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Note

On equitable and equitable list colorings of series–parallel graphs☆

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ABSTRACT

In this paper, we prove that every series–parallel graph with maximum degree Δ is equitably k-colorable and equitably k-choosable whenever $k \geq \max\{\Delta, 3\}$.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $d_G(v)$ (or $d(v)$ for brevity), we denote the degree of a vertex v in G. Throughout this paper, $|G|$ (or $|V(G)|$) stands for the order of G and a k-vertex is a vertex of degree k. For other undefined notations, we refer the readers to the reference book of West [12].

In graph theory, an equitable coloring is an assignment of colors to the vertices such that any two adjacent vertices receive different colors and the sizes of any two color classes differ by at most one. That is, the partition of vertices among the different colors is as uniform as possible. The motivations for equitable coloring concern some scheduling, partitioning and load balancing problems. Differing from some ordinary colorings such as proper vertex coloring, there are two kinds of chromatic number associated with equitable coloring. The equitable chromatic number $\chi_{eq}(G)$ of a graph G is the smallest number k such that G has an equitable coloring with k colors. However, an equitably k-colorable graph may admit no equitably k'-colorings for some $k' > k$. Turán graph $T_{n,k}$ (the balanced complete k-partite graph with n vertices) is such an example. The equitable chromatic threshold $\chi^*_{eq}(G)$ of G is the smallest k such that G has equitable colorings for any number of colors greater than or equal to k. For equitable coloring, in 1970, Hajnal and Szemerédi [2] answered a question of Erdős by proving every graph G with $\Delta(G) \leq r$ has an equitable $(r+1)$-coloring, which implies $\chi^*_{eq}(G) \leq \Delta(G) + 1$ for every graph G. Three years later, Meyer [10] considered an equitable version of Brook's Theorem and made the following Equitable Coloring Conjecture (ECC for short).

Conjecture 1. *For any connected graph G, except the complete graph and the odd cycle, $\chi_{eq}(G) \leq \Delta(G)$.*

Later in 1994, Chen et al. [1] put forth the following so-called Equitable Δ-coloring Conjecture, which is stronger than ECC.

Conjecture 2. *If G is a connected graph with maximum degree Δ other than $K_{\Delta+1}$, $K_{\Delta,\Delta}$ and odd cycle, then G is equitably Δ-colorable.*

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Now Equitable Δ-Coloring Conjecture has already been confirmed for outerplanar graphs [4,14], planar graphs with $\Delta \geq 13$ [13] and some other graph families (see [5,6,9]).

In 2003, Kostochka, Pelsmajer and West [7] introduced a list analogue of equitable coloring. A graph G is equitably k-choosable if, for any k-uniform list assignment L to each vertex $v \in V(G)$, G has a proper coloring such that each vertex is colored from its own list (i.e. G has an L-coloring) and each color appears on at most $\lceil \frac{|G|}{k} \rceil$ vertices. In [7], Kostochka et al. also conjectured the analogues of Hajnal and Szemerédi Theorem and Equitable Δ-coloring Conjecture.

Conjecture 3. *Every graph with maximum degree Δ is equitably k-choosable whenever $k \geq \Delta + 1$.*

Conjecture 4. *If G is a connected graph with maximum degree $\Delta \geq 3$ other than $K_{\Delta+1}$ and $K_{\Delta,\Delta}$, then G is equitably Δ-choosable.*

Note that the condition $\Delta \geq 3$ in Conjecture 4 cannot be weakened since any odd cycle is not 2-colorable. The above two conjectures have already been confirmed for some classes of graphs such as outerplanar graphs [16], 2-degenerate graphs with $\Delta \geq 5$ [7] and planar graphs without short cycles and with large maximum degree (see [8,15]).

A graph is a series–parallel graph if it can be reduced to the null graph by repeatedly applying the series–parallel reductions, by which we mean any of the following operations: (a) deletion of a loop; (b) deletion a vertex of degree at most one; (c) deletion of a parallel edge; (d) suppression a vertex of degree two. Series–parallel graphs are a useful class of graphs. On the one hand, they are fairly simple and reasonably well understood to allow easy proofs for many results. On the other hand, they are rich enough so that many problems are non-trivial even when restricted to this class. In the aspect of application, series–parallel graphs can be seen as a model of certain types of electric networks, and are of interest in computational complexity theory. Series–parallel graphs are also studied in the literature by some other mathematicians under the notion of the so-called K_4-minor-free graphs. One can trivially see that a subgraph of a series–parallel graph is still series–parallel graph.

In this paper, we prove that every series–parallel graph G is equitably k-colorable and equitably k-choosable whenever $k \geq \Delta(G) \geq 3$. As an immediate corollary, the four conjectures list in this section would have been confirmed for series–parallel graphs.

2. Main results and their proofs

Lemma 5. *Every connected series–parallel graph G with order at least three contains one of the following configurations:*

(a) *two nonadjacent 1-vertices u and v;*
(b) *a 1-vertex u adjacent to a 2-vertex v;*
(c) *a 2-vertex u adjacent to another 2-vertex v;*
(d) *a 3-vertex u adjacent to a 1-vertex v and a 2-vertex w;*
(e) *a 3-cycle $[uvw]$ such that v is a 2-vertex and w is a 3-vertex;*
(f) *a 3-cycle $[uvw]$ such that v and w are both 3-vertices, one of which is also adjacent to a 1-vertex x;*
(g) *a 3-cycle $[uvw]$ such that v is a 2-vertex and w is a 4-vertex which is also adjacent to a 1-vertex x;*
(h) *two intersecting 3-cycles $[ux_1y_1]$ and $[ux_2y_2]$ such that x_1, x_2 are both 2-vertices and u is a 4-vertex;*
(i) *two intersecting 3-cycles $[ux_1y_1]$ and $[ux_2y_2]$ such that x_1, x_2 are both 2-vertices and u is a 5-vertex who is also adjacent to a 1-vertex v;*
(j) *two intersecting 3-cycles $[ux_1y_1]$ and $[ux_2y_2]$ such that x_1 is a 2-vertices, u is a 4-vertex and x_2 is a 3-vertex who is also adjacent to a 1-vertex v;*
(k) *a 4-cycle $[uxvy]$ such that u is a 2-vertex and v is a 3-vertex who is also adjacent to a 1-vertex w;*
(l) *a 4-cycle $[uxvy]$ such that u and v are both 2-vertices.*

Proof. If $\delta(G) \geq 2$, then G contains one of the configurations: (c), (e), (h) and (l) (see Lemma 2.1 of [3]). So we suppose $\delta(G) = 1$ and select a vertex $v \in V(G)$ such that $d(v) = 1$. If G contains neither (a) nor (b), then one can easily see that $\delta(G - v) \geq 2$. This implies that one of the configurations (c), (e), (h) and (l) would occur in $G - v$. If all of the four configurations (c), (e), (h) and (l) are absent from G, then by adding back the 1-vertex v to $G - v$ one can also easily find that at least one of the configurations (d), (f), (g), (i), (j) and (k) would appear in G. $\square$

Lemma 6 (*[15]*). *Let $S = \{v_1, v_2, \ldots, v_k\}$ where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in graph G. If $G - S$ has an equitable k-coloring, and $|N_G(v_i) - S| \leq k - i (1 \leq i \leq k)$, then G has an equitable k-coloring.*

Lemma 7 (*[7]*). *Let G be a graph with a k-uniform listed assignment L. Let $S = \{v_1, v_2, \ldots, v_k\}$ where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in G. If $G - S$ has an equitable L-coloring, and $|N_G(v_i) - S| \leq k - i (1 \leq i \leq k)$, then G has an equitable L-coloring.*

Theorem 8. *Every series–parallel graph G is equitably k-colorable and equitably k-choosable whenever $k \geq \max\{\Delta(G), 3\}$.*

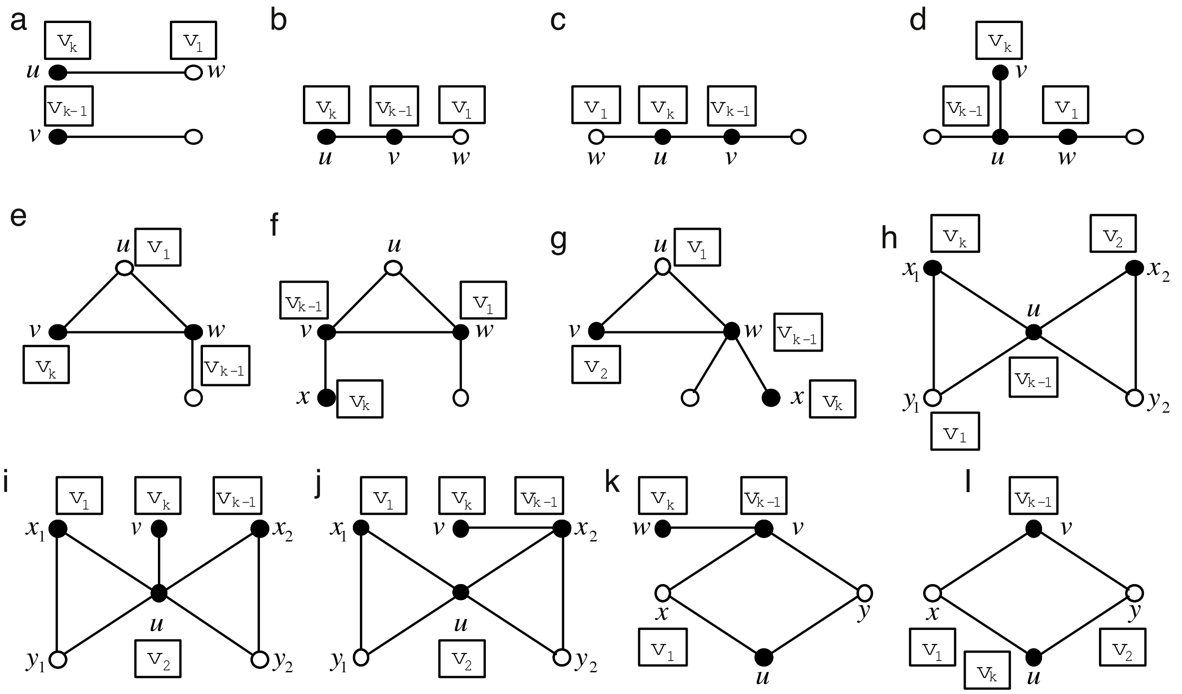

Fig. 1. Unavoidable configurations in series-parallel graphs.

Proof. If $\Delta(G) \le 2$, then G is equitably k-colorable and equitably k-choosable whenever $k \ge 3$ (see Theorem 1.1 of [11]). So we assume $\Delta(G) \ge 3$. This implies that G contains a component with order at least three and thus contains one of the configurations (a)–(l) by Lemma 5. In what follows, we will prove the theorem by induction on the order of G, via assigning k distinct vertices to S as described in Lemmas 6 and 7.

If G contains (a), then denote the neighbor of u by w and let $S' = \{w, v, u\}$. If G contains (b), then denote the other neighbor of v by w and let $S' = \{w, v, u\}$. If G contains (c), then denote the other neighbor of u by w and let $S' = \{w, v, u\}$. If G contains (d), then let $S' = \{w, u, v\}$. If G contains (e), then let $S' = \{u, w, v\}$. If G contains (f), then suppose $vx \in E(G)$ and then let $S' = \{w, v, x\}$. If G contains (k), then let $S' = \{x, v, w\}$. In each case discussed above, reset $S' = \{v_1, v_{k-1}, v_k\}$. That is to say, if $S' = \{w, v, u\}$ for example, then assign $v_1 := w$, $v_{k-1} := v$ and $v_k := u$ (see Fig. 1).

If G contains (g), then $k \ge 4$ and let $S' = \{u, v, w, x\}$. If G contains (h), then $k \ge 4$ and let $S' = \{y_1, x_2, u, x_1\}$. If G contains (i), then $k \ge 5$ and let $S' = \{x_1, u, x_2, v\}$. If G contains (j), then $k \ge 4$ and let $S' = \{x_1, u, x_2, v\}$. If G contains (l) and $k \ge 4$, then let $S' = \{x, y, v, u\}$. In each case discussed above, reset $S' = \{v_1, v_2, v_{k-1}, v_k\}$ (see Fig. 1).

Note that in each case we have ensured that $|N_G(v_i) - S| \le k - i$ for every $v_i \in S'$ and $S' \subseteq S \subseteq V(G)$ with $|S| = k$ by the choices of S'. So our next task is to fill the remaining unspecified positions in $S = \{v_1, v_2, \ldots, v_k\}$ as described in Lemmas 6 and 7 from highest to lowest indices properly. Indeed, one can easily complete it by choosing at each step a vertex of degree at most 2 in the graph obtained from G by deleting the vertices thus far chosen for S. Such vertices always exist because each series–parallel graph contains a vertex of degree at most 2 by Lemma 5. By doing so, we can guarantee that $|N_G(v_i) - S| \le k - i$ for all $1 \le i \le k$. Since $G - S$ is a series–parallel graph with smaller order than G, $G - S$ is equitably k-colorable and equitably k-choosable by induction hypothesis. Hence G is also equitably k-colorable and equitably k-choosable by Lemmas 6 and 7.

However, we have not completed the whole proof of this theorem yet since the last case when G contains (l) and $k = \Delta(G) = 3$ was not involved in the above discussion. Indeed, Lemmas 6 and 7 are disabled for this special case now.

Without loss of generality, we assume $\min\{d(x), d(y)\} \ge 3$ in this case. In the following, we only prove that G is equitably 3-choosable, since the proof of equitably 3-colorability is same but much easier indeed.

Denote $N(x) = \{u, v, x_1\}$ and $N(y) = \{u, v, y_1\}$, where x_1 and y_1 are not necessarily distinct. Let G' be a graph obtained from G via deleting u and v, contracting x with y into a common vertex (also say x for convenience) and then deleting any possible multi-edges. Then $d_{G'}(x) \le 2$ and $|G'| = |G| - 3$. Arbitrarily give a 3-uniform list assignment L to every vertex of G. Then by induction hypothesis, G' is equitably L-colorable when we restrict the list assignment of G to G'. That is to say, G' admits a 3-coloring c' such that $c'(v) \in L(v)$ for every $v \in V(G')$ and each color appears on at most $\lceil \frac{|G'|}{3} \rceil$ vertices. Now we extend the coloring c' of G' to a coloring c of G by soundly coloring u, v and y.

First suppose $c'(x) \in L(y)$. Recall that $c'(x) \ne c'(y_1)$ since $xy_1 \in E(G')$. So we can let $c(y) = c'(x)$, $c(u) \in L(u) \setminus \{c(y)\}$ and $c(v) \in L(v) \setminus \{c(u), c(y)\}$. Note that $c(y)$, $c(u)$ and $c(v)$ are pairwise different. So c is indeed an L-coloring of G such that each color appears on at most $\lceil \frac{|G'|}{3} \rceil + 1 = \lceil \frac{|G|}{3} \rceil$ vertices. Hence G is equitably 3-choosable. By a similar argument, if $c'(x) \notin L(u)$ or $c'(x) \notin L(v)$, then we can also prove the equitably 3-choosability of G. So at last we shall assume $c'(x) \in L(u) \cap L(v)$ and $c'(x) \notin L(y)$. Now recolor x by a new color $c(x) \in L(x) \setminus \{c'(x), c'(x_1)\}$ and then let $c(u) = c(v) = c'(x), c(y) \in L(y) \setminus \{c(x), c'(y_1)\}$. Note that $c(x), c(y)$ and $c'(x)$ are pairwise different. So c is also an L-coloring of G such that each color appears on at most $\lceil \frac{|G'|}{3} \rceil + 1 = \lceil \frac{|G|}{3} \rceil$ vertices. Hence G is equitably 3-choosable in final. □

To end this paper, we conjecture that the restriction on k in Theorem 8 can be weakened for equitable colorability. Note that the following Conjecture had already been proved for outerplanar graphs by Kostochka (see Theorem 1 of [4]).

Conjecture 9. *Every series–parallel graph G with maximum degree $\Delta \geq 3$ is equitably k-colorable whenever $k \geq \Delta/2$.*

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