqslant j\leqslant\lfloor\frac{M+2d-1}{2d}\rfloor$.

Proof of Theorem 0.3 Let G be a minimal counterexample in terms of $|V(G)|+|E(G)|$ to Theorem 0.3. In this theorem,

$$\begin{aligned} M &\geqslant \frac{d}{2d-1}\left(10d-8+\sqrt{(10d-2)^2-24(2d-1)\varepsilon}\right)+1 \\ &\geqslant 10d+1. \end{aligned}$$

Thus

$$\left\lfloor \frac{M+2d-1}{2d} \right\rfloor \geqslant 6.$$

In the following, we apply the discharging method to complete the proof by a contradiction. At the very beginning, we assign an initial charge $w(x)=d(x)-6$ for any $x \in V(G)$. By Euler's formula

$$|V|-|E|+|F|=\varepsilon,$$

we have

$$\begin{aligned} \sum_{x\in V} w(x) &= \sum_{x\in V}(d(x)-6) \\ &= -6\varepsilon - \sum_{x\in F}(2d(x)-6) \\ &\leqslant -6\varepsilon. \end{aligned}$$

The discharging rule is as follows.

(R1) each i-vertex (if exists) receives charge 1 from each of its j-master, where $3 \leqslant i \leqslant j \leqslant 5$.

If $M \geqslant \Delta+3$, then $\delta(G) \geqslant 6$. Otherwise, let $uv \in E(G)$ and $d(u) \leqslant 5$. Then

$$d(u)+d(v) \leqslant M-3+5 \leqslant M+2$$

and

$$d(u) \leqslant \left\lfloor \frac{M+2d-1}{2d} \right\rfloor \quad \text{as} \quad \left\lfloor \frac{M+2d-1}{2d} \right\rfloor \geqslant 6,$$

which is a contradiction to (C3). This obviously contradicts the fact $\delta(G) \leqslant 5$ for any planar graph. Proof of the theorem is completed. Next, we only consider the case $\Delta \leqslant M \leqslant \Delta+2$.

Claim 1 $\delta \geqslant M-\Delta+3$.

Proof If there is some $e=uv \in E(G)$ such that $d(v) \leqslant M-\Delta+2$, then

$$d(u)+d(v) \leqslant \Delta+(M-\Delta+2) \leqslant M+2$$

and

$$d(v) \leqslant 5 \leqslant \left\lfloor \frac{M+2d-1}{2d} \right\rfloor \quad \text{as} \quad \left\lfloor \frac{M+2d-1}{2d} \right\rfloor \geqslant 6,$$

a contradiction to (C3).

Let v be a k-vertex of G.

(a) If $3 \leqslant k \leqslant 5$, then

$$w'(v) = w(v) + \sum_{k\leqslant i\leqslant 5} 1 = (k-6)+(6-k) = 0$$

by (C4) and rule (R1);

(b) If $6 \leqslant k \leqslant M-3$, then for all $u \in N(v)$, $d(u) \geqslant 6$ by (C3). Therefore, v neither receives nor gives any charge by our rule, which implies that $w'(v) = w(v) = k-6 \geqslant 0$;

(c) If $M-2 \leqslant k \leqslant \Delta$.

Case 1 $M = \Delta + 2$. Then $\delta \geqslant 5$ by Claim 1. For $k = \Delta$, $w'(v) \geqslant w(v) - 3 = \Delta - 9 = M - 11$ by (C5) and (R1).

Case 2 $M = \Delta + 1$. Then $\delta \geqslant 4$ by Claim 1. For $k = \Delta - 1$, $w'(v) \geqslant w(v) - 3 = \Delta - 1 - 6 - 3 = M - 11$ by (C5) and rule (R1). For $k = \Delta$, $w'(v) \geqslant w(v) - 3 - 2 = \Delta - 6 - 3 - 2 = M - 12$ by (C5) and rule (R1).

Case 3 $M = \Delta$. Then $\delta(G) \geqslant 3$ by Claim 1. For $k = \Delta - 2$, $w'(v) \geqslant w(v) - 3 = \Delta - 2 - 6 - 3 = M - 11$ by (C5) and rule (R1). For $k = \Delta - 1$, $w'(v) \geqslant w(v) - 3 - 2 = \Delta - 1 - 6 - 3 - 2 = M - 12$ by (C5) and rule (R1). For $k = \Delta$, $w'(v) \geqslant w(v) - 3 - 2 - 1 = \Delta - 6 - 3 - 2 - 1 = M - 12$ by (C5) and rule (R1).

For all cases above, $w'(v) \geqslant M - 12 > 0$ for any $d(v) \geqslant \Delta - 2$ as $M \geqslant 10d + 1 \geqslant 21$.

Let $X = \{x \in V(G) \big| d_G(x) \leqslant \lfloor \frac{M+2d-1}{2d} \rfloor\}$. By (C3), X is an independent set of vertices.

Claim 2 The number of $\left(\lfloor \frac{M+2d-1}{2d} \rfloor + 1\right)^+$-vertex of G is at least $M - \lfloor \frac{M+2d-1}{2d} \rfloor + 3$. That is,

$$|V(G \backslash X)| \geqslant M - \left\lfloor \frac{M+2d-1}{2d} \right\rfloor + 3.$$

Proof Otherwise, let $Y = N_{x \in X}(x)$ and $B = B(X, Y)$ be the induced bipartite subgraph. For all $y \in Y$,

$$d_{G \backslash X}(y) \leqslant |Y| - 1 \leqslant M - \left\lfloor \frac{M+2d-1}{2d} \right\rfloor + 1.$$

Therefore,

$$d_B(y) = d_G(y) - d_{G \backslash X}(y) \geqslant d_G(y) + \left\lfloor \frac{M+2d-1}{2d} \right\rfloor - M - 1,$$

which implies B is a $\lfloor \frac{M+2d-1}{2d} \rfloor$-alternator of G, a contradiction to Lemma 2.5.

Since $M \geqslant 10d + 1$, it follows that

$$M - 12 > \left\lfloor \frac{M+2d-1}{2d} \right\rfloor - 5.$$

Thus,

$$w'(v) \geqslant \left\lfloor \frac{M+2d-1}{2d} \right\rfloor - 5$$

when $d_G(v) \geqslant \lfloor \frac{M+2d-1}{2d} \rfloor + 1$. Then

$$\begin{aligned}
\sum_{x \in V} w(x) &= \sum_{x \in V} w'(x) \\
&> \left(M - \left\lfloor \frac{M+2d-1}{2d} \right\rfloor + 3\right)\left(\left\lfloor \frac{M+2d-1}{2d} \right\rfloor - 5\right) \\
&\geqslant (2d-1)\left(\frac{M-1}{2d}\right)^2 - (10d-8)\frac{M-1}{2d} - 15
\end{aligned}$$

$$\geqslant -6\varepsilon$$

as

$$M \geqslant \frac{d}{2d-1}\left(10d-8+\sqrt{(10d-2)^2-24(2d-1)\varepsilon}\right)+1.$$

Then this contradiction completes the proof.

Proof of Theorem 0.4 Let G be a minimal counterexample in terms of $|V(G)|+|E(G)|$ to Theorem 0.4. In this theorem, $M \geqslant 5d+2$. We define the initial charge function $w(x) := d(x)-4$ for all element $x \in V \cup F$. By Euler's formula $|V|-|E|+|F| = \varepsilon$, we have

$$\sum_{x\in V\cup F} w(x) = \sum_{v\in V}(d(v)-4) + \sum_{f\in F}(d(f)-4) = -4\varepsilon < 0.$$

The transition rules are defined as follows.

(R1) Each 3-vertex (if exists) receives charge 1 from its 3-master.

(R2) Each k-vertex with $5 \leqslant k \leqslant 7$ transfer charge $\frac{k-4}{k}$ to each 3-face that incident with it.

(R3) Each 8^+-vertex transfer charge $\frac{1}{2}$ to each 3-face that incident with it.

Analogous with Claim 1 in the proof of Theorem 0.3, it is easy to prove that $\delta(G) \geqslant 3$ when $\Delta = M$ and $\delta(G) \geqslant 4$ otherwise. Let v be a k-vertex of G.

For $k = 3$, then $w'(v) = w(v)+1 = 3-4+1 = 0$ since it receives 1 from its 3-master;

For $k = 4$, then $w'(v) = w(v) = 0$ since we never change the charge by our rules;

For $5 \leqslant k \leqslant 7$, then $w'(v) \geqslant w(v) - k\frac{k-4}{k} = 0$ by (R2);

For $8 \leqslant k \leqslant M-1$, then $w'(v) \geqslant w(v) - k\frac{1}{2} \geqslant 0$ by (R3);

If $M > \Delta$, then $M-1 \geqslant \Delta$. Thus $w(v) \geqslant 0$ for all $v \in V(G)$. Otherwise, $\Delta = M$. Then for $k = \Delta$, $w'(v) \geqslant w(v) - \frac{1}{2}M - 1 = \frac{M}{2} - 5$ by (C5) and rules (R1), (R3). Since $M \geqslant 5d+2 \geqslant 12$, we have $w'(v) \geqslant \frac{M}{2} - 5 > 0$.

Let f be a k-face of G.

If $k \geqslant 4$, then $w'(f) = w(f) \geqslant 0$ since we never change the charge of them by our rules;

If $k = 3$, assume that $f = [v_1, v_2, v_3]$ with $d(v_1) \leqslant d(v_2) \leqslant d(v_3)$. It is easy to see $w(f) = -1$. Consider the subcases as follows.

(a) Suppose $d(v_1) = 3$. Then $M = \Delta$ and $d(v_2) = d(v_3) = \Delta$ by (C3). Thus, $w'(f) = w(f) + \frac{1}{2} \times 2 = 0$ by (R3);

(b) Suppose $d(v_1) = 4$. Then $d(v_3) \geqslant d(v_2) \geqslant M - 2d + 4 - d(v_1) \geqslant 3d+2 \geqslant 8$ by (C2). Therefore, $w'(f) = w(f) + \frac{1}{2} \times 2 = 0$ by (R3);

(c) Suppose $d(v_1) = 5$. Then $d(v_3) \geqslant d(v_2) \geqslant M - 2d + 4 - d(v_1) \geqslant 3d+1 \geqslant 7$ by (C2). Therefore, $w'(f) = w(f) + \frac{3}{7} \times 2 + \frac{1}{5} > 0$ by (R2).

(d) Suppose $d(v_1) = m \geqslant 6$. Then $d(v_3) \geqslant d(v_2) \geqslant 6$. Therefore, $w'(f) \geqslant w(f) + 3 \times \min\{\frac{m-4}{m}, \frac{1}{2}\} = 0$ by (R2) and (R3).

Thus, we have $\sum\limits_{x\in V\cup F} w'(x) \geqslant 0$ which is a contradiction with

$$\sum_{x\in V\cup F} w'(x) = \sum_{x\in V\cup F} w(x) < 0.$$

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