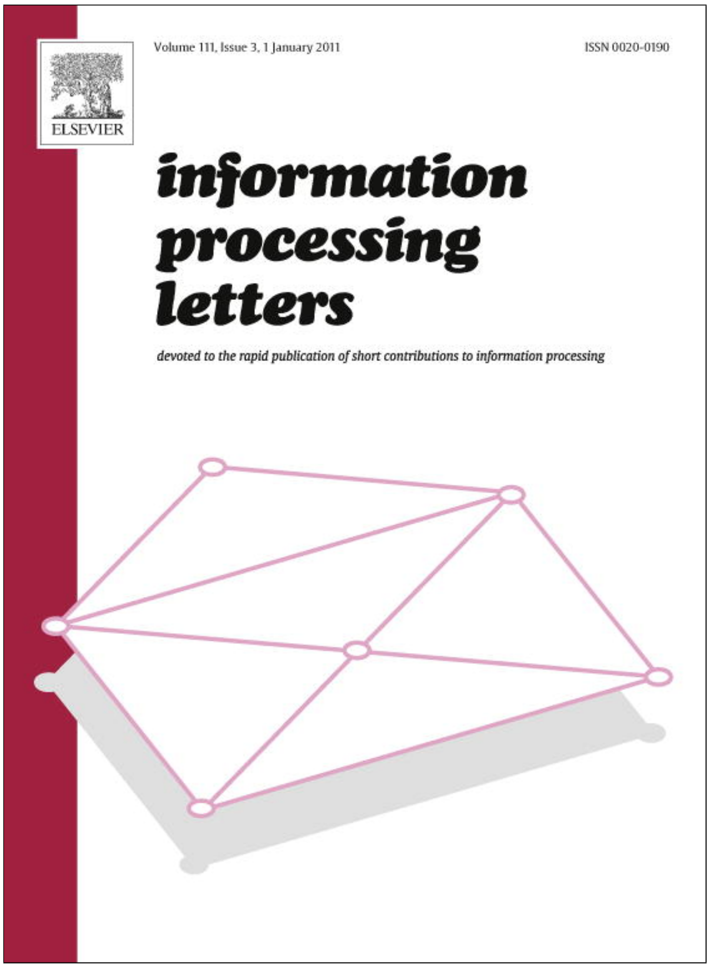

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A note on relaxed equitable coloring of graphs

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ABSTRACT

In this note we introduce the concept of equitable d-relaxed coloring. We prove that each graph with maximum degree at most r admits an equitable 1-relaxed r-coloring and provide a polynomial-time algorithm for constructing such a coloring.

1. Introduction

In this article we aim to introduce the notion of equitable relaxed coloring. We begin by reviewing the separate concepts of relaxed coloring and equitable coloring.

Let $G = (V, E)$ be a graph. A *d-relaxed k*-coloring, also known as a *d-defective* coloring, of G is a function f from V to $[k] := \{1, \ldots, k\}$ such that each color class $V_i := f^{-1}(i)$ induces a graph $G[V_i]$ with maximum degree $\Delta(G[V_i]) \leqslant d$. In this case we say that f is a (k, d)-coloring of G and that G is (k, d)-colorable. So a $(k, 0)$-coloring is just a (proper) k-coloring. Let $\chi^d(G)$ denote the least k such that G is (k, d)-colorable. We call an edge whose end points have the same color a *flaw*. This notion of coloring has been studied by many authors including [4–6,15]. For our purposes the following bound of Matsumoto [15] is a useful starting point.

Theorem 1. *Every graph G satisfies $\chi^1(G) \leqslant 1 + \lfloor \frac{\Delta(G)}{2} \rfloor = \chi^1(K_{\Delta(G)+1})$.*

A k-coloring f of G is *equitable* if every color class has size $\lfloor \frac{|V|}{k} \rfloor$ or $\lceil \frac{|V|}{k} \rceil$. Hajnal and Szemerédi [7] answered a question of Erdős by proving:

Theorem 2. *Every graph G with $\Delta(G) \leqslant r$ has an equitable $(r+1)$-coloring.*

Their proof was quite complicated, and did not yield a polynomial-time algorithm for producing the coloring. Recently Mydlarz and Szemerédi [17], and independently Kierstead and Kostochka [9], found simpler proofs that did yield polynomial-time algorithms. See [10] for an even simpler proof. These ideas were combined in [11] to obtain an $O(r|V|^2)$ time algorithm. Kierstead and Kostochka [8] also strengthened the Hajnal–Szemerédi Theorem by weakening the degree constraint:

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Theorem 3. *Let G be a graph with $d(x) + d(y) \leqslant 2r + 1$ for every edge xy. Then G has an equitable $(r + 1)$-coloring.*

The proof of Theorem 3 is more complicated than the new proofs of Theorem 2 in the sense that it does not yield a polynomial-time algorithm. This leads to the following open problem which first appeared in [11]:

Problem 4. Find a polynomial-time algorithm to equitably $(r+1)$-color any graph with $d(x)+d(y) \leqslant 2r+1$ for every edge xy.

It is natural to ask about an equitable version of the following theorem of Brooks [2]:

Theorem 5. *For any connected graph G, except the complete graph and the odd cycle, $\chi(G) \leqslant \Delta(G)$.*

Let $\chi_e(G)$ be the smallest integer k such that G has an equitable k-coloring. In 1973 Meyer [16] formulated:

Conjecture 6. *The statement of Theorem* 5 *holds with $\chi(G)$ replaced by $\chi_e(G)$.*

By Theorem 5, an equivalent version of this conjecture is that every $\Delta(G)$-colorable graph has an equitable coloring using at most $\Delta(G)$ colors. Here one should pay attention to an important difference between equitable coloring and normal coloring. Any k-colorable graph is k'-colorable for all $k' \geqslant k$ but this property does not necessarily hold for equitable coloring. For example, $H := K_{2m+1,2m+1}$ has no equitable $(2m + 1)$-coloring, although it satisfies $\Delta(H) = 2m + 1$ and $\chi_e(H) = 2$. This distinction motivates us to introduce another interesting parameter for equitable coloring. Let the *equitable chromatic threshold*, denoted by $\chi_e^*(G)$, of G be the smallest integer k such that G has an equitable k'-coloring for all $k' \geqslant k$. In these terms the Hajnal–Szemerédi Theorem states that $\chi_e^*(G) \leqslant \Delta(G) + 1$. In 1994 Chen, Lih, and Wu, strengthening Meyer's Conjecture, posed the most important open problem in the theory of equitable coloring:

Conjecture 7. *Every connected graph G, except the complete graph K_r and the odd cycle C_{2m+1} and the balanced complete bipartite graph $K_{2m+1,2m+1}$, has an equitable $\Delta(G)$-coloring.*

In our terms, and using the Hajnal–Szemerédi Theorem, the conjecture is equivalent to: Every connected graph G satisfies $\chi_e^*(G) \leqslant \Delta(G)$, except for K_r, C_{2m+1}, and $K_{2m+1,2m+1}$. Chen et al. [3] proved the case $\Delta(G) \leqslant 3$ and recently Kierstead and Kostochka [13] proved the case $\Delta(G) = 4$. They also phrased a similar conjecture without the connectedness condition of Chen et al. First, for every positive integer r they identified a small collection of *r-basic* graphs (if $r \geqslant 6$ then the only r-basic graph is K_r) with the property that every r-coloring of an r-basic graph has color classes of *equal* size, and they defined an r-colorable graph to be r-decomposable if its vertices could be partitioned into r-basic parts. It follows that every r-coloring of an r-decomposable graph is equitable. Moreover, if H is r-decomposable then $H + K_{r,r}$ has no equitable r-coloring. Kierstead and Kostochka [12] made the following conjecture, and proved that it is equivalent to Conjecture 7.

Conjecture 8. *If G satisfies $\Delta(G) \leqslant r$ and $\chi(G) \leqslant r$, then G has no equitable r-coloring if and only if r is odd, $K_{r,r} \subseteq G$ and $G - K_{r,r}$ is r-decomposable.*

Now we combine these lines of research. An *equitable (k,d)-coloring* is a (k,d)-coloring that is also equitable. Define the *equitable d-relaxed threshold*, denoted by $\chi_e^{*d}(G)$, of G to be the least k such that G has an equitable (k',d)-coloring for all $k' \geqslant k$.

Recall the following theorem of Vizing [18]:

Theorem 9. *For every graph G, there exists a $(\Delta(G) + 1)$-edge-coloring of G.*

With Theorem 9, the proof of our first result is easy.

Theorem 10. *Let G be a graph with $\Delta(G) \leqslant r$ and suppose $1 \leqslant d \leqslant r$. Then*:

(a) $\chi_e^{*d}(G) \leqslant r + 1 - d$;
(b) *an equitable $(r + 2 - d, d)$-coloring can be constructed in polynomial-time.*

Proof. (a) By Theorem 9, G has an $(r + 1)$-edge-coloring g with edge-color classes $F_1, \ldots, F_{r+1}$. Let R be the spanning subgraph of G with $E(R) := \bigcup_{i=1}^{d} F_i$. Let R^+ be an edge-maximal graph such that $R \subseteq R^+ \subseteq G$ and $\Delta(R^+) \leqslant d$; set $H := G - E(R^+)$. For any edge $xy \in E(H)$, $g(xy) \in \{d + 1, \ldots, r + 1\}$; so $\Delta(H) \leqslant r + 1 - d$. Furthermore, either x or y must have degree d in R^+ by the edge maximality of R^+. Therefore

$$d_H(x) + d_H(y) \leqslant \Delta(G) - d + \Delta(H) \leqslant 2(r - d) + 1.$$

Thus by Theorem 3, H has an equitable $(r+1-d)$-coloring f, and f is an $(r + 1 - d, d)$-coloring of G, since all flaws are edges in R^+ and $\Delta(R^+) \leqslant d$.

(b) Let H be as in the proof of part (a). $\Delta(H) \leqslant r+1-d$ so H has an equitable $(r+2-d)$-coloring f by Theorem 2, and f is an $(r + 2 - d, d)$-coloring of G, as required. Any standard proof of Vizing's Theorem yields a polynomial-time algorithm. Moreover as remarked above [9,11,15] all give polynomial-time algorithms for equitable coloring. So it follows that we can construct f in polynomial-time. $\square$

Now we restrict our attention to the case $d = 1$. Notice that our proof of Theorem 10(a) does not yield a polynomial-time algorithm because the proof of Theorem 3 does not. In the next section we give an alternative proof of Theorem 10(a) in the case $d = 1$ that does yield a polynomial-time algorithm. We are also hopeful that it may lead to improvements for the upper bound. This proof uses techniques from the new proofs of the Hajnal–Szemerédi Theorem discussed above, but also has interesting new twists.

For any class of graphs $\mathcal{C}$ set $\chi_e^{*d}(\mathcal{C}) := \max_{G\in\mathcal{C}} \chi_e^{*d}(G)$, etc. Let $\mathcal{C}_r$ be the class of graphs with maximum degree at most r. Using Theorems 1 and 10, we have the natural problem:

Problem 11. Close the gap $\lfloor \frac{r}{2} \rfloor + 1 = \chi^1(\mathcal{C}_r) \leqslant \chi_e^{*1}(\mathcal{C}_r) \leqslant r$.

It is tempting to conjecture that the lower bound is tight, but one must be careful, at least for small values of r and special graphs. For example, if $r = 3$ then $\lfloor \frac{r}{2} \rfloor + 1 = 2$, but the Petersen graph has no equitable $(2, 1)$-coloring.

Notation. Most of our notation is standard or defined above; possible exceptions include the following. For a vertex y and sets of vertices X and Y, $N_X(y)$ denotes the set of neighbors of y contained in X and $d_X(y)$ denotes the size of $N_X(y)$. The set of edges with one end in X and the other in Y is denoted by $E(X, Y)$. An (X, Y)-path is a path that starts with a vertex in X and ends with a vertex in Y. We abbreviate $(X, \{y\})$-path with (X, y)-path, etc. $|G|$ is the *order* (the number of vertices) of the graph G. Recall (see p. 12 of [14]) that for a family $\mathcal{S}$ of sets, $\bigcup\mathcal{S} := \{x\colon\ \exists S(S \in \mathcal{S} \wedge x \in S)\}$. For basic undefined concepts we refer the reader to [1].

2. A polynomial-time algorithm for Theorem 10(a), case $d = 1$

In this section we first give another proof of Theorem 10(a) in the case $d = 1$, and show that it yields a polynomial-time algorithm.

Algorithmic proof of Theorem 10(a), case $d = 1$. We argue by induction on $|V|$. The base case is trivial so assume $|V| > 1$. Let $v \in V$. By the induction hypothesis, there exists an equitable $(r, 1)$-coloring f of $G - v$. Let $s := \lceil \frac{|V|}{r} \rceil$. If $\lceil \frac{|V-v|}{r} \rceil < s$ then $|V - v| = r(s - 1)$. If $\lceil \frac{|V-v|}{r} \rceil = s$ then $r(s - 1) \leqslant |V - v| \leqslant rs - 1$. In any case, every color class of f has order s or $s - 1$. Since $d(v) \leqslant r$ we can pick a color class X of f in which v has at most one neighbor. Extend f to G by moving v to X and let $X' = X + v$. If f is not an $(r, 1)$-coloring of G, there must exist $u \in N_{X'}(v)$ such that $d_{X'}(u) = 2$. Then $d_{V\setminus X'}(u) \leqslant r - 2$ and we can move u to a color class in which it has no neighbors. Since the order of exactly one color class has increased, if f is not already equitable then there exists exactly one color class of order $s + 1$, at least one color class of order $s - 1$, and all other color classes have order s. Call such an $(r, 1)$-coloring a *nearly equitable* coloring of G.

Consider any nearly equitable $(r, 1)$-coloring f. Define an auxiliary digraph $\mathcal{H} := \mathcal{H}(f)$ on the color classes of f by $XY \in E(\mathcal{H})$ if and only if some vertex $x \in X$ has no neighbors in Y. In this case we say that x *witnesses* XY. If $P := X_1 X_2 \cdots X_t$ is a path in $\mathcal{H}$ and x_i $(1 \leqslant i \leqslant t - 1)$ is a vertex in X_i such that x_i witnesses $X_i X_{i+1}$, then switching witnesses along P means moving x_i to X_{i+1} for every $1 \leqslant i \leqslant t - 1$. This operation decreases $|X_1|$ by one and increases $|X_t|$ by one, while leaving the sizes of the interior vertices (color classes) unchanged. Call a color class X *small* if $|X| = s - 1$ and let $\mathcal{A}_0$ be the set of small classes and $\mathcal{A} \supseteq \mathcal{A}_0$ be the set of color classes X for which there exists an $(X, \mathcal{A}_0)$-path in $\mathcal{H}$. Let $V^+ := V^+(f)$ be the unique color class of size $s + 1$. Define $\mathcal{B} := \mathcal{B}(f)$ to be the set of color classes X such that there exists a (V^+, X)-path in $\mathcal{H}$. If $X \in \mathcal{A} \cap \mathcal{B}$ then there exists a (V^+, X)-path and an $(X, \mathcal{A}_0)$-path in $\mathcal{H}$; joining these paths at X to obtain a $(V^+, \mathcal{A}_0)$-path, and then switching witnesses, yields an equitable $(r, 1)$-coloring of G. Otherwise $\mathcal{A}$ and $\mathcal{B}$ are disjoint, and we must work harder. In this case, let $\mathcal{M}$ be the set of classes not in $\mathcal{A} \cup \mathcal{B}$, and set: $A := \bigcup\mathcal{A}$, $B := \bigcup\mathcal{B}$, $M := \bigcup\mathcal{M}$, $a := |\mathcal{A}|$, $b := |\mathcal{B}|$ and $m := |\mathcal{M}| = r - a - b$. Since $\mathcal{A}_0 \subseteq \mathcal{A}$ and $V^+ \in \mathcal{B}$ we have

$$a(s-1) \leqslant |A| \leqslant as - 1,$$
$$ms = |M| \quad \text{and} \quad bs + 1 = |B|. \tag{2.1}$$

Call an edge $zy \in E(A, B)$ a *solo* edge if $z \in Z \in \mathcal{A}$ and $d_Z(y) = 1$. Solo edges exist: Otherwise every $y \in B$ has at least two neighbors in every class of $\mathcal{A}$ and at least one neighbor in every class of $\mathcal{M}$; so $2a + m \leqslant d_{A\cup M}(y) \leqslant r$. Let $X \in \mathcal{A}_0$. If $v \in M$ then $d_X(v) \geqslant 1$. Thus by (2.1),

$$ms + 2(bs + 1) = |M| + 2|B| \leqslant \left|E(X, B \cup M)\right| \leqslant r|X|$$
$$= r(s - 1).$$

So $m + 2b < r$, yielding the contradiction: $2r > (2a + m) + (m + 2b) = 2r$.

Fix a solo edge zy with $z \in Z \in \mathcal{A}$ and $y \in Y \in \mathcal{B}$, and let P_z be a $(Z, \mathcal{A}_0)$-path and P_y be a (V^+, Y)-path in $\mathcal{H}$. We will improve f to an $(r, 1)$-coloring f' so that either f' is equitable (possibly with one additional flaw) or f' is nearly equitable, but has less flaws. More formally, argue by secondary induction on the number γ of flaws in f. First, move y to Z and switch witnesses along P_y to form an equitable $(b, 1)$-coloring of $B - y$. If z is not incident to a flaw in f then this yields an equitable or nearly equitable $(r, 1)$-coloring of G. If the coloring is not already equitable, the large class is now $Z + y$ and we obtain an equitable $(r, 1)$-coloring f' of G by switching witnesses along P_z. In particular, this proves the base case $\gamma = 0$. If z is incident to a flaw in f, then $d_{Z+y}(z) = 2$. Thus we can move z to some color class in which it has no neighbors. This yields an equitable or nearly equitable $(r, 1)$-coloring f' of G; moreover we have removed a flaw incident to z without introducing any new flaws, so we are done by the secondary induction hypothesis. □

Our next goal is to implement an algorithm based on the preceding proof.

Proposition 12. *There is an $O(n^2 r)$ time algorithm to construct an equitable $(r, 1)$-coloring for any graph on n vertices with maximum degree at most $r \geqslant 1$.*[5]

Proof. The algorithm will receive a graph G on n vertices $x_1, \ldots, x_n$. Let $V_k := \{x_i\colon\ i \leqslant k\}$ and $G_k := G[V_k]$. We start by finding (trivially) an equitable r-coloring of the 1-vertex

[5] Our model of computation assumes that array entries can be written and read in one step, and all numbers and addresses are $O(n)$.

graph G_1. Now suppose that we have constructed an $(r,1)$-coloring of G_k, and consider stage $k+1$, in which we color G_{k+1}. First we add x_{k+1} to a color class in which it has at most one neighbor. This can create at most one flaw, and the result is an equitable or nearly equitable $(r,1)$-coloring of G_{k+1}, unless a flaw $x_k y$ is created and y is already incident to a flaw. In this case, we can move y to a class in which it has no flaws, and again obtain an equitable or nearly equitable coloring of G_{k+1}. If we are left with a nearly equitable coloring, then we improve the situation by recoloring to get either an equitable $(r,1)$-coloring, possibly with one more flaw, or a nearly equitable $(r,1)$-coloring with fewer flaws. This is the most time consuming operation, but it is performed at most $3n$ times throughout the entire process: at most $2n$ times it removes a flaw, since at each stage we introduce at most 2 flaws, and at most n times it produces an equitable $(r,1)$-coloring.

We will use the following global data structures:

- L is an $n \times r$-array in which $L[v,i]$ is the i-th neighbor of vertex v (we assume this is how G is received).
- F is an n-array, where $F[v]$ is the color of vertex v.
- N is an $n \times r$-array, where for any vertex v and color class X, $N[v,X] = d_X(v)$.
- C is an r-array, where $C[X]$ is a linked-list of vertices in color class X.
- S is an r-array, where $S[X]$ is the number of vertices in color class X.
- P is an n-array, where $P[v]$ is the node of v in the list $C[F[v]]$ (this array is needed for $O(1)$ removal from $C[F[v]]$).
- H is an $r \times r$ array representing the adjacency matrix of the digraph $\mathcal{H}$.

We will show that the procedures listed below meet the specified performance bounds in the order given. Recall that $s = \lceil \frac{n}{r} \rceil$. In the top-down analysis that follows, we will assuming that the specified bounds for the lower procedures hold. Since MAIN-PROCEDURE produces an equitable $(r,1)$-coloring of G, this will complete the proof. In the pseudo-code below, lines that begin with $\triangleright$ represent a comment.

Procedure name	Performance bound
MAIN-PROCEDURE	$O(n^2 r)$
IMPROVE-NEARLY-EQUITABLE-COLORING	$O(r^2 s)$
CONSTRUCT-H	$O(r^2 s)$
MOVE-WITNESSES-ALONG-PATH	$O(r^2 + rs)$
MOVE-VERTEX-TO-COLOR-CLASS	$O(r)$

MAIN-PROCEDURE()
1 Initialize arrays F, N, C, P and S to represent an empty coloring
2 For any color class X call MOVE-VERTEX-TO-COLOR-CLASS(x_1, X)
3 **for** $k \leftarrow 2$ **to** n
  do $\triangleright$ G_{k-1} is equitably $(r,1)$-colored
4   $s \leftarrow \lceil \frac{k}{r} \rceil$
5   Find a color class X such that $N[x_k, X] \leq 1$
6   MOVE-VERTEX-TO-COLOR-CLASS(x_k, X)
7   **if** there exists $u \in N_{V_k}(x_k)$ such that $F[u] = X$ and $N[u,X] = 2$
8   **then** find a color class Y such that $N[u,Y] = 0$
9   MOVE-VERTEX-TO-COLOR-CLASS(u, Y)
10   **while** there exists a color class of size $s+1$
11   **do** IMPROVE-NEARLY-EQUITABLE-COLORING(k, s)
  $\triangleright$ G_k is equitably $(r,1)$-colored

Since each array has at most nr elements, initialization takes $O(nr)$ steps. The searches on lines 5, 7 and 8 each take $O(r)$ steps. Therefore the lines 2 through 9 consume $O(nr)$ steps throughout the entire procedure since each line completes in $O(r)$ steps. We can perform the check on line 10 by searching the array S in $O(r)$ steps. As was argued above, because IMPROVE-NEARLY-EQUITABLE-COLORING either removes a flaw or produces an equitable $(r,1)$-coloring of G_k, IMPROVE-NEARLY-EQUITABLE-COLORING is called at most $3n$ times. Therefore the entire routines run in $O(nr^2 s) = O(n(rs)r) = O(n^2 r)$ steps, because $rs \leqslant n + r$.

IMPROVE-NEARLY-EQUITABLE-COLORING(k, s)
 $\triangleright$ Let γ be the number of flaws
1 Find the color class V^+ such that $|V^+| = s + 1$
2 CONSTRUCT-H()
3 Mark color classes belonging to $\mathcal{A}$ and $\mathcal{B}$ using breadth-first search
4 **if** $\mathcal{A} \cap \mathcal{B} \neq \emptyset$
5   **then** MOVE-WITNESSES-ALONG-PATH$(V^+, \mathcal{A}_0)$
6   **else** Search all edges in $E(G_k)$ to find a solo edge zy with $z \in A$ and $y \in B$
7   MOVE-WITNESSES-ALONG-PATH$(V^+, \{F[y]\})$
8   MOVE-VERTEX-TO-COLOR-CLASS$(y, F[z])$
    $\triangleright$ Edge zy is now a flaw, so there are now $\gamma + 1$ flaws
9   **if** $N[z, F[z]] = 1$
10   **then** MOVE-WITNESSES-ALONG-PATH$(F[z], \mathcal{A}_0)$
11   **else** Find a color class X such that $N[z, X] = 0$
12   MOVE-VERTEX-TO-COLOR-CLASS(z, X)
        $\triangleright$ Moving z removed 2 flaws, so there are now $\gamma - 1$ flaws

V^+ can be found by searching the array S, so line 1 takes $O(r)$ steps. Starting with V^+, a breadth-first search of $\mathcal{H}$ is used to mark all color classes in $\mathcal{B}$. To compute $\mathcal{A}$, we perform a breadth-first search from each color class in $\mathcal{A}_0$, but with the sense of the edges in $\mathcal{H}$ reversed. Note that in either search the edges incident to a color class need only be considered once. Therefore, since there are at most r^2 edges in $\mathcal{H}$, computing $\mathcal{A}$ or $\mathcal{B}$ takes $O(r^2)$ steps. The search for a solo edge on line 6 takes $O(r^2 s)$ steps, because there are at most $r^2 s$ edges in G_k. The search on line 11 takes $O(r)$ steps. Therefore every line in IMPROVE-NEARLY-EQUITABLE-COLORING completes in $O(r^2 s)$ steps.

CONSTRUCT-H()
1 **for** every ordered pair of color classes (X, Y)
2   **do if** there exists $x \in X$ such that $N[x, Y] = 0$
3   **then** $H[X, Y] \leftarrow 1$
4   **else** $H[X, Y] \leftarrow 0$

There are r^2 pairs of color classes to be considered. For each ordered pair (X, Y), the search of $C[X]$ on line 2 takes $O(s)$ steps. Therefore CONSTRUCT-H runs in $O(r^2 s)$ steps.

MOVE-WITNESSES-ALONG-PATH$(U, \mathcal{V})$
1 Find a $(U, \mathcal{V})$-path $X_1 X_2 \cdots X_t$ in $\mathcal{H}$ using a breadth-first search
2 **for** $i \leftarrow 1$ **to** $t - 1$
3   **do** find $x \in X_i$ such that $N[x, X_{i+1}] = 0$
4   MOVE-VERTEX-TO-COLOR-CLASS(x, X_{i+1})

It takes $O(r^2)$ steps to find a path in $\mathcal{H}$ using breadth-first search since there are at most r^2 edges in $\mathcal{H}$. For each of the $t-1$ edges in the path, it takes $O(r+s)$ steps to find and move a witness. Since $t - 1 \leqslant r$ the running time of MOVE-WITNESSES-ALONG-PATH is $O(r^2 + rs)$.

```
MOVE-VERTEX-TO-COLOR-CLASS(v, Y)
if v has been colored
   then X ← F[v]
        Remove node P[v] from linked list C[X]
        S[X] ← S[X] − 1
        for each neighbor u of v
            do N[u, X] ← N[u, X] − 1
F[v] ← Y
Insert node P[v] into linked list C[Y]
S[Y] ← S[Y] + 1
for each neighbor u of v
    do N[u, Y] ← N[u, Y] + 1
```

It takes $O(1)$ steps to remove $P[v]$ from $C[X]$ and add $P[v]$ to $C[Y]$; and $O(r)$ steps to update $N[u, X]$ and $N[u, Y]$. So the entire routines run in $O(r)$ steps. This completes the proof. $\square$

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