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On edge colorings of 1-planar graphs ☆

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ABSTRACT

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is shown that every 1-planar graph with maximum degree $\Delta \geqslant 10$ can be edge-colored with Δ colors.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [1]. For a real number x, let $\lfloor x \rfloor$ be the greatest integer not larger than x. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and its edge set, respectively. Let $v \in V(G)$. If $uv \in E(G)$, then the vertex u is said to be a *neighbor* of v in G. We denote by $N_G(v)$ the set of neighbors of v in G and by $d_G(v)$ the degree of v in G ($d_G(v) = |N_G(v)|$). We use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and the maximum degree of G, respectively. If G is a planar graph, we assume that G has always been embedded in the plane. Let G be a planar graph. We denote by $F(G)$ the face set of G. The degree of a face f in G, denote by $d_G(f)$, is the number of edges incident with it, where each cut-edge is counted twice. Throughout this paper, a k-, k^+- and k^--vertex (or face) in a planar graph is a vertex (or face) of degree k, at least k and at most k, respectively.

A graph is k *edge-colorable* if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The *edge chromatic number* (or *chromatic index*) of G, denoted by $\chi'(G)$, is the smallest integer k such that G is k edge-colorable. For edge coloring, Vizing's theorem states that for any graph G, $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$. A graph G is said to be of class one if $\chi'(G) = \Delta(G)$, and of class two if $\chi'(G) = \Delta(G)+1$. A graph G is critical if G is a connected graph of class two and $\chi'(G-e) < \chi'(G)$ for every edge e of G. If a graph G is critical and its maximum degree is Δ, we say that G is Δ-critical. Edge coloring of planar graphs was investigated in many papers, such as [6] and [7].

A graph G is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planar graphs was introduced by Ringel [5] in connection with the problem of simultaneous coloring of adjacent/incidence vertices and faces of plane graphs. Compared to the family of planar graphs, 1-planar graphs have not been extensively studied in the literature. Ringel conjectured that each 1-planar graph is 6-colorable, which was confirmed by Borodin [2]. Since there exists a 7-regular 1-planar graph, the bound 6 here is sharp. For 1-planar graphs with girth at least 5, Fabrici and Madaras [4] showed that five colors suffices for properly coloring edges. In [3], Borodin et al. also proved that each 1-planar graph is acyclically 20-colorable but they did not claim that this bound is tight. Although it would be natural to consider other kinds of colorings (and other questions concerning

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standard vertex coloring) of 1-planar graphs, it appears that no other work on this has been done. This paper is devoted to prove that every 1-planar graph with maximum degree at least 10 is of class one.

2. Main results and their proofs

In the following, we always assume that every 1-planar graph G is drawn on a plane properly and optimally, that is,

(1) Properly. Every edge is crossed by at most one other edge;
(2) Optimally. The number of crossings is as small as possible.

Such a 1-planar drawing of G is called 1-*plane graph*. So for each pair x_1y_1, x_2y_2 of edges in a 1-plane graph G that cross each other at a crossing point z, their end vertices are pairwise distinct. Let $C(G)$ be the set of all crossing points and $E_0(G)$ be the non-crossed edges in G. Then the *associated plane graph* $G^\times$ of G is the plane graph such that $V(G^\times) = V(G) \cup C(G)$ and $E(G^\times) = E_0(G) \cup \{xz, yz \mid xy \in E(G)\backslash E_0(G)$ and z is the crossing point on $xy\}$. Thus in $G^\times$, each of the crossing point becomes an actual vertex of degree four. For convenience, we still call the new vertices in $G^\times$ crossing vertices, and the edge in $E(G)$ which contains a crossing vertex is called a crossing edge. The notion of "associated plane graph" is very useful when we deal with the coloring problem of 1-planar graphs by using discharging.

Now, we begin to give some basic properties on a 1-plane graph G and its associated plane graph $G^\times$. For a vertex $v \in V(G)$, we use $f_k(v)$ to denote the number of k-faces that are incident with it in $G^\times$ and use $n_c(v)$ to denote the number of crossing vertices that are adjacent to it in $G^\times$. In the following, one should be clear that $d_G(v) = d_{G^\times}(v)$ holds for every vertex $v \in V(G)$ by the definition of $G^\times$.

Lemma 1. *Let G be a 1-plane graph and $G^\times$ be its associated plane graph. Then the following results hold.*

(a) *For any two crossing vertices u and v in $G^\times$, $uv \notin E(G^\times)$.*
(b) *If there is a 3-face $uvwu$ in $G^\times$ such that $d_G(v) = 2$, then u and w are not crossing vertices.*
(c) *If $d_G(u) = 3$ and v is a crossing vertex in $G^\times$, then either $uv \notin E(G^\times)$ or uv is not incident with two 3-faces.*
(d) *If a 3-vertex v in G is incident with two 3-faces and adjacent to two crossing vertices in $G^\times$, then v must also be incident with a 5^+-face.*

Proof. (a) follows from (1).

For (b), if u is a crossing vertex of $G^\times$ that is crossed by two edges vv_1 and ww_1 in G, then we can draw ww_1 into the face incident with w, v, w_1 such that ww_1 is not crossed by the path wvv_1, a contradiction to (2). So u is not a crossing vertex. Similarly, we can also prove that w is not a crossing vertex either.

For (c), suppose, to the contrary, that there are two 3-faces, say $uvxu$ and $uvyu$ sharing a common edge uv such that $d_G(u) = 3$ and v is a crossing vertex. Then the path xvy in $G^\times$ corresponds to the crossing edge xy in G and v is a crossing point on xy. Now we can draw xy into the face incident with u, x, y such that xy is not a crossing edge, which contradicts to (2).

For (d), let x, y and z be the neighbors of v in $G^\times$. Since $f_3(v) \geqslant 2$ and $n_c(v) = 2$, without loss of generality, we assume $xy, xz \in E(G^\times)$. It follows that y and z must be crossing vertices by (a) and (c). Now we claim that the face $f \in F(G^\times)$ that contains the edges vy and vz is a 5^+-face. Suppose $d_{G^\times}(f) = 4$ and thus denote f by $vyuzv$. So by the definition of $G^\times$, both the path xyu and the path xzu are the crossing edges of G connecting x to u, which implies that there is a multiple edge in G, a contradiction. □

Lemma 2. *Let G be a 1-plane graph. Then for every vertex $v \in V(G)$, we have*

$$f_3(v) + n_c(v) \leqslant \begin{cases} 3, & \text{if } d_G(v) = 3 \text{ and } f_3(v) \neq 2; \\ 4, & \text{if } d_G(v) = 3 \text{ and } f_3(v) = 2; \\ 5, & \text{if } d_G(v) = 4; \\ \lfloor \frac{3d_G(v)}{2} \rfloor, & \text{if } d_G(v) \geqslant 5. \end{cases}$$

Proof. If $d_G(v) = 3$, let x, y and z be the neighbors of v in $G^\times$ (recall that $d_{G^\times}(v) = d_G(v) = 3$). Suppose $f_3(v) = 3$. That is, $xy, yz, zx \in E(G^\times)$. If x is a crossing vertex, then there exists two edge joining y and z, which cannot occur. Similarly, y and z are not crossing vertices. So $n_c(v) = 0$. Suppose $f_3(v) = 2$. That is, $xy, yz \in E(G^\times)$ but $zx \notin E(G^\times)$ without loss of generality. Then by Lemma 1(a), $n_c(v) \leqslant 2$. Suppose $f_3(v) = 1$. Without loss of generality, we assume $xy \in E(G^\times)$. Then x and y cannot be crossing vertices simultaneously by Lemma 1(a). So $n_c(v) \leqslant 2$. Suppose $f_3(v) = 0$. Then we have $n_c(v) \leqslant 3$ trivially. In each case we all have $f_3(v) + n_c(v) \leqslant 4$, and the strict inequality holds if $f_3(v) \neq 2$.

If $d_G(v) = 4$, let x, y, z and w be the neighbors of v in $G^\times$ cyclicity. Suppose $f_3(v) = 4$. That is, $xy, yz, zw, wx \in E(G^\times)$. Then by Lemma 1(a), $n_c(v) \leqslant 2$. If $n_c(v) = 2$, without loss of generality, we assume x, z are crossing vertices. But in this case, both the paths wxy and wzy in $G^\times$ correspond to a crossing edge in G that joins w and y, a contradiction to the fact that G admits no multiple edge. So $n_c(v) \leqslant 1$. Similarly, we can prove that if $f_3(v) = i$, then $n_c(v) \leqslant 5 - i$ for each $0 \leqslant i \leqslant 3$. Thus, we have $f_3(v) + n_c(v) \leqslant 5$ if $d_G(v) = 4$.

If $d_G(v) \geqslant 5$, then v is incident with $(d_G(v) - f_3(v))$ faces having degree 4 or larger. So by Lemma 1, the largest possible number of crossing vertices that are adjacent to v is $d_G(v) - f_3(v) + \lfloor \frac{d_G(v)}{2} \rfloor = \lfloor \frac{3d_G(v)}{2} \rfloor - f_3(v)$. Thus, $f_3(v) + n_c(v) \leqslant \lfloor \frac{3d_G(v)}{2} \rfloor$. □

Let G be a 1-plane graph and $G^\times$ be its associated plane graph. A 3-face in $G^\times$ is *special* if it is incident with one crossing vertex. Otherwise we call it a *normal* 3-face. For every vertex $v \in V(G)$, let $s(v)$ be the number of the special 3-faces that are incident with v in $G^\times$. By Lemmas 1 and 2, one can easily have the following corollary. Here we also mark this corollary as a lemma since it will be useful in the following.

Lemma 3. *Let G be a 1-plane graph and v be a vertex in G. Then $s(v) \leqslant 2n_c(v)$. Furthermore, if $d_G(v) = 3$ and $f_3(v) = 3$, then $s(v) = 0$; if $d_G(v) = 3$, $f_3(v) = 2$, then $s(v) = 2$ only if v is incident with a 5^+-face; if $d_G(v) = 4$ and $f_3(v) = 4$, then $s(v) \leqslant 2$. If $d_G(v) \geqslant 5$, then $s(v) \leqslant 2\lfloor \frac{d_G(v)}{2} \rfloor$.*

The following well-known lemma can be found in [8].

Lemma 4 *(Vizing's Adjacency Lemma). Let G be a Δ-critical graph and let v, w be adjacent vertices of G with $d_G(v) = k$. Then*

(1) *if $k < \Delta$, then w is adjacent to at least $(\Delta - k + 1)$ Δ-vertices*;
(2) *if $k = \Delta$, then w is adjacent to at least two Δ-vertices*;
(3) *G contains at least $(\Delta - \delta(G) + 2)$ Δ-vertices.*

Lemma 5. *(See [6].) No Δ-critical graph G has distinct vertices x, y, z such that x is adjacent to both y and z, $d_G(z) < 2\Delta - d_G(x) - d_G(y) + 2$, and xz is in at least $d_G(x) + d_G(y) - \Delta - 2$ triangles not containing y.*

Let v be a non-crossing 4-vertex in $G^\times$ with x, y, z and u being its neighbors in $G^\times$ cyclically. If x, z are crossing vertices, $xy, yz, zu \in E(G^\times)$ and v is also incident with a 4-face $xvuw$ where $d_G(w) < \Delta$, then we call such a vertex v a *special* 4-vertex. Otherwise we call it a *non-special* 4-vertex.

Lemma 6. *Let G be a Δ-critical 1-plane graph. If three 4-vertices in G are adjacent to a vertex $v \in V(G)$ with $d_G(v) = f_3(v) = \Delta \geqslant 10$, then none of them is special.*

Proof. We prove it by contradiction. By Vizing's Adjacency Lemma (VAL for short), v is also adjacent to $(\Delta - 3)$ Δ-vertices in G. Choose u to be a special 4-vertex adjacent to v in G, that is to say, $uv \in E(G)$. Suppose $uv \in E(G^\times)$. Then since $d_G(v) = f_3(v)$, one can easily observe that uv is incident with at least two triangles in G, say uvx and uvy. By VAL, x and y cannot be both 4-vertices. So there exists another 4-vertex in G, say w, that is a neighbor of v. Consider the three vertices u, v, w. We have that $d_G(u) = 4 < \Delta - 2 = 2\Delta - d_G(v) - d_G(w) + 2$ and uv is in at least $d_G(v) + d_G(w) - \Delta - 2$ triangles not containing w. By Lemma 5, such three vertices cannot appear in G, a contradiction. So $uv \notin E(G^\times)$, that is to say uv is a crossing edge in G. We assume that uv is crossed by xy at a point z in G. So z is a crossing vertex in $G^\times$. Since $d_G(v) = f_3(v)$, we can also deduce that $vx, vy \in E(G)$. Suppose $ux, uy \in E(G)$. Then $uvxu$ and $uvyu$ are two triangles in G. By the same argument as above, this is impossible. Without loss of generality, we assume that $ux \notin E(G)$. Denote the face incident with the two edges xz, uz in $G^\times$ to be f. We then have $d_{G^\times}(f) \geqslant 4$. Since u is special, we also have $d_{G^\times}(f) \leqslant 4$. So $d_{G^\times}(f) = 4$ and $d_G(x) < \Delta$. Thus we must have $d_G(x) = 4$ (notice that x is not a crossing vertex and does not need to be a special vertex). Note that $vx \in E(G^\times)$, $uv \in E(G)$ and $d_G(u) = 4$. By the same argument as before, this is again impossible. This contradiction just completes the proof of this lemma. □

Theorem 7. *Each 1-planar graph with maximum degree $\Delta \geqslant 10$ can be edge-colored with Δ colors.*

Proof. Suppose that G is a counterexample to the theorem with the smallest number of edges. Then G is a Δ-critical 1-plane graph. By VAL, we have $\delta(G) \geqslant 2$. In the following, we apply the discharging method on the associated planar graph $G^\times$ of G and complete the proof by a contradiction.

Since $G^\times$ is a plane graph, we have

$$\sum_{v \in V(G^\times)} \big(d_{G^\times}(v) - 4\big) + \sum_{f \in F(G^\times)} \big(d_{G^\times}(f) - 4\big) = -8$$

by Euler's formula. Note that $d_{G^\times}(v) - 4 = 0$ for every $v \in V(G^\times) \setminus V(G)$ and $d_G(v) = d_{G^\times}(v)$ for every $v \in V(G)$. We can rewrite the above equation in a much more convenient form:

$$\sum_{v \in V(G)} \big(d_G(v) - 4\big) + \sum_{f \in F(G^\times)} \big(d_{G^\times}(f) - 4\big) = -8.$$

Now we define $ch(x)$ to be the initial charge of $x \in V(G) \cup F(G^\times)$. Let $ch(v) = d_G(v) - 4$ for each vertex $v \in V(G^\times)$ and let $ch(f) = d_{G^\times}(f) - 4$ for each face $f \in F(G^\times)$. It follows that $\sum_{x \in V(G) \cup F(G^\times)} ch(x) = -8$. We now redistribute the initial charge $ch(x)$ and form a new charge $ch'(x)$ for each $x \in V(G) \cup F(G^\times)$ by discharging method. Since our rules only move charge around, and do not affect the sum, we have $\sum_{x \in V(G) \cup F(G^\times)} ch'(x) = \sum_{x \in V(G) \cup F(G^\times)} ch(x) = -8$. We use $\tau(x_1 \to x_2)$ to denote the charge move from x_1 to x_2. Our discharging rules are defined as follows.

R1. Suppose $f = uvw$ is a normal 3-face in $G^\times$.
- R1-1. If $2 \leqslant d_G(u) \leqslant 5$, then $\tau(u \to f) = 0$ and $\tau(v \to f) = \tau(w \to f) = \frac{1}{2}$.
- R1-2. If $\min\{d_G(u), d_G(v), d_G(w)\} \geqslant 6$, then $\tau(u \to f) = \tau(v \to f) = \tau(w \to f) = \frac{1}{3}$.

R2. Suppose $f = uvw$ is a special 3-face in $G^\times$ and u is the crossing vertex of f, then $\tau(u \to f) = 0$ and $\tau(v \to f) = \tau(w \to f) = \frac{1}{2}$.

R3. Suppose f is a face in $G^\times$ with $d_{G^\times}(f) \geqslant 5$ and v is a vertex in G that is incident with f with $3 \leqslant d_G(v) \leqslant 4$, then $\tau(f \to v) = \frac{1}{2}$.

R4. Suppose $d_G(u) = 7$ and $uv \in E(G)$. If $d_G(v) = 6$, then $\tau(u \to v) = \frac{1}{10}$.

R5. Suppose $d_G(u) = 8$ and $uv \in E(G)$.
- R5-1. If $d_G(v) = 4$, then $\tau(u \to v) = \frac{3}{8}$.
- R5-2. If $d_G(v) = 5$, then $\tau(u \to v) = \frac{1}{8}$.
- R5-3. If $d_G(v) = 6$, then $\tau(u \to v) = \frac{2}{15}$.

R6. Suppose $d_G(u) = 9$ and $uv \in E(G)$.
- R6-1. If $d_G(v) = 3$, then $\tau(u \to v) = \frac{1}{2}$.
- R6-2. If $d_G(v) = 4$, then $\tau(u \to v) = \frac{3}{8}$.
- R6-3. If $d_G(v) = 5$, then $\tau(u \to v) = \frac{1}{6}$.
- R6-4. If $d_G(v) = 6$, then $\tau(u \to v) = \frac{3}{20}$.
- R6-5. If $d_G(v) = 7$, then $\tau(u \to v) = \frac{1}{30}$.

R7. Suppose $10 \leqslant d_G(u) \leqslant \Delta$ and $uv \in E(G)$.
- R7-1 If $2 \leqslant d_G(v) \leqslant 9$ and v is non-special when $d_G(v) = 4$, then $\tau(u \to v) = \frac{1}{d_G(v) - 1}$.
- R7-2. If $d_G(v) = 4$ and v is special, $\tau(u \to v) = \frac{3}{8}$.

R7-3. If u is incident with a 4-face $f = uvwx$ in $G^\times$, then $\tau(u \to w) = \frac{1}{6}$.

Let f be a face of $G^\times$. If $d_{G^\times}(f) = 3$, then $ch'(f) \geqslant ch(f) + \min\{2 \times \frac{1}{2}, 3 \times \frac{1}{3}\} = 3 - 4 + 1 = 0$ by R1 and R2. If $d_{G^\times}(f) = 4$, then $ch'(f) = ch(f) = 0$. Suppose $d_{G^\times}(f) \geqslant 5$. Since any two 4^--vertices in G are not adjacent by VAL, f gives out at most $\frac{1}{2}\lfloor \frac{d_{G^\times}(f)}{2} \rfloor$ by R3. So $ch'(f) \geqslant d_{G^\times}(f) - 4 - \frac{1}{2}\lfloor \frac{d_{G^\times}(f)}{2} \rfloor \geqslant 0$.

Let v be a vertex of G. If $d_G(v) = 2$, then v is adjacent to two Δ-vertices in G by VAL. By Lemma 1(b), v cannot be incident with a special 3-face in $G^\times$. It follows that $ch'(v) = ch(v) + 2 \times 1 = 0$ by R1-1 and R7-1.

Suppose $d_G(v) = 3$. By R6-1, R7-1 and VAL, v receives $\frac{1}{2}$ from each of its neighbors of G. At the same time, v gives out some charge if and only if v is incident with special 3-faces in $G^\times$ by R2. If $s(v) \leqslant 1$, then $ch'(v) \geqslant ch(v) - \frac{1}{2} + 3 \times \frac{1}{2} = 0$. Otherwise, by Lemmas 1(c) and 1(d), $s(v) = 2$ and v is incident with a 5^+-face f in $G^\times$. So v receives $\frac{1}{2}$ from f by R3 and it follows that $ch'(v) \geqslant ch(v) - 2 \times \frac{1}{2} + \frac{1}{2} + 3 \times \frac{1}{2} = 0$.

Suppose $d_G(v) = 4$. By VAL and R4–R7, v receives at least $4 \times \frac{1}{3}$ from its neighbors in G. At the same time by R2, v needs to send $\frac{1}{2}$ to its incident special 3-faces in $G^\times$ (if such special 3-faces exist). Thus if $s(v) \leqslant 2$, then $ch'(v) \geqslant ch(v) - 2 \times \frac{1}{2} + 4 \times \frac{1}{3} = \frac{1}{3} > 0$. Otherwise $s(v) \geqslant 3$. By Lemma 2, we must have $s(v) = f_3(v) = 3$ and $n_c(v) = 2$. So v gives out $3 \times \frac{1}{2} = \frac{3}{2}$ to the special 3-faces that are incident with v in $G^\times$ by R2. Denote by f the non-special face that is incident with v. Then $d_{G^\times}(f) \geqslant 4$. If $d_{G^\times}(f) \geqslant 5$, then v receives $\frac{1}{2}$ from f by R3 and we have $ch'(v) \geqslant ch(v) - \frac{3}{2} + \frac{1}{2} + 4 \times \frac{1}{3} = \frac{1}{3} > 0$. So assume that $d_{G^\times}(f) = 4$ and thus we denote $f = vxyzv$. If v is a non-special vertex, then $d_G(y) = \Delta$ and v receives $\frac{1}{6}$ from y by R7-3. So $ch'(v) \geqslant ch(v) - \frac{3}{2} + \frac{1}{6} + 4 \times \frac{1}{3} = 0$. If v is a special 4-vertex, then v is adjacent to at least two Δ-vertices in G by VAL. So $ch'(v) = ch(v) - \frac{3}{2} + 4 \times \frac{3}{8} = 0$ by R5-1, R6-2 and R7-2.

Suppose $d_G(v) = 5$. By Lemma 3, $s(v) \leqslant 4$. So v gives out at most $4 \times \frac{1}{2} = 2$ by R1-1 and R2. Denote the smallest degree among all the neighbors of v in G by d. Then by VAL we have $d \geqslant 7$, since $\Delta \geqslant 10$. If $d = 7$, then v must be adjacent to at least four Δ-vertices in G from which v receives $4 \times \frac{1}{4} = 1$ by R7-1. So $ch'(v) \geqslant ch(v) - 2 + 1 = 0$. If $d = 8$, then v must be adjacent to at least three Δ-vertices in G from which v receives $3 \times \frac{1}{4} = \frac{3}{4}$ by R7-1 and adjacent to two 8^+-vertices in G from which v receives at least $2 \times \frac{1}{8} = \frac{1}{4}$ by R5-2, R6-3 and R7-1. So $ch'(v) \geqslant ch(v) - 2 + \frac{3}{4} + \frac{1}{4} = 0$. If $d = 9$, then v must be adjacent to at least two Δ-vertices in G from which v receives $2 \times \frac{1}{4} = \frac{1}{2}$ by R7-1 and adjacent to three 9^+-vertices in G from which v receives at least $3 \times \frac{1}{6} = \frac{1}{2}$ by R6-3 and R7-1. So $ch'(v) \geqslant ch(v) - 2 + \frac{1}{2} + \frac{1}{2} = 0$. If $d \geqslant 10$, then by R7-1 it is trivial that $ch'(v) \geqslant ch(v) - 2 + 5 \times \frac{1}{4} > 0$.

Suppose $d_G(v) = 6$. By R1 and R2, v gives out at most $6 \times \frac{1}{2} = 3$. Denote the smallest degree among all the neighbors of v in G by d. Then by VAL we have $d \geqslant 6$ since $\Delta \geqslant 10$. If $d = 6$, then v must be adjacent to five Δ-vertices in G from which v receives $5 \times \frac{1}{5} = 1$ by R7-1. So $ch'(v) \geqslant ch(v) - 3 + 1 = 0$. If $d = 7$, then v must be adjacent to at least four Δ-vertices in G from which v receives $4 \times \frac{1}{5} = \frac{4}{5}$ by R7-1 and adjacent to two 7^+-vertex in G from which v receives $2 \times \frac{1}{10} = \frac{1}{5}$ by R5-3, R6-4 and R7-1. So $ch'(v) \geqslant ch(v) - 3 + \frac{4}{5} + \frac{1}{5} = 0$. If $d = 8$, then v must be adjacent to at least three Δ-vertices in G from which v receives $3 \times \frac{1}{5} = \frac{3}{5}$ by R7-1 and adjacent to three 8^+-vertices in G from which v receives at least $3 \times \frac{2}{15} = \frac{2}{5}$ by R5-3, R6-4 and R7-1. So $ch'(v) \geqslant ch(v) - 3 + \frac{3}{5} + \frac{2}{5} = 0$. If $d = 9$, then v must be adjacent to at least two Δ-vertices in G from which v receives $2 \times \frac{1}{5} = \frac{2}{5}$ by (R7-1) and adjacent to four 9^+-vertices in G from which v receives at least $4 \times \frac{3}{20} = \frac{3}{5}$ by R6-4. So $ch'(v) \geqslant ch(v) - 3 + \frac{2}{5} + \frac{3}{5} = 0$. If $d \geqslant 10$, then by R7-1 it is trivial that $ch'(v) \geqslant ch(v) - 3 + 6 \times \frac{1}{5} > 0$.

Suppose $d_G(v) = 7$. By R1 and R2, v gives out at most $7 \times \frac{1}{2} = \frac{7}{2}$ to the 3-faces incident with it. Denote the smallest degree among all the neighbors of v in G by d. If $d \leqslant 6$, then by VAL v is adjacent to at least five Δ-vertices in G from which v receives $5 \times \frac{1}{6} = \frac{5}{6}$ by R7-1. Meanwhile, v may be adjacent to two 6^+-vertices in G, in which case v must give out at most $2 \times \frac{1}{10} = \frac{1}{5}$ by R4. So $ch'(v) \geqslant ch(v) - \frac{7}{2} + \frac{5}{6} - \frac{1}{5} = \frac{2}{15} > 0$. If $7 \leqslant d \leqslant 8$, then by VAL v is adjacent to at least three Δ-vertices in G from which v receives $3 \times \frac{1}{6} = \frac{1}{2}$ by R7-1. So $ch'(v) \geqslant ch(v) - \frac{7}{2} + \frac{1}{2} = 0$. If $d = 9$, then v must be adjacent to at least two Δ-vertices in G from which v receives $2 \times \frac{1}{6} = \frac{1}{3}$ by R7-1 and adjacent to five 9^+-vertices in G from which v receives at least $5 \times \frac{1}{30} = \frac{1}{6}$ by R6-5 and R7-1. So $ch'(v) \geqslant ch(v) - \frac{7}{2} + \frac{1}{3} + \frac{1}{6} = 0$. If $d \geqslant 10$, then by R7-1, $ch'(v) \geqslant ch(v) - \frac{7}{2} + 7 \times \frac{1}{6} > 0$.

Suppose $d_G(v) = 8$. By R1 and R2, v gives out at most $8 \times \frac{1}{2} = 4$ to the 3-faces incident with it. Denote the smallest degree among all the neighbors of v in G by d. By VAL we have $d \geqslant 4$. If $d \geqslant 7$, then by R5 v does not give out any charge to its neighbors in G. So $ch'(v) \geqslant ch(v) - 4 = 0$. If $d = 6$, then by VAL v is adjacent to at least five Δ-vertices in G from which v receives $5 \times \frac{1}{7} = \frac{5}{7}$ by R7-1. Meanwhile v may be adjacent to three 6^+-vertices in G to which v may give out at most $3 \times \frac{2}{15} = \frac{2}{5}$ by R5-3. So $ch'(v) \geqslant ch(v) - 4 + \frac{5}{7} - \frac{2}{5} > 0$. If $d = 5$, then by VAL v is adjacent to at least six Δ-vertices in G from which v receives $6 \times \frac{1}{7} = \frac{6}{7}$ by R7-1. Meanwhile v may be adjacent to two 5^+-vertices in G to which v may give out at most $2 \times \frac{1}{8} = \frac{1}{4}$ by R5-2. So $ch'(v) \geqslant ch(v) - 4 + \frac{6}{7} - \frac{1}{4} > 0$. If $d = 4$, then by VAL v is adjacent to at least seven Δ-vertices in G from which v receives $7 \times \frac{1}{7} = 1$ by R7-1. Meanwhile v may be adjacent to one 4^+-vertex in G to which v may give out at most $\frac{3}{8}$. So $ch'(v) \geqslant ch(v) - 4 + 1 - \frac{3}{8} > 0$.

Suppose $d_G(v) = 9$. By R1 and R2, v gives out at most $9 \times \frac{1}{2} = \frac{9}{2}$ to the 3-faces incident with it. By VAL, v is adjacent to at least two Δ-vertices in G from which v receives $2 \times \frac{1}{8} = \frac{1}{4}$ by R7-1. Meanwhile, v gives out at most $\max\{\frac{1}{2}, 2 \times \frac{3}{8}, 3 \times \frac{1}{6}, 4 \times \frac{3}{20}, 5 \times \frac{1}{30}\} = \frac{3}{4}$ to its neighbors in G by VAL and R6. So $ch'(v) \geqslant ch(v) - \frac{9}{2} + \frac{1}{4} - \frac{3}{4} = 0$.

Suppose $10 \leqslant d_G(v) \leqslant \Delta$. If $f_3(v) \leqslant d_G(v) - 1$, then v gives out at most $\frac{d_G(v)-1}{2} + \frac{1}{6}$ by R1 and R7-3. And v gives out at most $\max\{1, 2 \times \frac{1}{2}, 3 \times \frac{3}{8}, 4 \times \frac{1}{4}, 5 \times \frac{1}{5}, 6 \times \frac{1}{6}, 7 \times \frac{1}{7}, 8 \times \frac{1}{8}\} = \frac{9}{8}$ to its neighbors in G by VAL, R7-1 and R7-2. So $ch'(v) \geqslant ch(v) - \frac{d_G(v)-1}{2} - \frac{1}{6} - \frac{9}{8} = \frac{12d_G(v)-115}{24} > 0$ since $d_G(v) \geqslant 10$. Now we assume $f_3(v) = d_G(v)$ in which case v gives out at most $\frac{d_G(v)}{2}$ to the faces incident with v by R1. Denote the smallest degree among all the neighbors of v in G by d. If $d = 2$, then by VAL $d_G(v) = \Delta$. Moreover, v is adjacent to $(\Delta - 1)$ Δ-vertices and only one 2-vertex in G. So $ch'(v) \geqslant ch(v) - \frac{\Delta}{2} - 1 = \frac{\Delta-10}{2} \geqslant 0$ by R7-1 since $\Delta \geqslant 10$. If $d = 3$, then by VAL $d_G(v) \geqslant \Delta - 1$. Moreover, if $d_G(v) = \Delta - i$ $(i = 0, 1)$, then v is adjacent to $(\Delta - 2)$ Δ-vertices and $(2 - i)$ 3-vertices in G. In either case we have $ch'(v) \geqslant \max\{\Delta - 1 - 4 - \frac{\Delta-1}{2} - \frac{1}{2}, \Delta - 4 - \frac{\Delta}{2} - 2 \times \frac{1}{2}\} = \frac{\Delta-10}{2} \geqslant 0$ by R7-1. If $d \geqslant 5$, then by VAL, v gives out at most $\max\{4 \times \frac{1}{4}, 5 \times \frac{1}{5}, 6 \times \frac{1}{6}, 7 \times \frac{1}{7}, 8 \times \frac{1}{8}\} = 1$ to its neighbors in G by R7-1. So $ch'(v) \geqslant ch(v) - \frac{d_G(v)}{2} - 1 = \frac{d_G(v)-10}{2} \geqslant 0$. The last case is when $d = 4$. In this case, by VAL we have $d_G(v) \geqslant \Delta - 2$. Moreover, if $d_G(v) = \Delta - j$ $(j = 0, 1, 2)$, then $\Delta \geqslant 10 + j$ and v is adjacent to $(\Delta - 3)$ Δ-vertices and $(3 - j)$ 4^+-vertices in G. Suppose $j = 2$, then by R7-1 and R7-2, $ch'(v) \geqslant \Delta - 2 - 4 - \frac{\Delta-2}{2} - \frac{3}{8} = \frac{2\Delta-23}{4} > 0$ since $\Delta \geqslant 10 + 2 = 12$ when this case occurs. Suppose $j = 1$, then by R7-1 and R7-2, $ch'(v) \geqslant \Delta - 1 - 4 - \frac{\Delta-1}{2} - 2 \times \frac{3}{8} = \frac{2\Delta-21}{4} > 0$ since $\Delta \geqslant 10 + 1 = 11$ when this case occurs. At last suppose $j = 0$. Recall that now v is a Δ-vertex being adjacent to $(\Delta - 3)$ Δ-vertices and three 4^+-vertices in G, say x, y, z. If $\max\{d_G(x), d_G(y), d_G(z)\} \geqslant 5$, then v totally gives to x, y, z at most $2 \times \frac{3}{8} + \frac{1}{4} = 1$ by R7-1 and R7-2. If $d_G(x) = d_G(y) = d_G(z) = 4$, then by Lemma 6, none of them can be special. So v totally gives to x, y, z at most $3 \times \frac{1}{3} = 1$ by R7-1. In each case we have $ch'(v) \geqslant ch(v) - \frac{\Delta}{2} - 1 = \frac{\Delta-10}{2} \geqslant 0$ in final, since $\Delta \geqslant 10$.

Hence the proof is complete since

$$-8 = \sum_{x \in V(G) \cup F(G^\times)} ch(x) = \sum_{x \in V(G) \cup F(G^\times)} ch'(x) \geqslant 0,$$

which is a contradiction. □

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