Chapter 6: Residue Theory

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Introduction

- In the previous chapters, we have seen how the theory of contour integration lends great insight into the properties of analytic functions.
- The goal this chapter is to explore another dividend of this theory, namely, its usefulness in evaluating certain real integrals.
- We shall begin by presenting a technique for evaluating contour integrals that is known as residue theory.
- Then we will introduce some application of the theory to the evaluating the real integrals.

The Residue Theorem

- If \( f(z) \) is analytic on and inside a simple closed positively oriented contour \( \Gamma \) except a single isolated singularity, \( z_0 \), lying interior to \( \Gamma \), \( f(z) \) has a Laurent series expansion
  \[
  f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j
  \]
  converging to some punctured neighborhood of \( z_0 \).
- In particular, the above equation is valid for all \( z \) on the small positively oriented circle \( C \) continuously deformed from \( \Gamma \) (as shown in Fig. 6.1).
The Residue Theorem (Cont’d)

According to the Continuous Deformation Invariance Theorem (page 231), we have
\[ \int_{\Gamma} f(z) \, dz = \int_{C} f(z) \, dz \]

The last integral can be computed by termwise integration of the series along \( C \). For all \( j \neq -1 \) the integral is zero, and for \( j = -1 \) we obtain the value \( 2\pi i a_{-1} \)

Consequently we have
\[ \int_{\Gamma} f(z) \, dz = 2\pi i a_{-1} \]

Thus the constant \( a_{-1} \) plays an important role in contour integration. Accordingly, we adopt the following terminology

**Definition**
If \( f \) has an isolated singularity at the point \( z_0 \), then the coefficient \( a_{-1} \) of \( (z - z_0)^{-1} \) in the Laurent expansion for \( f \) around \( z_0 \) is called the **residue** of \( f \) at \( z_0 \) and is denoted by
\[ \text{Res}(f; z_0) \text{ or Res}(z_0) \]

How to Compute the Residue (Cont’d)

- If \( f \) has a **removable singularity** at \( z_0 \), all the coefficients of the negative powers of \( (z - z_0) \) in its Laurent expansion are zero, and so, in particular, the residue at \( z_0 \) is zero
- If \( f \) has an **essential singularity** at \( z_0 \), we have to use its Laurent expansion to find the residue at \( z_0 \) (See Example 1 on page 308)
- If \( f \) has a pole of order \( m \) at \( z_0 \), we have the following theorem to find the residue

**Theorem**
**If \( f \) has a pole of order \( m \) at \( z_0 \), then**
\[ \text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \]

Example 2 gives us another way to compute the residue when \( f \) is a rational polynomial
- Let \( f(z) = P(z)/Q(z) \), where the functions \( P(z) \) and \( Q(z) \) are both analytic at \( z_0 \) and \( Q \) has a simple zero at \( z_0 \), while \( P(z_0) \neq 0 \). Then we have
\[ \text{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)} \]
How to Compute the Residue (Cont’d)

- When there are a finite number of isolated singularities inside the simple closed positively oriented contour \( \Gamma \), we have the following theorem

**Theorem**

If \( \Gamma \) is a simple closed positively oriented contour and \( f \) is analytic inside and on \( \Gamma \) except at the points \( z_1, z_2, \ldots, z_n \) inside \( \Gamma \), then

\[
\int_{\Gamma} f(z) \, dz = 2\pi i \sum_{j=1}^{n} \text{Res}(z_j)
\]

Trigonometric Integrals Over \([0, 2\pi]\) (Cont’d)

- Also taking \( dz = ie^{i\theta}d\theta = izd\theta \) into account, Eq. (1) can be transformed into a complex contour integration as

\[
\int_{0}^{2\pi} U(\cos \theta, \sin \theta) d\theta = \oint_{|z|=1} F(z) dz
\]

where the new integrand \( F \) is

\[
F(z) := U \left[ \frac{1}{2} \left( z + \frac{1}{z} \right), \frac{1}{2i} \left( z - \frac{1}{z} \right) \right] \cdot \frac{1}{iz}
\]

Trigonometric Integrals Over \([0, 2\pi]\)

- Our goal of this section is to apply the residue theory to evaluate real integrals of the form

\[
\int_{0}^{2\pi} U(\cos \theta, \sin \theta) d\theta \quad (1)
\]

- We use \( z = e^{i\theta} \) \((0 \leq \theta \leq 2\pi)\) to parameterize the closed positively oriented contour \( |z| = 1 \). Then a contour integral can be transformed into a real integral

- According to Euler’s equation, we have

\[
\begin{align*}
\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = (z + z^{-1})/2 \\
n\sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = (z - z^{-1})/2i
\end{align*}
\]

- Of course, the function \( F \) must be a rational function of \( z \)

- Hence, it has only removable singularities (which can be ignored in evaluation integrals) or poles

- Consequently, by the residue theorem, our trigonometric integral equals \( 2\pi i \) time the sum of the residues at those poles of \( F \) that lie inside the unite circle
Improper Integrals of Certain Functions Over \((\infty, \infty)\)

- Given any function \(f\) continuous on \((\infty, \infty)\), the limit
  \[
  \lim_{\rho \to \infty} \int_{-\rho}^{\rho} f(x) \, dx
  \]
  is called the Cauchy principal value of the integral of \(f\) over \((\infty, \infty)\), and we write
  \[
  \text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx := \lim_{\rho \to \infty} \int_{-\rho}^{\rho} f(x) \, dx
  \]

- We shall now show how the theory of residue can be used to compute p.v. integrals for certain functions of \(f\).
- See Example 1 on page 319 to learn the basic idea of the algorithm.

**Lemma**

If \(f(z) = P(z)/Q(z)\) is the quotient of two polynomials such that

\[
\text{degree } Q \geq 2 + \text{degree } P
\]

then

\[
\lim_{\rho \to \infty} \int_{C_\rho^+} f(z) \, dz = 0
\]

where \(C_\rho^+\) is the upper half-circle of radius \(\rho\) defined in Eq. (4) on page 320 as shown in Figure 6.4.

Improper Integrals Involving Trigonometric Functions

- The purpose of this section is to use residue theory to evaluate integrals of the general forms:
  \[
  \text{p.v.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \, dx, \quad \text{p.v.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \, dx
  \]

- If we obtain the value of the integral
  \[
  \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx} \, dx
  \]
  the above two integrals can be obtained by computing the real and imaginary parts.
**Lemma**

If \( m > 0 \) and \( P/Q \) is the quotient of two polynomials such that

\[
\text{degree } Q \geq 1 + \text{degree } P
\]

then

\[
\lim_{\rho \to \infty} \int_{C_\rho^+} e^{imx} \frac{P(x)}{Q(x)} \, dz = 0
\]

where \( C_\rho^+ \) is the upper half-circle of radius \( \rho \)

Then the improper integral \( \int_{-\infty}^{\infty} f(x) \, dx \) can be computed as follows

\[
p.v. \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} \, dx = \lim_{\rho \to \infty} 2\pi i \sum (\text{residues inside } \Gamma_\rho)
\]

Thus

\[
p.v. \int_{-\infty}^{\infty} \cos mx \frac{P(x)}{Q(x)} \, dx = \Re \left\{ p.v. \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} \, dx \right\}
\]

\[
p.v. \int_{-\infty}^{\infty} \sin mx \frac{P(x)}{Q(x)} \, dx = \Im \left\{ p.v. \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} \, dx \right\}
\]