Degree conditions for the partition of a graph into triangles and quadrilaterals *

Xin Zhang, Jian-Liang Wu † and Jin Yan
School of Mathematics, Shandong University, Jinan, 250100, China

Abstract

For two positive integers r and s with $r \geq 2s-2$, if G is a graph of order $3r+4s$ such that $d(x)+d(y) \geq 4r+4s$ for every $xy \notin E(G)$, then G independently contains r triangles and s quadrilaterals, which partially prove the El-Zahar's Conjecture.

Keywords: degree, partition, triangle, quadrilateral

1 Introduction

In this paper, all graphs are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G. If $uv \in E(G)$, then u is said to be the *neighbor* of v. We use $N(v)$ to denote the set of neighbors of a vertex v. The *degree* $d(v) = |N(v)|$. A k*-vertex* is a vertex of degree k. For a subgraph(or a subset) H of G, we denote $N(v,H) = N(v) \cap V(H)$ and let $d(v,H) = |N(v,H)|$. The minimum degree sum $\sigma_2(G) = \min\{d(x)+d(y)|x,y \in V(G), xy \notin E(G)\}$(When G is a complete graph, we define $\sigma_2(G) = \infty$). For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. For subsets L and M of $V(G)$, if $L \cap M = \emptyset$, we say that L and M are *independent*, and let $E(L,M) = \{uv \in E(G) : u \in L, v \in M\}$ and $e(L,M) = |E(L,M)|$. The graph P_k is a path with k vertices, and C_k a cycle with k vertices. We call C_3 a triangle and C_4 a quadrilateral. We use mQ to represent m copies of graph Q. Other notations can be found in [3].

Degree conditions which guarantee that disjoint cycles with specified length exist in a graph, especially small cycles, are investigated in lots of paper. El-Zahar [6] gave the following conjecture.

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†Corresponding author. E-mail: jlwu@sdu.edu.cn.

Conjecture 1.1. *Let G be a graph. If $|V(G)| = n_1 + \cdots + n_k$ and $\delta(G) \geq \lceil n_1/2 \rceil + \cdots + \lceil n_k/2 \rceil$ where $n_i \geq 3 (1 \leq i \leq k)$. Then G contains k disjoint cycles of length $n_1, \cdots, n_k$, respectively.*

He also proved it for $k = 2$. The earlier result given by Corrádi and Hajnal [5] states that every graph of order at least $3k$ and the minimum degree at least $2k$ contains k disjoint cycles. In fact, this result just proves Conjecture 1 when $n_1 = \cdots = n_k = 3$. The case $n_1 = \cdots = n_k = 4$ is also called Erdös conjecture [7]. Randerath *et al* [9] proved that if a graph G has order $4k$ and $\delta(G) \geq 2k$, then G contains $k-1$ disjoint quadrilaterals and a subgraph of order 4 with at least four edges such that all the quadrilaterals are disjoint to the subgraph. It is very close to Erdös conjecture. Other corresponding results can be found in [2] and [10].

Here we consider the case $n_i \in \{3, 4\}$ of Conjecture 1.1. Aigner and Brandt [1] proved that if G is a graph such that $|V(G)| = 3r + 4s$ and $\delta(G) \geq 2r + \frac{8s}{3}$, then G contains r triangles and s quadrilaterals, all vertex disjoint. Brandt et al [4] proved that for two positive integers r and s, if G is a graph of order $n \geq 3r + 4s$ and $\sigma_2(G) \geq n + r$, then G contains r triangles and s cycles with length at most 4 which are vertex disjoint. Recently, Yan [11] improved the result and proved that if G is a graph of order $n \geq 3r + 4s + 3$ and $\sigma_2(G) \geq n + r$, then G contains r triangles and s quadrilaterals, all vertex disjoint.

In the paper , we prove that for two positive integers r and s, if G is a graph of order $n = 3r+4s$ and $\sigma_2(G) \geq n+r$, then $G \supseteq rC_3 \cup (s-1)C_4 \cup D$, where D is a graph of order four with at least four edges; moreover, if $r \geq 2s-2$, then $G \supseteq rC_3 \cup sC_4$. The result partially proves Conjecture 1.1 under the condition $n_i \in \{3, 4\}$ and $r \geq 2s - 2$, and at the same time, the corresponding result in [9] as described above is generalized.

2 Some Useful Lemmas

Lemma 2.1. [8] *Let P and Q be two disjoint paths where $P = P_3$. If $Q = P_2$ and $|E(P, Q)| \geq 3$, then $G[V(P \cup Q)] \supseteq C_4$. If $Q = P_3$ and $|E(P, Q)| \geq 4$, then $G[V(P \cup Q)] \supseteq C_4$.*

Lemma 2.2. [9] *Let $C = a_1a_2a_3a_4a_1$ be a quadrilateral of G and u, v be two non-adjacent vertices such that $\{u, v\} \subseteq V(G) - V(C)$. If $d(u, C) + d(v, C) \geq 5$, then $G[V(C) \cup \{u, v\}]$ contains a quadrilateral C' and an edge e such that C' and e are disjoint and e is incident with exactly one of u and v.*

Lemma 2.3. *Let $C = a_1a_2a_3a_1$ be a triangle of G and u, v be two non-adjacent vertices such that $\{u, v\} \subseteq V(G) - V(C)$. If $d(u, C) + d(v, C) \geq 5$,* *then $G[V(C) \cup \{u, v\}]$ contains a triangle $C^{'}$ and an edge e such that $C^{'}$ and e are disjoint and e is incident with exactly one of u and v.*

Proof. Without loss of generality, assume that $N(u, C) = \{a_1, a_2, a_3\}$ and $N(v, C) = \{a_1, a_2\}$. Then we choose $C^{'} = va_1a_2v$ and $e = ua_3$. □

Lemma 2.4. [9] *Let $C = a_1a_2a_3a_4a_1$ be a quadrilateral of G and M_1, M_2 be two paths in G with order 2. Suppose C, M_1, M_2 are disjoint and $e(C, M_1 \cup M_2) \geq 9$. Then $G[V(C \cup M_1 \cup M_2)] \supseteq C_4 \cup P_4$.*

Lemma 2.5. *Let $C = a_1a_2a_3a_1$ be a triangle of G and let M_1, M_2 be two paths in G with order 2. Suppose C, M_1, M_2 are disjoint and $e(C, M_1 \cup M_2) \geq 9$. Then $G[V(C \cup M_1 \cup M_2)] \supseteq C_3 \cup D$ where $|D| = 4$ and $|E(D)| \geq 4$.*

Proof. Let $M_1 = uv$ and $M_2 = xy$. Without loss of generality, we assume that $d(u, C) \geq d(v, C)$, $d(x, C) \geq d(y, C)$ and $d(u, C) + d(v, C) \geq d(x, C) + d(y, C)$. Since $d(u, C) + d(v, C) + d(x, C) + d(y, C) = e(C, M_1 \cup M_2) \geq 9$, we have that $d(u, C) = 3$, $d(u, C) + d(v, C) \geq 5$ and $d(x, C) + d(y, C) \geq 3$. Suppose $d(x, C) + d(y, C) = 3$. Then $d(x, C) \geq 2$ and $d(u, C) = d(v, C) = 3$. Without loss of generality, we assume that $a_1, a_2 \in N(x)$. Thus we can choose $C_3 = uva_3u$ and $F = G[\{x, y, a_1, a_2\}]$. Suppose $d(x, C) + d(y, C) \geq 4$. Then x and y have the same neighbor in C, say the neighbor is a_1. Thus we can choose $C_3 = xya_1x$ and $D = G[\{u, v, a_2, a_3\}]$. □

Lemma 2.6. [9] *Let C be a quadrilateral and P a path of order 4 in G such that C and P are independent. $G[V(C \cup P)]$ does not contain a quadrilateral C' and a path P' of order 4 such that C' and P' are independent and $e(G[V(C')]) > e(G[V(C)])$. If $e(C, P) \geq 9$, then $G[V(C \cup P)] \supseteq C_4 \cup D$ where $|D| = 4$ and $|E(D)| \geq 4$.*

Let F_4 be the graph such that $|F_4| = 4$, $e(F_4) = 4$ and $F_4 \not\supseteq C_4$. In fact, F_4 can be got from a claw by adding a new edge. From now on, we always write the only 3-vertex of F_4 as u_0, the two 2-vertices as u_1, u_2, the only 1-vertex as u_3, and $U = \{u_1, u_2, u_3\}$.

Lemma 2.7. [10] *Let Q be a quadrilateral. If $Q \cap F_4 = \emptyset$ and $e(U, Q) \geq 9$, then $G[V(Q \cup F_4)] \supseteq 2C_4$.*

Lemma 2.8. *Let T be a triangle and $T \cap F_4 = \emptyset$. If $d(u_3, T) \geq 2$, or $e(U, T) \geq 6$ and $d(u_3, T) > 0$, then $G[V(T \cup F_4)] \supseteq C_3 \cup C_4$.*

Proof. Write $T = c_1c_2c_3c_1$. If $d(u_3, T) \geq 2$, then $G[\{u_1, u_2, u_0\}] = C_3$ and $G[\{u_3, c_1, c_2, c_3\}] \supseteq C_4$. So we may assume that $d(u_3, T) = 1$, without loss of generality, we assume that $u_3c_1 \in E(G)$. Then $d(u_1, T) + d(u_2, T) \geq e(U, T) - d(u_3, T) \geq 5$. By the symmetry of u_1 and u_2, we just consider the case when $d(u_1, T) = 3$. If $u_2c_1 \in E(G)$, then we can choose $C_3 = u_1c_2c_3u_1$ and $C_4 = u_2c_1u_3u_0u_2$. Otherwise $d(u_2, T) = 2$, $u_2c_2, u_2c_3 \in E(G)$, and it follows that we choose $C_3 = u_2c_2c_3u_2$ and $C_4 = c_1u_1u_0u_3c_1$. □

Lemma 2.9. *Let T be a triangle. If $T \cap F_4 = \emptyset$ and $e(U,T) \geq 7$, then $G[V(T \cup F_4)] \supseteq C_3 \cup C_4$.*

Proof. Since $e(U,T) \geq 7$, $e(U,T) \geq 6$ and $d(u_3,T) > 0$. By Lemma 2.8, the lemma is true. □

3 Main Results and their Proofs

Lemma 3.1. *For two positive integers r and s, if G is a graph of order $n = 3r + 4s$ and $\sigma_2(G) \geq n + r$, then $G \supseteq rC_3 \cup (s-1)C_4$.*

This lemma follows from Yan's result [11] described in the introduction.

Theorem 3.2. *For two positive integers r and s, if G is a graph of order $n = 3r + 4s$ and $\sigma_2(G) \geq n + r$, then $G \supseteq rC_3 \cup (s-1)C_4 \cup D$, where D is a graph of order four with at least four edges.*

Proof. By Lemma 3.1, G independently contains r triangles $T_1, \cdots, T_r$ and $s-1$ quadrilaterals $Q_1, \cdots, Q_{s-1}$. Let $H_T = G[\bigcup_{i=1}^{r} V(T_i)]$, $H_Q = G[\bigcup_{i=1}^{s-1} V(Q_i)]$, $H = H_T \bigcup H_Q$ and $D = G - V(H)$. Then $|D| = 4$.

First, we can choose D such that D contains two independent edges. Suppose that there are two vertices x and y in $V(D)$ such that $xy \notin E(G)$ and $d(x,D) = d(y,D) = 0$. Then $d(x,H) + d(y,H) \geq \sigma_2(G) \geq 4r + 4s > 4(r+s-1)$, and it follows that there exists a cycle $C \in \{T_1, \cdots, T_r, Q_1, \cdots, Q_{s-1}\}$ in H such that $e(\{x,y\}, C) \geq 5$. By Lemma 2.2 and Lemma 2.3, we have $G[V(C) \cup \{x,y\}] \supseteq C' \cup K$ where $|C'| = |C|$ and K is an edge. We replace C by C' in H. Thus the new H independently contains r triangles and $s-1$ quadrilaterals and the new D satisfies $|E(D)| \geq 1$. So we assume that $|E(D)| \geq 1$. Let $uv \in E(D)$ and $\{z,w\} = V(D) - \{u,v\}$. If D does not contain two independent edges, then $zw \notin E(D)$ and $e(\{z,w\}, uv) \leq 2$. So $d(z,H) + d(w,H) \geq \sigma_2(G) - e(\{z,w\}, uv) \geq 4r + 4s - 2 > 4(r+s-1)$. By the similar argument as above, we can find a new H and D such that D contains two independent edges.

Next, we can properly choose H such that D contains a path of order 4. Let xy and zw are two independent edges. If $e(xy, zw) > 0$, then $D \supseteq P_4$. Otherwise $\Sigma_{v \in V(D)} d(v,D) = 4$ and it follows that $e(xy \cup zw, H) \geq 2\sigma_2(G) - e(xy \cup zw, D) \geq 8r + 8s - 4 > 8(r+s-1)$. So there exists a cycle $C \in \{T_1, \cdots, T_r, Q_1, \cdots, Q_{s-1}\}$ in H such that $e(xy \cup zw, C) \geq 9$. Then by Lemma 2.4 and Lemma 2.5, we have $G[xy \cup zw \cup C] \supseteq C' \cup P_4$ where $|C'| = |C|$. Then we replace C by C' in H and then D contains a path of order 4.

Finally, we can properly choose H such that D has at least four edges. Now we choose $T_1, \cdots, T_r$ and $Q_1, \cdots, Q_{s-1}$ such that $M = \Sigma_{i=1}^{r} e(G[V(T_i)])+$

$\Sigma_{i=1}^{s-1}e(G[V(Q_i)])$ is maximal and D contains a path of order 4. Suppose $|E(D)| = 3$, that is, D is a path of order 4. Then $e(D,H) \geq 2\sigma_2(G) - 2|E(D)| \geq 8r + 8s - 6 > 8(r + s - 1)$. So there exists a cycle $C \in \{T_1, \cdots, T_r, Q_1, \cdots, Q_{s-1}\}$ in H such that $e(D,C) \geq 9$. By Lemma 2.6, Lemma 2.5 and the maximality of M, we have $G[V(C \cup D)] \supseteq C^{'} \cup D^{'}$ where $|C^{'}| = |C|$, $|D^{'}| = 4$ and $e(D^{'}) \geq 4$. Now we replace C by C' in H and then D contains a subgraph of order 4 with at least four edges. Hence no matter cases, D has at least four edges. We complete the proof of the theorem. □

Theorem 3.3. *For two positive integers r and s with $r \geq 2s - 2$, if G is a graph of order $n = 3r + 4s$ and $\sigma_2(G) \geq n + r$, then $G \supseteq rC_3 \cup sC_4$.*

Proof. By Theorem 3.2, $G \supseteq T_1 \bigcup \cdots \bigcup T_r \bigcup Q_1 \bigcup \cdots \bigcup Q_{s-1} \bigcup D$ where $T_1, \cdots, T_r$ are triangles, $Q_1, \cdots, Q_{s-1}$ are quadrilaterals, and D is a subgraph of order 4 with at least four edges. Let $H_T = G[\bigcup_{i=1}^{r} V(T_i)]$, $H_Q = G[\bigcup_{i=1}^{s-1} V(Q_i)]$, $H = H_T \bigcup H_Q$.

Suppose $D \not\supseteq C_4$. Then $D = F_4$ and $u_1u_3 \notin E(G)$, $u_2u_3 \notin E(G)$. So we have $d(u_1) + d(u_2) + 2d(u_3) \geq 2\sigma_2(G) \geq 8r + 8s$. If there exists a quadrilateral Q_i in H_Q such that $e(U, Q_i) \geq 9$, then we have $G[V(U \cup Q_i)] \supseteq 2C_4$ by Lemma 2.7. This implies that $G \supseteq rC_3 \cup sC_4$. So we may assume that $e(U, H_Q) \leq 8(s - 1)$. On the other hand, if there exists a triangle T_i in H_T such that $e(U, T_i) \geq 7$, then we have $G[V(U \cup T_i)] \supseteq C_3 \cup C_4$ by Lemma 2.9. This also implies that $G \supseteq rC_3 \cup sC_4$. So we may assume that $e(U, T_i) \leq 6$ for every triangle T_i in H_T. Suppose there are r_1 triangles satisfying $e(U, T_i) \leq 5$ and r_2 triangles satisfying $e(U, T_i) = 6$. Then $r_1 + r_2 = r$, $e(U, H_T) \leq 5r_1 + 6r_2$. If there are some $T_i (1 \leq i \leq r)$ such that $d(u_3, T_i) > 1$, or $d(u_3, T_i) > 0$ and $e(U, T_i) = 6$, then $G[V(T_i \cup F_4)] \supseteq C_3 \cup C_4$ by Lemma 2.8 and it follows that $G \supseteq rC_3 \cup sC_4$. So we can assume $d(u_3, H_T) \leq r_1$. Here $d(u_3) = e(U, H_Q) + e(U, H_T) - (d(u_1) - 2) - (d(u_2) - 2) + 1 = e(U, H_Q) + e(U, H_T) - (d(u_1) + d(u_2) + 2d(u_3)) + 2d(u_3) + 5$. $d(u_3) \geq (8r + 8s) - 8(s - 1) - (5r_1 + 6r_2) - 5 = 8r - 5r_1 - 6r_2 + 3$. So we have $d(u_3, H_Q) = d(u_3) - d(u_3, H_T) - 1 \geq (8r - 5r_1 - 6r_2 + 3) - (r_1 + 1) = 2r + 2 \geq 4s - 2 > 4(s - 1)$ which contradicts to the fact $d(u_3, H_Q) \leq |H_Q| = 4(s - 1)$. Hence $D \supseteq C_4$. □

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