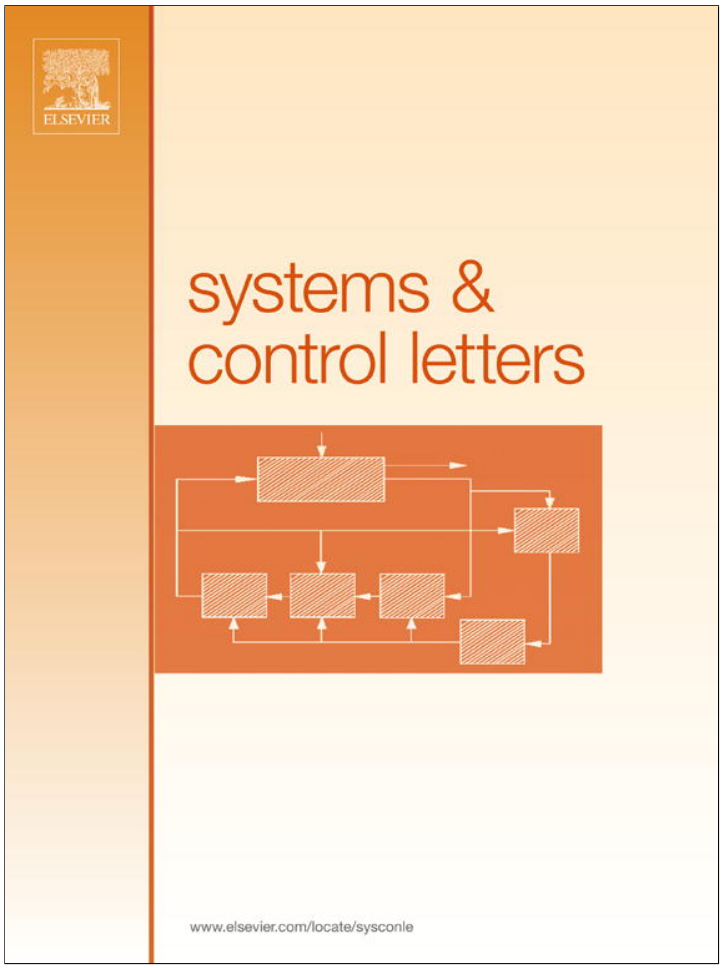

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Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements[☆]

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ABSTRACT

This paper studies the finite-time consensus problem of heterogeneous multi-agent systems composed of first-order and second-order integrator agents. By combining the homogeneous domination method with the adding a power integrator method, we propose two classes of consensus protocols with and without velocity measurements. First, we consider the protocol with velocity measurements and prove that it can solve the finite-time consensus under a strongly connected graph and leader-following network, respectively. Second, we consider the finite-time consensus problem of heterogeneous multi-agent systems, for which the second-order integrator agents cannot obtain the velocity measurements for feedback. Finally, some examples are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

Distributed systems and networks have attracted much attention in the last few years because of their flexibility and computational performance. As a fundamental of distributed coordination, the consensus problem which means that a group of autonomous agents reaches an agreement upon some parameters has been widely studied by multi-disciplinary researchers. This is partly due to its broad applications of multi-agent systems in many areas, such as the formation control of robotic systems, the cooperative control of unmanned aerial vehicles, the attitude alignment of satellite clusters, the target tracking of sensor networks, the congestion control of communication networks, and so on [1–3].

1.1. Related work

The study of multi-agent consensus is expected to establish the effective consensus protocols with required performance. The convergence speed is an active topic and can reflect the performance of the proposed consensus protocols. Up to now, most of the existing consensus protocols for multi-agent systems are convergent asymptotically, i.e., the systems have at most exponential convergence rates [4,5]. For such consensus protocols, lots of researchers found that the convergence rates can be influenced by the weights of networks (the second smallest eigenvalue of the interaction graph Laplacian) [6,7]. One can increase convergence speed with respect to the linear protocols, but the consensus can never be reached in a finite time. However, in many practical situations, it is required that the consensus can be reached in a finite time, such as when high precision performance and stringent convergence time are required. Cortés [8] proposed two normalized and signed gradient flows of a differential function and used these protocols to solve the finite-time consensus problem. Hui [9] used the notion of finite-time semistability to develop the finite-time rendezvous problem. Chen et al. [10] studied the finite-time consensus of a multi-agent system using a binary consensus protocol. However, the aforementioned consensus protocols involved discontinuous dynamical systems, which may lead to chattering or excite high frequency dynamics in applications involving flexible structures [11]. To avoid these negative effects, some continuous consensus protocols are proposed for multi-agent systems which can reach the finite-time consensus [12–18]. Based on the Lyapunov method, Wang and Xiao [12,13] showed that the multi-agent systems could solve the finite-time consensus problem for both the bidirectional and unidirectional interaction cases. Jiang and Wang [14,15] investigated the finite-time consensus of multi-agent systems with respect to a monotonic function under fixed and switching topologies. Zheng et al. [16] studied the finite-time

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consensus of stochastic multi-agent systems with a general protocol. For second-order multi-agent systems, Wang and Hong [17] gave some protocols and showed that these protocols can reach the finite-time consensus based on homogeneous method. Based on the adding a power integrator method [19], Li et al. [18] designed a protocol and discussed the finite-time consensus of leaderless and leader-following multi-agent systems with external disturbances.

Unfortunately, all the aforementioned multi-agent systems were homogeneous, that is, all the agents have the same dynamics behavior. However, the dynamics of the agents are quite different because of various restrictions or the common goals with mixed agents in the practical systems. Liu and Liu [20] studied the stationary consensus of discrete-time heterogeneous multi-agent systems with communication delays. In [21,22], the authors considered the consensus of heterogeneous multi-agent systems with and without velocity measurements. To our best knowledge, there is no literature researching the finite-time consensus of heterogeneous multi-agent systems.

1.2. Our results

Inspired by the recent developments in heterogeneous multi-agent systems, we decide to investigate the finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements under two classes of specific directed networks in this paper, in which the agents are governed by first-order and second-order integrators, respectively. By combining the homogeneous domination method with the adding a power integrator method, we first propose a continuous protocol with velocity measurements for the heterogeneous multi-agent systems and show that these systems can achieve the finite-time consensus. For the second-order integrator agents, the velocity information sometimes is unmeasurable because of technology limitations or environmental disturbances [23–25]. Therefore, we design a continuous finite-time consensus protocol for the heterogeneous multi-agent systems in which the second-order integrator agents cannot obtain the velocity information. By using the graph theory, Lyapunov theory and the property of a homogeneous function, we prove that the heterogeneous multi-agent systems converge in finite time under two classes of directed networks: (1) strongly connected and satisfies the detailed balance condition, (2) leader-following network and the network among the followers is strongly connected and satisfies the detailed balance condition. Finally, some simulation examples are presented to show the effectiveness of our proposed protocols.

This paper is organized as follows. In Section 2, we present some notions in graph theory and formulate the model to be studied, and assemble some key lemmas. In Sections 3 and 4, we propose the continuous protocols with and without velocity measurements and show that the heterogeneous multi-agent system can achieve the finite-time consensus, respectively. In Section 5, simulation examples are given to illustrate the effectiveness of our proposed protocols. Finally, some conclusions are drawn in Section 6.

Notation: Throughout this paper, we let $\mathbb{R}, \mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the set of real numbers, positive real numbers and non-negative real numbers, $\mathbb{R}^n$ is the n-dimensional real vector space, $\mathcal{I}_n = \{1, 2, \ldots, n\}$. For a given vector or matrix X, X^T denotes its transpose. $\mathbf{1}_n$ is a vector with elements being all ones. A is said to be non-negative (resp. positive) if all entries a_{ij} are non-negative (resp. positive), denoted by $A \geq 0$ (resp. $A > 0$). $\mathrm{sig}(x)^\alpha = \mathrm{sign}(x)|x|^\alpha$, where $\mathrm{sign}(\cdot)$ is the sign function.

2. Preliminaries

2.1. Graph theory

The network formed by multi-agent systems can always be represented by a graph. Thus, graph theory is an important tool to analyze the consensus problem for multi-agent systems. In this subsection, some basic concepts and properties are presented in graph theory [26].

Let $\mathcal{G}(\mathcal{A}) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order n with a vertex set $\mathcal{V} = \{s_1, s_2, \ldots, s_n\}$, an edge set $\mathcal{E} = \{e_{ij} = (s_i, s_j)\} \subset \mathcal{V} \times \mathcal{V}$ and a non-negative asymmetric matrix $\mathcal{A} = [a_{ij}]$. $(s_j, s_i) \in \mathcal{E} \Leftrightarrow a_{ij} > 0 \Leftrightarrow$ agent i and j can communicate with each other, namely, they are adjacent. Moreover, we assume $a_{ii} = 0$. A is called the weighted matrix and a_{ij} is the weight of $e_{ij} = (s_i, s_j)$. The set of neighbors of s_i is denoted by $\mathcal{N}_i = \{s_j : e_{ji} = (s_j, s_i) \in \mathcal{E}\}$. A path that connects s_i and s_j in the directed graph $\mathcal{G}$ is a sequence of distinct vertices $s_{i_0}, s_{i_1}, s_{i_2}, \ldots, s_{i_m}$, where $s_{i_0} = s_i, s_{i_m} = s_j$ and $(s_{i_r}, s_{i_{r+1}}) \in \mathcal{E}$, $0 \leq r \leq m-1$. A directed graph is said to be strongly connected if there exists a path between any two distinct vertices of the graph. If a directed graph has the property that $(s_i, s_j) \in \mathcal{E} \Leftrightarrow (s_j, s_i) \in \mathcal{E}$, the directed graph is called undirected. It is easy to see that adjacency matrix $\mathcal{A}$ is symmetric if $\mathcal{G}$ is an undirected graph. The degree matrix $\mathcal{D} = [d_{ij}]_{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j:s_j\in\mathcal{N}_i} a_{ij}$ and the Laplacian matrix of the graph is defined as $\mathcal{L} = [l_{ij}]_{n\times n} = \mathcal{D} - \mathcal{A}$. It has been shown in [26] that $\mathcal{L}\mathbf{1}_n = 0$. The directed graph $\mathcal{G}(\mathcal{A})$ is said to satisfy the detailed balance condition if there exist some scalars $\omega_i > 0$ $(i = 1, 2, \ldots, n)$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$ [1]. In the multi-agent systems, we refer to the agent as the leader if it only sends information to other agents and cannot receive any from other agents, i.e., $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$ and $\bar{a} = [a_{1n}, a_{2n}, \ldots, a_{(n-1)n}]^T \geq 0$ if the agent n is the leader.

2.2. Heterogeneous multi-agent systems

Suppose that the heterogeneous multi-agent system consists of first-order and second-order integrator agents. The number of agents is n, labeled 1 through n. In this subsection, we propose the heterogeneous multi-agent system and define the concept of finite-time consensus.

Suppose that the number of second-order integrator agents is m $(m < n)$. Each second-order agent dynamics is given as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \quad i \in \mathcal{I}_m, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position, velocity and control input, respectively, of agent i. The initial conditions are $x_i(0) = x_{i0}$, $v_i(0) = v_{i0}$. Each first-order agent dynamics is given as follows:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n/\mathcal{I}_m, \tag{2}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position and control input, respectively, of agent i. The initial conditions is $x_i(0) = x_{i0}$. Let $x_0 = [x_{10}, x_{20}, \ldots, x_{n0}]$, $v_0 = [v_{10}, v_{20}, \ldots, v_{m0}]$.

Definition 1. The heterogeneous multi-agent system (1)–(2) is said to reach consensus asymptotically if for any initial conditions x_0 and v_0, we have $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ $(i, j \in \mathcal{I}_n)$ and $\lim_{t\to\infty}(v_i(t) - v_j(t)) = 0$ $(i, j \in \mathcal{I}_m)$.

Definition 2. The heterogeneous multi-agent system (1)–(2) is said to reach consensus in finite time if for any initial conditions x_0 and v_0, there exists a finite settling time $T(x_0, v_0)$ such that $\lim_{t\to T(x_0,v_0)^-}(x_i(t) - x_j(t)) = 0$ $(i, j \in \mathcal{I}_n)$ and $\lim_{t\to T(x_0,v_0)^-}(v_i(t) - v_j(t)) = 0$ $(i, j \in \mathcal{I}_m)$.

2.3. Key lemmas

In this subsection, some key lemmas are given to be used to prove our main results.

Lemma 1 ([19]). *Let $q \geq 1$ be an odd integer or a ratio of odd integers. Then, the following inequality holds:*

$$|a-b|^q \leq 2^{q-1}|a^q - b^q|, \quad \forall a \in \mathbb{R},\ b \in \mathbb{R}.$$

Lemma 2 ([19]). *Suppose $n, m \in \mathbb{R}_{>0}$ and $a \geq 0$, $b \geq 0$ and $\pi \geq 0$ are continuous functions. Then, for any constant $c > 0$*

$$a^n b^m \pi \leq c \cdot a^{n+m} + \frac{m}{n+m}\left[\frac{n}{c(n+m)}\right]^{n/m} b^{n+m}\pi^{(n+m)/m}.$$

Lemma 3 ([19]). *Let the real number $r \in (0,1)$ be a ratio of odd integers. Then, the following inequality holds for any real numbers $0 < l < 1$ and t*

$$t^r + (1-t)^r + l^2 t^{1+r} \geq (2^r - 1)l^{1-r}.$$

Consider the autonomous system

$$\dot{x} = f(x), \tag{3}$$

where $f : \mathbf{D} \to \mathbb{R}^n$ is a continuous function with $\mathbf{D} \subset \mathbb{R}^n$.

A vector field $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is homogenous of degree $\sigma > 0$ with dilation $(r_1, r_2, \ldots, r_n)$, $r_i > 0 (i \in \mathcal{I}_n)$, if

$$f_i(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^{\sigma+r_i} f_i(x), \quad i \in \mathcal{I}_n,\ \varepsilon > 0.$$

Lemma 4 ([27]). *Suppose that the system (3) is homogeneous of degree σ with dilation $(r_1, r_2, \ldots, r_n)$, function $f(x)$ is continuous and $x = 0$ is its asymptotically stable equilibrium. If homogeneity degree $\sigma < 0$, the equilibrium of the system (3) is finite-time stable.* □

Remark 1. Similar to the analysis of Ref. [17], it is easy to see that the heterogeneous multi-agent system (1)–(2) can achieve consensus in finite time, if the system (1)–(2) with $(x_1, \ldots, x_n, v_1, \ldots, v_m)$ is homogeneous of degree $\sigma < 0$ with dilation $(\underbrace{r_1, \ldots, r_1}_{n}, \underbrace{r_2, \ldots, r_2}_{m})$ and can achieve consensus asymptotically.

3. Finite-time consensus protocol with velocity information

In this section, we first propose the protocol with velocity information for the heterogeneous multi-agent system (1)–(2) as follows:

$$u_i = \begin{cases} k_1\left[k_2^{\alpha_1}\left(\sum_{j=1}^n a_{ij}(x_j - x_i)\right) - v_i^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1}, & i \in \mathcal{I}_m, \\ k_2\left(\sum_{j=1}^n a_{ij}(x_j - x_i)\right)^{\frac{1}{\alpha_1}}, & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{4}$$

where $\mathcal{A} = [a_{ij}]_{n\times n}$ is the aforementioned weighted adjacency matrix associated with the graph $\mathcal{G}$, $1 < \alpha_1 < 2$ is a ratio of odd integers, $k_2 > \frac{(D+nA)}{(1+\alpha_1)w} + \frac{2^{1-1/\alpha_1}\alpha_1}{1+\alpha_1}$, $k_1 > (2 - 1/\alpha_1)2^{1-1/\alpha_1}k_2^{\alpha_1}\left(\frac{2k_2 D\alpha_1 + 2^{1-1/\alpha_1}(D + D\alpha_1 + d\alpha_1 + ma + k_2 W)}{1+\alpha_1}\right)$ are the feedback gains, $A = \max_{i,j\in\mathcal{I}_n}\{a_{ij}\}$, $a = \max_{i,j\in\mathcal{I}_m}\{a_{ij}\}$, $D = \max_{i\in\mathcal{I}_n}\{d_{ii}\}$, $d = \max_{i\in\mathcal{I}_m}\{d_{ii}\}$, $W = \max_{i\in\mathcal{I}_m}\{\omega_i\}$ and $w = \min_{i\in\mathcal{I}_m}\{\omega_i\}$. Then, we show that the heterogeneous multi-agent system (1)–(2) with protocol (4) reaches consensus in finite time under two classes of directed networks.

Theorem 1. *Suppose the graph $\mathcal{G}(\mathcal{A})$ is strongly connected and satisfies the detailed balance condition. Then the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (4).*

Proof. The graph $\mathcal{G}(\mathcal{A})$ is strongly connected and satisfies the detailed balance condition, i.e., there exists a vector $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T \in \mathbb{R}^n_{>0}$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$. Let $y_i = \sum_{j=1}^n a_{ij}(x_j - x_i)$, $i \in \mathcal{I}_n$. Take a Lyapunov function as

$$V_1(t) = V_0(t) + \sum_{i=1}^m \frac{1}{(2 - 1/\alpha_1)2^{1-1/\alpha_1}k_2^{\alpha_1+1}} \times \int_{v_i^*}^{v_i} (s^{\alpha_1} - v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}} ds,$$

where

$$V_0(t) = \sum_{i=1}^n \sum_{j=1}^n \omega_i a_{ij} \frac{(x_j - x_i)^2}{4}$$

and $v_i^* = k_2 y_i^{1/\alpha_1}$ $(i \in \mathcal{I}_m)$, $\int_{v_i^*}^{v_i}(s^{\alpha_1} - v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}} ds$ are positive definite and proper. Thus, $V_1(t)$ is positive definite with respect to $x_i(t) - x_j(t)$ $(\forall i \neq j,\ i, j \in \mathcal{I}_n)$ and $v_i(t)$ $(i \in \mathcal{I}_m)$. For the sake of simplicity, let $\xi_i = v_i^{\alpha_1} - v_i^{*\alpha_1}$ $(i \in \mathcal{I}_m)$ and $\alpha_2 = 1 + 1/\alpha_1$. Due to $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$, we have

$$\begin{aligned} \dot{V}_0(t) &= \sum_{i=1}^n \sum_{j=1}^n \omega_i a_{ij} \frac{(x_j - x_i)(\dot{x}_j - \dot{x}_i)}{2} \\ &= -\sum_{i=1}^n \sum_{j=1}^n \omega_i a_{ij}(x_j - x_i)\dot{x}_i = -\sum_{i=1}^n \omega_i \dot{x}_i y_i \\ &= -k_2 \sum_{i=1}^n \omega_i y_i^{\alpha_2} - \sum_{i=1}^m \omega_i y_i (v_i - v_i^*) \\ &\leq -k_2 \sum_{i=1}^n \omega_i y_i^{\alpha_2} + \sum_{i=1}^m \omega_i |y_i|\,|v_i - v_i^*|. \end{aligned}$$

By Lemmas 1 and 2, we have

$$\begin{aligned} \dot{V}_0(t) &\leq -k_2 \sum_{i=1}^n \omega_i y_i^{\alpha_2} + \sum_{i=1}^m \omega_i 2^{1-1/\alpha_1}|y_i|\,|v_i^{\alpha_1} - v_i^{*\alpha_1}|^{1/\alpha_1} \\ &\leq -k_2 \sum_{i=1}^n \omega_i y_i^{\alpha_2} \\ &\quad + \sum_{i=1}^m \omega_i 2^{1-1/\alpha_1}\left(\frac{\alpha_1|y_i|^{\alpha_2}}{1+\alpha_1} + \frac{|\xi_i|^{\alpha_2}}{1+\alpha_1}\right). \end{aligned} \tag{5}$$

Due to the fact that $1 < \alpha_1 < 2$ is a ratio of odd integers, $\frac{2}{\alpha_1} - 1$ is also a ratio of odd integers. Thus, it can be shown that

$$\begin{aligned} &\frac{d}{dt}\int_{v_i^*}^{v_i}(s^{\alpha_1} - v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}} ds \\ &= (v_i^{\alpha_1} - v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}}\frac{dv_i}{dt} \\ &\quad - \left(2 - \frac{1}{\alpha_1}\right)\frac{dv_i^{*\alpha_1}}{dt}\int_{v_i^*}^{v_i}(s^{\alpha_1} - v_i^{*\alpha_1})^{1-\frac{1}{\alpha_1}} ds \\ &\leq \xi_i^{2-1/\alpha_1}(-k_1\xi_i^{2/\alpha_1-1}) - (2 - 1/\alpha_1)k_2^{\alpha_1}\sum_{j=1}^n a_{ij} \\ &\quad \times (\dot{x}_j - \dot{x}_i)\int_{v_i^*}^{v_i}(s^{\alpha_1} - v_i^{*\alpha_1})^{1-\frac{1}{\alpha_1}} ds. \end{aligned} \tag{6}$$

Note that by Lemma 1, we have

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}(\dot{x}_j-\dot{x}_i) &\le \left|\sum_{j=1}^{n} a_{ij}(\dot{x}_j-\dot{x}_i)\right| \\ &\le \sum_{j=1}^{m} a_{ij}\left|\dot{x}_j-\dot{x}_i\right| + \sum_{j=m+1}^{n} a_{ij}\left|\dot{x}_j-\dot{x}_i\right| \\ &\le d_{ii}\left|v_i\right| + \sum_{j=1}^{m} a_{ij}\left|v_j\right| + \sum_{j=m+1}^{n} k_2 a_{ij}\left|y_j\right|^{1/\alpha_1} \\ &\le d_{ii}\left(\left|v_i-v_i^*\right|+\left|v_i^*\right|\right) + \sum_{j=1}^{m} a_{ij} \\ &\quad\times\left(\left|v_j-v_j^*\right|+\left|v_j^*\right|\right) + \sum_{j=m+1}^{n} k_2 a_{ij}\left|y_j\right|^{1/\alpha_1} \\ &\le d_{ii}\left(2^{1-1/\alpha_1}\left|\xi_i\right|^{1/\alpha_1} + k_2\left|y_i\right|^{1/\alpha_1}\right) + 2^{1-1/\alpha_1} \\ &\quad\times\sum_{j=1}^{m} a_{ij}\left|\xi_j\right|^{1/\alpha_1} + \sum_{j=1}^{n} k_2 a_{ij}\left|y_j\right|^{1/\alpha_1}\end{aligned}$$

and $\int_{v_i^*}^{v_i}(s^{\alpha_1}-v_i^{*\alpha_1})^{1-\frac{1}{\alpha_1}}ds \le \left|\int_{v_i^*}^{v_i}(s^{\alpha_1}-v_i^{*\alpha_1})^{1-\frac{1}{\alpha_1}}ds\right| \le |v_i - v_i^*|\,|\xi_i|^{1-1/\alpha_1} \le 2^{1-1/\alpha_1}|\xi_i|$ for all $i \in \mathcal{I}_m$. Using Lemma 2, we have $|\xi_i|\,|y_i|^{1/\alpha_1} \le \frac{|y_i|^{\alpha_2}}{1+\alpha_1} + \frac{\alpha_1|\xi_i|^{\alpha_2}}{1+\alpha_1}$, $|\xi_i|\,|\xi_j|^{1/\alpha_1} \le \frac{|\xi_j|^{\alpha_2}}{1+\alpha_1} + \frac{\alpha_1|\xi_i|^{\alpha_2}}{1+\alpha_1}$ and $|\xi_i|\,|y_j|^{1/\alpha_1} \le \frac{|y_j|^{\alpha_2}}{1+\alpha_1} + \frac{\alpha_1|\xi_i|^{\alpha_2}}{1+\alpha_1}$. Substituting aforementioned inequalities into (6) yields

$$\begin{aligned}&\frac{d}{dt}\int_{v_i^*}^{v_i}(s^{\alpha_1}-v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}}ds \\ &\le -k_1\xi_i^{\alpha_2} + (2-1/\alpha_1)k_2^{\alpha_1}\left|\sum_{j=1}^{n} a_{ij}(\dot{x}_j-\dot{x}_i)\right| \\ &\quad\times\left|\int_{v_i^*}^{v_i}(s^{\alpha_1}-v_i^{*\alpha_1})^{1-\frac{1}{\alpha_1}}ds\right| \\ &\le -k_1\xi_i^{\alpha_2} + k_3\Bigg(2^{1-1/\alpha_1}d_{ii}\left|\xi_i\right|^{\alpha_2} + d_{ii}k_2|\xi_i|\,|y_i|^{1/\alpha_1} \\ &\quad + 2^{1-1/\alpha_1}\sum_{j=1}^{m} a_{ij}|\xi_i|\left|\xi_j\right|^{1/\alpha_1} + \sum_{j=1}^{n} k_2 a_{ij}|\xi_i|\left|y_j\right|^{1/\alpha_1}\Bigg) \\ &\le -k_1\xi_i^{\alpha_2} + k_3k_4|y_i|^{\alpha_2} + k_3k_5|\xi_i|^{\alpha_2} \qquad (7)\end{aligned}$$

where $k_3 = (2-1/\alpha_1)2^{1-1/\alpha_1}k_2^{\alpha_1}$, $k_4 = \frac{k_2(D+nA)}{1+\alpha_1}$ and $k_5 = 2^{1-1/\alpha_1}D + \frac{2k_2D\alpha_1}{1+\alpha_1} + \frac{2^{1-1/\alpha_1}d\alpha_1}{1+\alpha_1} + \frac{2^{1-1/\alpha_1}ma}{1+\alpha_1}$. Because $1<\alpha_1<2$ is a ratio of odd integers, it is not difficult to show that $\xi_i^{\alpha_2} = |\xi_i|^{\alpha_2}$ and $y_i^{\alpha_2} = |y_i|^{\alpha_2}$.

Combining (5) with (7), we have

$$\begin{aligned}\dot{V}_1(t) &\le -k_2\sum_{i=1}^{n}\omega_i y_i^{\alpha_2} + \sum_{i=1}^{m}\omega_i 2^{1-1/\alpha_1}\left(\frac{\alpha_1|y_i|^{\alpha_2}}{1+\alpha_1} + \frac{|\xi_i|^{\alpha_2}}{1+\alpha_1}\right) \\ &\quad + \sum_{i=1}^{m}\frac{1}{k_2k_3}\left(-k_1\xi_i^{\alpha_2} + k_3k_4|y_i|^{\alpha_2} + k_3k_5|\xi_i|^{\alpha_2}\right) \\ &\le -\sum_{i=1}^{m}\left(k_2\omega_i - \frac{(D+nA)}{1+\alpha_1} - \frac{2^{1-1/\alpha_1}\omega_i\alpha_1}{1+\alpha_1}\right)|y_i|^{\alpha_2} \\ &\quad - \sum_{i=m+1}^{n} k_2\omega_i\left|y_i\right|^{\alpha_2} - \sum_{i=1}^{m}\left(\frac{k_1}{k_2k_3} - \frac{k_5}{k_2} - \frac{2^{1-1/\alpha_1}\omega_i}{1+\alpha_1}\right)|\xi_i|^{\alpha_2}.\end{aligned}$$

Due to $k_2 > \frac{(D+nA)}{(1+\alpha_1)w} + \frac{2^{1-1/\alpha_1}\alpha_1}{1+\alpha_1}$ and $k_1 > (2-1/\alpha_1)2^{1-1/\alpha_1}k_2^{\alpha_1}\left(\frac{2k_2D\alpha_1+2^{1-1/\alpha_1}(D+D\alpha_1+d\alpha_1+ma+k_2W)}{1+\alpha_1}\right)$, we have $k_2\omega_i - \frac{(D+nA)}{1+\alpha_1} - \frac{2^{1-1/\alpha_1}\omega_i\alpha_1}{1+\alpha_1} > 0$, $k_2\omega_i > 0$ and $\frac{k_1}{k_2k_3} - \frac{k_5}{k_2} - \frac{2^{1-1/\alpha_1}\omega_i}{1+\alpha_1} > 0$ for all $i \in \mathcal{I}_n$, which implies that $\dot{V}_1(t) < 0$. Thus, we have $\lim_{t\to\infty}(x_i(t)-x_j(t)) = 0$ $(i,j \in \mathcal{I}_n)$ and $\lim_{t\to\infty}(v_i(t)-v_j(t)) = 0$ $(i,j \in \mathcal{I}_m)$.

Next, note that the system (1)–(2) under protocol (4) is a homogeneous system of degree $\sigma = 1-\alpha_1 < 0$ with dilation $(\underbrace{\alpha_1,\ldots,\alpha_1}_{n},\underbrace{1,\ldots,1}_{m})$. Therefore, the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (4) by Lemma 4. □

Remark 2. In fact, we have $\dot{V}_1(t) \le -\mathcal{K}_1\sum_{i=1}^{n}|y_i|^{\alpha_2} - \mathcal{K}_1\sum_{i=1}^{m}|\xi_i|^{\alpha_2}$, where $\mathcal{K}_1 = \max_{i\in\mathcal{I}_n}\left\{k_2\omega_i - \frac{(D+nA)}{1+\alpha_1} - \frac{2^{1-1/\alpha_1}\omega_i\alpha_1}{1+\alpha_1}, k_2\omega_i, \frac{k_1}{k_2k_3} - \frac{k_5}{k_2} - \frac{2^{1-1/\alpha_1}\omega_i}{1+\alpha_1}\right\} > 0$. Similar to the analysis of Ref. [18], we get $V_1^{\alpha_2/2}(t) \le \mathcal{K}_2^{\alpha_2/2}\left(\sum_{i=1}^{n}|y_i|^{\alpha_2} + \sum_{i=1}^{m}|\xi_i|^{\alpha_2}\right)$, where $\mathcal{K}_2 = \max\left\{\frac{W^2}{2\lambda_2}, \frac{1}{(2-1/\alpha_1)k_2^{\alpha_1+1}}\right\}$, λ_2 is the second smallest eigenvalue of matrix $\mathrm{diag}\{\omega_1,\omega_2,\ldots,\omega_n\}\cdot\mathscr{L}$. Thus, $\dot{V}_1(t) + \frac{\mathcal{K}_1}{2\mathcal{K}_2^{\alpha_2/2}}V_1^{\alpha_2/2}(t) \le 0$. From Theorem 4.2 in Ref. [11], the inequality leads to the settling-time estimation given by $T \le \frac{2\mathcal{K}_2^{\alpha_2/2}}{\mathcal{K}_1(1-\alpha_2/2)}V_1^{1-\alpha_2/2}(0)$.

Remark 3. Note that the heterogeneous multi-agent system (1)–(2) is a second-order multi-agent system when $m = n$, and the second-order multi-agent system (1) with protocol (4) has been studied under an undirected connected graph in [18]. Moreover, the system (1)–(2) is a first-order multi-agent system when $m = 0$. Owing to $\mathrm{sig}(x)^{\frac{1}{\alpha_1}} = (x)^{\frac{1}{\alpha_1}}$ when $1<\alpha_1<2$ is a ratio of odd integers, the first-order multi-agent system (2) with protocol (4) is a special case of the multi-agent system in [13]. Therefore, the heterogeneous multi-agent system (1)–(2) with protocol (4) presents a unified viewpoint to solve the finite-time consensus problem for both the first-order multi-agent system and the second-order multi-agent system.

Theorem 2. *Suppose that the heterogeneous multi-agent system (1)–(2) has a leader with first-order dynamics (labeled as n) and $n-1$ followers (labeled as $1,\ldots,n-1$), and the network among the followers is strongly connected and satisfies the detailed balance condition. Then the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (4).*

Proof. The network among the followers is strongly connected and satisfies the detailed balance condition, i.e., there exists a vector $\omega = [\omega_1,\omega_2,\ldots,\omega_{n-1}]^T \in \mathbb{R}_{>0}^{n-1}$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i,j \in \mathcal{I}_{n-1}$. We rewrite the protocol (4) as follows:

$$u_i(t) = \begin{cases} k_1\left[k_2^{\alpha_1}\left(\sum_{j=1}^{n-1}a_{ij}(x_j-x_i) + a_{in}(x_n-x_i)\right) - v_i^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1}, \\ \quad i \in \mathcal{I}_m, \\ k_2\left(\sum_{j=1}^{n-1}a_{ij}(x_j-x_i) + a_{in}(x_n-x_i)\right)^{\frac{1}{\alpha_1}}, \\ \quad i \in \mathcal{I}_{n-1}/\mathcal{I}_m, \\ 0, \quad i = n. \end{cases} \qquad (8)$$

Let $\bar{x}_i(t) = x_i(t) - x_n(t),\ i \in \mathcal{I}_{n-1}$. Without loss of generality, we assume $(n-m) > 1$. Then, we have

$$\begin{cases} \dot{\bar{x}}_i(t) = v_i(t), \quad i \in \mathcal{I}_m, \\ v_i(t) = k_1\left[k_2^{\alpha_1}\left(\sum_{j=1}^{n-1} a_{ij}(\bar{x}_j - \bar{x}_i) - a_{in}\bar{x}_i\right) - v_i^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1}, \\ \quad i \in \mathcal{I}_m, \\ \dot{\bar{x}}_i(t) = k_2\left(\sum_{j=1}^{n-1} a_{ij}(\bar{x}_j - \bar{x}_i) - a_{in}\bar{x}_i\right)^{\frac{1}{\alpha_1}}, \quad i \in \mathcal{I}_{n-1}/\mathcal{I}_m. \end{cases} \tag{9}$$

Let $\bar{y}_i = \sum_{j=1}^{n-1} a_{ij}(\bar{x}_j - \bar{x}_i) - a_{in}\bar{x}_i,\ i \in \mathcal{I}_{n-1}$. Take a Lyapunov function for (9) as

$$V_3(t) = V_2(t) + \sum_{i=1}^{m} \frac{1}{(2-1/\alpha_1)2^{1-1/\alpha_1}k_2^{\alpha_1+1}} \int_{\bar{v}_i^*}^{v_i} \times (s^{\alpha_1} - \bar{v}_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}}\, ds,$$

where

$$V_2(t) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \omega_i a_{ij} \frac{(\bar{x}_j - \bar{x}_i)^2}{4} + \sum_{i=1}^{n-1} \omega_i a_{in} \frac{\bar{x}_i^2}{2}$$

and $\bar{v}_i^* = k_2 \bar{y}_i^{1/\alpha_1}$ $(i \in \mathcal{I}_m)$, $\int_{\bar{v}_i^*}^{v_i} (s^{\alpha_1} - \bar{v}_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}} ds$ are positive definite and proper.

Similar to the analysis of Theorem 1, we obtain $\dot{V}_3(t) < 0$. Therefore, the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (4) by Lemma 4. □

Remark 4. If the leader is a second-order dynamics agent, i.e., the leader's dynamics is given as follows:

$$\begin{cases} \dot{x}^d(t) = v^d(t), \\ \dot{v}^d(t) = u^d(t). \end{cases}$$

Then, the protocol (4) is replaced by

$$u_i = \begin{cases} k_1\left[k_2^{\alpha_1}\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i)\right) - \left(v_i - v^d\right)^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1} + u^d, \\ \quad i \in \mathcal{I}_m, \\ k_2\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i)\right)^{\frac{1}{\alpha_1}} + v^d, \quad i \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{10}$$

Let $\bar{x}_i(t) = x_i(t) - x^d(t)$ $(i \in \mathcal{I}_n)$ and $\bar{v}_i(t) = v_i(t) - v^d(t)$ $(i \in \mathcal{I}_m)$. The heterogeneous multi-agent system with protocol (10) can be rewritten as follows:

$$\begin{cases} \dot{\bar{x}}_i(t) = \bar{v}_i(t), \quad i \in \mathcal{I}_m, \\ \bar{v}_i(t) = k_1\left[k_2^{\alpha_1}\left(\sum_{j=1}^{n-1} a_{ij}(\bar{x}_j - \bar{x}_i)\right) - \bar{v}_i^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1}, \quad i \in \mathcal{I}_m, \\ \dot{\bar{x}}_i(t) = k_2\left(\sum_{j=1}^{n-1} a_{ij}(\bar{x}_j - \bar{x}_i)\right)^{\frac{1}{\alpha_1}}, \quad i \in \mathcal{I}_{n-1}/\mathcal{I}_m. \end{cases} \tag{11}$$

From the results of Theorem 1, we know that the heterogeneous multi-agent system with protocol (10) reaches consensus in finite time.

4. Finite-time consensus protocol without velocity information

In this section, first, we propose the protocol without velocity information for the heterogeneous multi-agent system (1)–(2) as follows:

$$u_i = \begin{cases} k_1\left[k_2^{\alpha_1}\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i)\right) + \dot{\hat{v}}_i^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1}, & i \in \mathcal{I}_m, \\ k_2\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i)\right)^{\frac{1}{\alpha_1}}, & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{12}$$

where

$$\dot{\hat{v}}_i = -\left[k_6\left(x_i + \hat{v}_i\right)\right]^{\frac{1}{\alpha_1}}, \quad i \in \mathcal{I}_m, \tag{13}$$

and $\mathcal{A} = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix associated with the graph $\mathcal{G}$, $1 < \alpha_1 < 2$ is a ratio of odd integers, $k_2 > \frac{1}{1-c}\left(\frac{(D+nA)}{(1+\alpha_1)w} + \frac{2^{1-1/\alpha_1}\alpha_1}{1+\alpha_1} + \frac{1}{w}\right)$, $k_1 > \frac{1}{1-2c}(2-1/\alpha_1)2^{1-1/\alpha_1}(k_2^{\alpha_1} \cdot \frac{2k_2 D\alpha_1 + 2^{1-1/\alpha_1}(D+D\alpha_1+d\alpha_1+ma+k_2W)}{1+\alpha_1} + \frac{2^{1-1/\alpha_1}}{\alpha_2} + 2^{2\alpha_1-1/\alpha_1})$ are the feedback gains, $0 < c < \frac{1}{2}$, $A = \max_{i,j\in\mathcal{I}_n}\{a_{ij}\}$, $a = \max_{i,j\in\mathcal{I}_m}\{a_{ij}\}$, $D = \max_{i\in\mathcal{I}_n}\{d_{ii}\}$, $d = \max_{i\in\mathcal{I}_m}\{d_{ii}\}$, $W = \max_{i\in\mathcal{I}_m}\{\omega_i\}$ and $w = \min_{i\in\mathcal{I}_m}\{\omega_i\}$, k_6 is a large enough constant, $\hat{v}_i(0) = \hat{v}_{i0}$ for $i \in \mathcal{I}_m$. Then, we show that the heterogeneous multi-agent system (1)–(2) with protocol (12)–(13) reaches consensus in finite time under two classes of directed networks.

Theorem 3. *Suppose the graph $\mathcal{G}(\mathcal{A})$ is strongly connected and satisfies the detailed balance condition. Then the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (12)–(13).*

Proof. The graph $\mathcal{G}(\mathcal{A})$ is strongly connected and satisfies the detailed balance condition, i.e., there exists a vector $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T \in \mathbb{R}^n_{>0}$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$. Let $y_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i)$ $(i \in \mathcal{I}_n)$, $\xi_i = v_i^{\alpha_1} - v_i^{*\alpha_1}$ $(i \in \mathcal{I}_m)$ and $\alpha_2 = 1 + 1/\alpha_1$. Take a Lyapunov function as

$$V_5(t) = V_1(t) + V_4(t),$$

where $V_4(t) = \sum_{i=1}^{m} \frac{1}{k_2^{\alpha_1+1}} \frac{e_i^2}{2}$ and $e_i = v_i^{\alpha_1} + \dot{\hat{v}}_i^{\alpha_1}$. $V_5(t)$ is positive definite with respect to $x_i(t) - x_j(t)$ $(\forall i \neq j,\ i, j \in \mathcal{I}_n)$, $v_i(t)$ $(i \in \mathcal{I}_m)$ and $e_i(t)$ $(i \in \mathcal{I}_m)$.

Similar to the analysis and using the identical symbol of Theorem 1, we have

$$\dot{V}_0(t) \le -k_2 \sum_{i=1}^{n} \omega_i y_i^{\alpha_2} + \sum_{i=1}^{m} \omega_i 2^{1-1/\alpha_1}\left(\frac{\alpha_1|y_i|^{\alpha_2}}{1+\alpha_1} + \frac{|\xi_i|^{\alpha_2}}{1+\alpha_1}\right), \tag{14}$$

and

$$\frac{d}{dt}\left(\int_{v_i^*}^{v_i} (s^{\alpha_1} - v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}} ds\right) \le (\xi_i)^{2-\frac{1}{\alpha_1}} \frac{dv_i}{dt} + k_3 k_4 |y_i|^{\alpha_2} + k_3 k_5 |\xi_i|^{\alpha_2}. \tag{15}$$

Next, we use Lemmas 1 and 2 to estimate $(\xi_i)^{2-\frac{1}{\alpha_1}} \frac{dv_i}{dt}$. For $i \in \mathcal{I}_m$, we have

$$(\xi_i)^{2-\frac{1}{\alpha_1}} \frac{dv_i}{dt} = (\xi_i)^{2-\frac{1}{\alpha_1}} u_i = k_1(\xi_i)^{2-\frac{1}{\alpha_1}}\left[k_2^{\alpha_1}\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i)\right) + \dot{\hat{v}}_i^{\alpha_1}\right]^{\frac{2}{\alpha_1}-1}$$

$$
\begin{aligned}
&= k_1(\xi_i)^{2-\frac{1}{\alpha_1}}\,(e_i-\xi_i)^{\frac{2}{\alpha_1}-1}\\
&= -k_1\xi_i^{\alpha_2} + k_1(\xi_i)^{2-\frac{1}{\alpha_1}}\left((\xi_i)^{\frac{2}{\alpha_1}-1} + (e_i-\xi_i)^{\frac{2}{\alpha_1}-1}\right)\\
&\le -k_1\xi_i^{\alpha_2} + k_1|\xi_i|^{2-\frac{1}{\alpha_1}}|e_i|^{\frac{2}{\alpha_1}-1}2^{2-\frac{2}{\alpha_1}}\\
&\le -k_1\xi_i^{\alpha_2} + ck_1|\xi_i|^{\alpha_2} + k_7|e_i|^{\alpha_2}\\
&= -(1-c)k_1|\xi_i|^{\alpha_2} + k_7|e_i|^{\alpha_2}, \qquad (16)
\end{aligned}
$$

where $0 < c < \frac{1}{2}$ and $k_7 = k_1\frac{2-\alpha_1}{\alpha_1+1}\left(\frac{2\alpha_1-1}{c(\alpha_1+1)}\right)^{(2\alpha_1-1)/(2-\alpha_1)} 2^{(2\alpha_1^2-2)/(2\alpha_1-\alpha_1^2)}$.

On the other hand, the time derivative of $\frac{e_i^2}{2}$ is

$$
\frac{d}{dt}\left(\frac{e_i^2}{2}\right) = e_i\dot{e}_i = e_i\left(\alpha_1 v_i^{\alpha_1-1}\frac{dv_i}{dt} + \frac{d\dot{\hat{v}}_i^{\alpha_1}}{dt}\right).
$$

Applying Lemmas 1 and 2, we have

$$
\begin{aligned}
e_i\left(v_i^{\alpha_1-1}\frac{dv_i}{dt}\right) &= e_i\left(v_i^{\alpha_1-1}u_i\right) \le |e_i|\left(|v_i|^{\alpha_1-1}1^{2-\alpha_1}|u_i|\right)\\
&\le |e_i|\left(|v_i| + k_8|u_i|^{1/(2-\alpha_1)}\right)\\
&\le |e_i|\left(2^{1-1/\alpha_1}|\xi_i|^{1/\alpha_1} + k_2|y_i|^{1/\alpha_1} + k_1k_8|e_i-\xi_i|^{1/\alpha_1}\right)\\
&\le 2^{1-1/\alpha_1}|e_i||\xi_i|^{1/\alpha_1} + k_2|e_i||y_i|^{1/\alpha_1}\\
&\quad + k_1k_8|e_i|^{\alpha_2} + k_1k_8|e_i||\xi_i|^{1/\alpha_1}\\
&\le \frac{ck_2^{\alpha_1+2}\omega_i}{\alpha_1}|y_i|^{\alpha_2} + \left(\frac{2^{1-1/\alpha_1}}{\alpha_1\alpha_2} + \frac{ck_1}{\alpha_1(2-1/\alpha_1)2^{1-1/\alpha_1}}\right)\\
&\quad \times |\xi_i|^{\alpha_2} + k_9|e_i|^{\alpha_2}, \qquad (17)
\end{aligned}
$$

where $k_8 = (2-\alpha_1)(\alpha_1-1)^{(\alpha_1-1)/(2-\alpha_1)}$, $k_9 = k_1k_8 + \frac{2^{1-1/\alpha_1}}{\alpha_2} + k_2\frac{1}{\alpha_2}\left(\frac{1}{ck_2^{\alpha_1+1}\omega_i\alpha_2}\right)^{1/\alpha_1} + k_1k_8^{\alpha_2}\frac{1}{\alpha_2}\left(\frac{(2-1/\alpha_1)2^{1-1/\alpha_1}}{c\alpha_2}\right)^{1/\alpha_1}$.

Then, we use Lemma 3 $\left(t = \left(\frac{v_i^{\alpha_1}}{e_i}\right), r = 1/\alpha_1, l = k_6^{\frac{-1}{2}}\right)$ to estimate $e_i\frac{d\dot{\hat{v}}_i^{\alpha_1}}{dt}$. For $i \in \mathcal{I}_m$,

$$
\begin{aligned}
e_i\frac{d\dot{\hat{v}}_i^{\alpha_1}}{dt} &= -k_6e_i\frac{d\left(x_i+\hat{v}_i\right)}{dt} = -k_6e_i\left(v_i + \left(e_i - v_i^{\alpha_1}\right)^{1/\alpha_1}\right)\\
&\le v_i^{\alpha_1+1} - \left(2^{1/\alpha_1}-1\right)k_6^{1-(\alpha_1-1)/2\alpha_1}e_i^{\alpha_2}.
\end{aligned}
$$

Due to $v_i^{\alpha_1+1} \le \left(|v_i - v_i^*| + |v_i^*|\right)^{\alpha_1+1} \le 2^{2\alpha_1-\frac{1}{\alpha_1}}|\xi_i|^{\alpha_2} + k_2^{\alpha_1+1}|y_i|^{\alpha_2}$, we get

$$
\begin{aligned}
e_i\frac{d\dot{\hat{v}}_i^{\alpha_1}}{dt} &\le 2^{2\alpha_1-\frac{1}{\alpha_1}}|\xi_i|^{\alpha_2} + k_2^{\alpha_1+1}|y_i|^{\alpha_2}\\
&\quad - \left(2^{1/\alpha_1}-1\right)k_6^{1-(\alpha_1-1)/2\alpha_1}|e_i|^{\alpha_2}. \qquad (18)
\end{aligned}
$$

Thus, differentiating $V_5(t)$ and substituting (14)–(18) for it,

$$
\begin{aligned}
\dot{V}_5(t) &= \dot{V}_1(t) + \dot{V}_4(t)\\
&= \frac{dV_0(t)}{dt} + \sum_{i=1}^{m}\frac{1}{k_2k_3}\frac{d}{dt}\left(\int_{v_i^*}^{v_i}(s^{\alpha_1} - v_i^{*\alpha_1})^{2-\frac{1}{\alpha_1}}ds\right)\\
&\quad + \sum_{i=1}^{m}\frac{1}{k_2^{\alpha_1+1}}\frac{d}{dt}\left(\frac{e_i^2}{2}\right)\\
&\le -\sum_{i=1}^{m}\left((1-c)k_2\omega_i - \frac{(D+nA)}{1+\alpha_1} - \frac{2^{1-1/\alpha_1}\omega_i\alpha_1}{1+\alpha_1} - 1\right)\\
&\quad \times |y_i|^{\alpha_2} - \sum_{i=m+1}^{n}k_2\omega_i|y_i|^{\alpha_2}\\
&\quad - \sum_{i=1}^{m}\left(\frac{(1-2c)k_1}{k_2k_3} - \frac{k_5}{k_2} - \frac{2^{1-1/\alpha_1}\omega_i}{1+\alpha_1}\right.\\
&\quad \left. - \frac{2^{1-1/\alpha_1}}{\alpha_2k_2^{\alpha_1+1}} - \frac{2^{2\alpha_1-\frac{1}{\alpha_1}}}{k_2^{\alpha_1+1}}\right)|\xi_i|^{\alpha_2}\\
&\quad - \sum_{i=1}^{m}\left(\frac{\left(2^{1/\alpha_1}-1\right)k_6^{1-(\alpha_1-1)/2\alpha_1} - k_9\alpha_1}{k_2^{\alpha_1+1}} - \frac{k_7}{k_2k_3}\right)|e_i|^{\alpha_2}.
\end{aligned}
$$

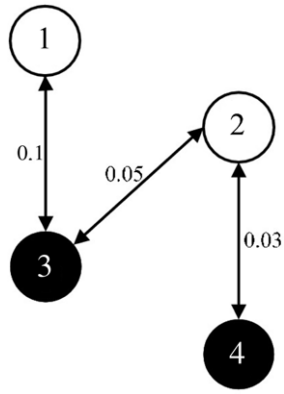

Fig. 1. A strongly connected graph $\mathcal{G}(\mathcal{A})$.

By the definition of k_1, k_2, k_6 and an easy calculation, we have $\dot{V}_5(t) < 0$. Thus, we have $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ $(i, j \in \mathcal{I}_n)$ and $\lim_{t\to\infty}(v_i(t) - v_j(t)) = 0$ $(i, j \in \mathcal{I}_m)$.

Note that the system (1)–(2) under protocol (12)–(13) with $(x_1, \ldots, x_n, \hat{v}_1, \ldots, \hat{v}_m, v_1, \ldots, v_m)$ is a homogeneous system of degree $\sigma = 1 - \alpha_1 < 0$ with dilation $(\underbrace{\alpha_1, \ldots, \alpha_1}_{n}, \underbrace{\alpha_1, \ldots, \alpha_1}_{m}, \underbrace{1, \ldots, 1}_{m})$. Therefore, the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (12)–(13) by Lemma 4. □

Similar to the analysis of Theorems 2 and 3, we can get the following result.

Theorem 4. *Suppose that the heterogeneous multi-agent system (1)–(2) has a leader with first-order dynamics (labeled as n) and $n-1$ followers (labeled as $1, \ldots, n-1$), and the network among the followers is strongly connected and satisfies the detailed balance condition. Then the heterogeneous multi-agent system (1)–(2) reaches consensus in finite time with protocol (12)–(13).* □

5. Simulations

In this section, we begin with a numerical simulation in Example 1 to illustrate the effectiveness of the theoretical result in Section 3. In Example 2, we provide an illustration of the theoretical result in Section 4.

Fig. 1 shows a strongly connected graph with weight which satisfies the detailed balance condition $\omega = \mathbf{1}_4$. Suppose that the vertices 1 and 2 denote the second-order integrator agents and the vertices 3 and 4 denote the first-order integrator agents. In Fig. 2, the agent 4 is a leader and the followers are strongly connected and satisfy the detailed balance condition. Let $x(0) = [8, 5, 1, 3]$ and $v(0) = [1, -5]$.

Example 1. Assume that $\alpha_1 = \frac{9}{7}$, $k_2 = 2$ and $k_1 = 30$. It is easy to calculate that k_1 and k_2 satisfy the conditions in Section 3. Using consensus protocol (4), the state trajectories of all the agents reach

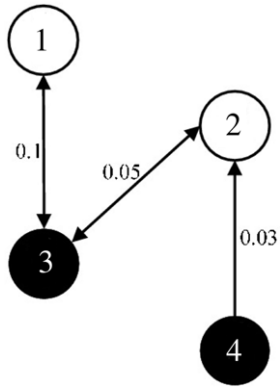

Fig. 2. A leader-following network $\mathscr{G}(\mathscr{A})$.

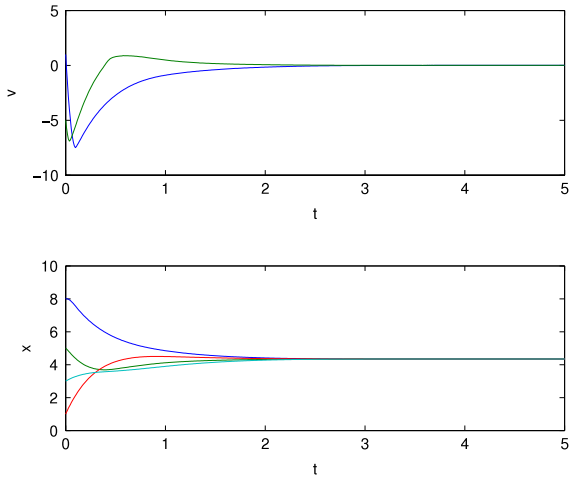

Fig. 3. Simulation results with the network depicted in Fig. 1 and consensus protocol (4).

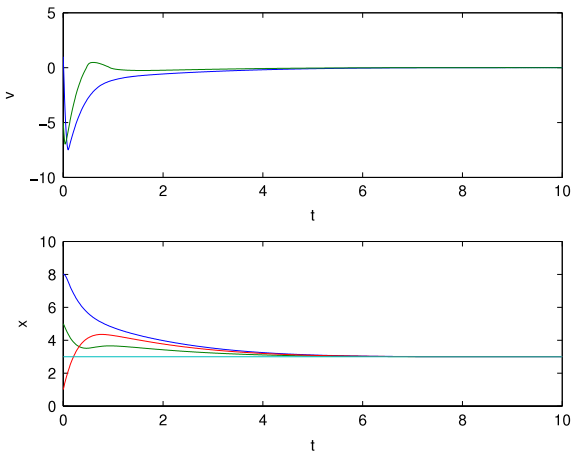

Fig. 4. Simulation results with the network depicted in Fig. 2 and consensus protocol (4).

consensus as shown in Fig. 3 with the network depicted in Fig. 1, which is consistent with the result of Theorem 1. In Fig. 4, it is the state trajectories of all the agents with the network depicted in Fig. 2, which is consistent with the result of Theorem 2.

Example 2. Assume that $\alpha_1 = \frac{9}{7}$, $\hat{v}_{i0} = 0$ $(i = 1, 2)$, $k_2 = 4$, $k_1 = 250$ and $k_3 = 5000$ which satisfy the conditions in Section 4. Using consensus protocol (12)–(13), the heterogeneous multi-agent system (1)–(2) achieves consensus as shown in Fig. 5

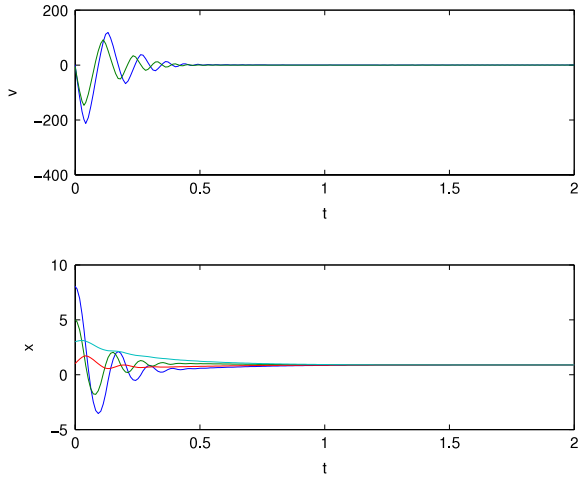

Fig. 5. Simulation results with the network depicted in Fig. 1 and consensus protocol (12)–(13).

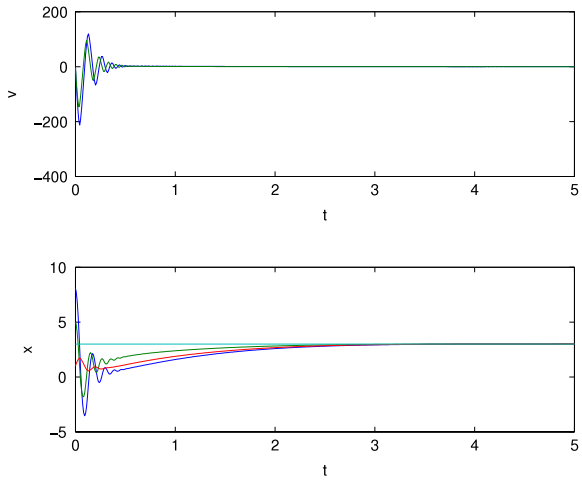

Fig. 6. Simulation results with the network depicted in Fig. 2 and consensus protocol (12)–(13).

with the network depicted in Fig. 1, which accords with the result established in Theorem 3. In Fig. 6, the heterogeneous multi-agent system (1)–(2) achieves consensus with the network depicted in Fig. 2, which accords with the result established in Theorem 4.

6. Conclusions

In this paper, the finite-time consensus problem of the heterogeneous multi-agent system with agents modeled by first-order and second-order integrators was considered. Two kinds of consensus protocols with and without velocity measurements were proposed. Based on the graph theory, the Lyapunov theory and the homogeneous domination method, we proved that the protocols can solve the finite-time consensus under two classes of special directed networks, respectively. Future work may focus on the more complex consensus problem of heterogeneous multi-agent systems, for example, heterogeneous multi-agent systems with switching topologies/heterogeneous networks etc.

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