Principles of Communications

Chapter VIII: Optimum Receiving of Digital Signal

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8.1 Statistical Characteristics of Digital Signal

Derivations of likelihood function

- During $[0,\ T]$, sample signal and noise by rate f_s. We obtain k samples, $s_1, \cdots, s_k$ and $n_1, \cdots, n_k$ where $k = f_s T$. Assume that the channel is AWGN channel. $r_i = s_i + n_i$.
- Then, the k-dims pdf

$$f(n_1, \cdots, n_k) = f(n_1) \cdots f(n_k) = \frac{1}{(\sqrt{2\pi}\sigma_n)^k} \exp(-\frac{1}{2\sigma_n^2} \sum_{i=1}^{k} n_i^2).$$

- For discrete signal n_i, its power $\sigma_n^2 = \frac{1}{2} n_0 f_s$, where n_0 is the power spectral density of the noise.
- When increasing k,

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{k} n_i^2 = \frac{1}{T} \int_0^T n^2(t) dt \to \lim_{k \to +\infty} \sum_{i=1}^{k} n_i^2 = f_s \int_0^T n^2(t) dt.$$

- Then we obtain that

$$f(\mathbf{n}) = \frac{1}{(\sqrt{2\pi}\sigma_n)^k}\exp\Big(- \frac{1}{n_0}\int_0^T n^2(t)dt\Big).$$

- In the receiver side, the received signal is $r_i(t) = s_i(t) + n(t)$, $i = 1, \cdots, m$.
- For the fixed $s_i(t)$, we have the conditional probability function

$$f_{s_i}(r) = \frac{1}{(\sqrt{2\pi}\sigma_n)^k}\exp\Big(- \frac{1}{n_0}\int_0^T [r(t) - s_i(t)]^2 dt\Big)$$

, where $i = 1, \cdots, m$ $f_{s_i}(\mathbf{r}$ is named likelihood function.

8.2 Optimum Receiving Criterion

- In the procedure of transmission, while $s_i(t)$ is transmitted, the receiver in some cases cannot make the decision of $s_i(t)$ because of the distortion and interference.
- In the digital communication system, the optimum receiving criterion is the minimum of error symbol rate.
- This criterion is equivalent to maximize a posteriori (MAP), for binary case, i.e.,

$$\begin{aligned} P(s_1(t)|r) \geq P(s_2|r), \quad \text{is } s_1(t), \\ P(s_1(t)|r) < P(s_2|r), \quad \text{is } s_2(t) \end{aligned}$$

- For the transmitted signals $s_1(t)$ or $s_2(t)$ and their likelihood functions, there are

$$P(s_i|r) \propto f_{s_i}(r) = \frac{1}{(\sqrt{2\pi}\sigma_n)^k} \exp\Big(-\frac{1}{n_0}\int_0^T [r(t)-s_i(t)]^2 dt\Big)$$

- According to Bayesian equation

$$P(s_i|r) = \frac{P(r|s_i)P(s_i)}{P(r)}, \quad i = 1,\ 2.$$

So we have

$$P(s_i|r) \propto f_{s_i}(r)P(s_i(t)).$$

- Map rule is reduced to

$$f_{s_1}(r)P(s_1) \geq f_{s_2}(r)P(s_2) - -s_1(t),$$
$$f_{s_1}(r)P(s_1) \leq f_{s_2}(r)P(s_2) - -s_2(t).$$

- Specially, if $P(s_1) = P(s_2)$, the above decision rule can be simplified according to the values of the likelihood functions, $f_{s_1}(r)$ or $f_{s_2}(r)$. (Maximum likelihood rule — ML)
- For M-ary case, we can derive the similar rules.

8.3 Optimum Receiver for Deterministic Digital Signal

In $[0, T]$, assume the transmitted signals $s_1(t)$ and $s_2(t)$, $E_{s_1} = E_{s_2} = E_b$, and the received signal is

$$r(t) = \begin{cases} s_1(t) + n(t), & s_1(t) \text{ is transmitted} \\ s_2(t) + n(t), & s_2(t) \text{ is transmitted} \end{cases}.$$

According to the MAP criterion,

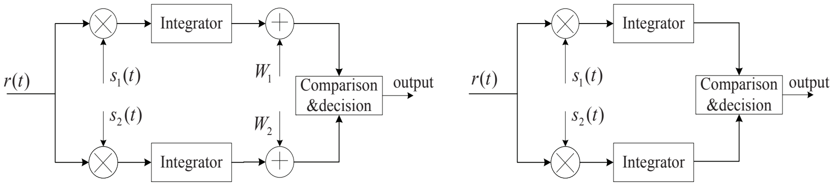
$$\frac{P(s_1) f_{s_1}(r)}{P(s_2) f_{s_2}(r)} = \frac{P(s_1)\frac{1}{(\sqrt{2\pi}\sigma_n)^k}\exp\Big(-\frac{1}{n_0}\int_0^T [r(t) - s_1(t)]^2 dt\Big)}{P(s_2)\frac{1}{(\sqrt{2\pi}\sigma_n)^k}\exp\Big(-\frac{1}{n_0}\int_0^T [r(t) - s_2(t)]^2 dt\Big)} \gtrless 1.$$

While “$>$”, obtain $s_1(t)$. Otherwise, obtain $s_2(t)$.

Let $W_1 = \frac{n_0}{2}\ln P(s_1(t))$ and $W_2 = \frac{n_0}{2}\ln P(s_2(t))$, MAP criterion can be reduced to

$$W_1 + \int_0^T r(t)s_1(t)dt \lessgtr W_2 + \int_0^T r(t)s_2(t)dt.$$

While ">", obtain $s_1(t)$. Otherwise, obtain $s_2(t)$. So we can obtain the following block diagram of the optimum receiver (a). Further, if $P(s_1) = P(s_2)$, the block diagram (a) can be simplified to the (b).



(a) MAP Receiver (b) ML Receiver

The symbol error probability is $P_e = P(s_1)P_{s_1}(s_2) + P(s_2)P_{s_2}(s_1)$. If $P(s_1) = P(s_2)$, we can verify that $P_e = \frac{1}{2}\mathrm{erfc}(\sqrt{(\frac{E_b(1-\rho)}{2n_0}})})$, where E_b is the average energy of $s_1(t)$ and $s_2(t)$, and $\rho = \frac{\int_0^T s_1(t)s_2(t)dt}{E_b}$.

Discussions:

1. The symbol error probability is related to the correlation coefficient and the signal-to-noise ratio $\frac{E_b}{n_0}$.
2. Correlation coefficient ρ

	correlation coefficient	P_e
2PSK	-1	$P_e = \frac{1}{2}\mathrm{erfc}(\sqrt{\frac{E_b}{n_0}})$
2FSK	0	$P_e = \frac{1}{2}\mathrm{erfc}(\sqrt{\frac{E_b}{2n_0}})$
2ASK	0	$P_e = \frac{1}{2}\mathrm{erfc}(\sqrt{\frac{E_b}{4n_0}})$

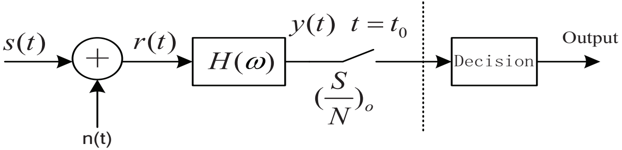
3. $\frac{S}{N}$ and $\frac{E_b}{n_0}$: $\frac{S}{N} = \frac{S}{n_0 B} = \frac{E_b}{n_0} \cdot \frac{1}{BT} = \frac{E_b}{n_0} \cdot \frac{R_b}{B}$. For common receivers in chapter 6, either 2ASK or 2PSK, $B = 2R_b$. So the optimum receiver has 3dB advantage over common receiver.

- 8.5 Optimum Receiving of Random Phase Digital Signal (不讲)
- 8.6 Optimum Receiving of Fluctuation Digital Signal (不讲)
- 8.7 Performance Comparison of Practical Receiver and Optimum Receiver

	P_e of common receiver	P_e of optimum receiver
2PSK	$P_e = \frac{1}{2}\text{erfc}(\sqrt{r})$	$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{n_0}})$
2FSK	$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{r}{2}})$	$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{2n_0}})$
2ASK	$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{r}{4}})$	$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{4n_0}})$

8.8 Matched Filtering Receiving Principle of Digital Signal

8.8.1 Matched Filtering Receiving of Digital Signal



- Aim: Design the optimal $H(\omega)$ to maximize the output SNR S_o/N_o.
- Input signal: $r(t) = s(t) + n(t)$.
- Output signal: $y(t) = s_o(t) + n_o(t)$.

- At time t_0, the output SNR $S_o/N_o = \frac{|s_o(t_0)|^2}{N_0}$ of $H(\omega)$

$$s_o(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_o(\omega)e^{j\omega t_0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega$$

$$N_o = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{n_o}(\omega)d\omega = \frac{n_0}{4\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

$$r_o = \frac{S_o}{N_o} = \frac{|s_o(t_0)|^2}{N_0} = \frac{|\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)H(\omega)e^{j\omega t_0}d\omega|^2}{\frac{n_0}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega}$$

$$\leq \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|S(\omega)e^{j\omega t_0}|^2 d\omega \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega}{\frac{n_0}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} = \frac{2E}{n_o} \quad (*)$$

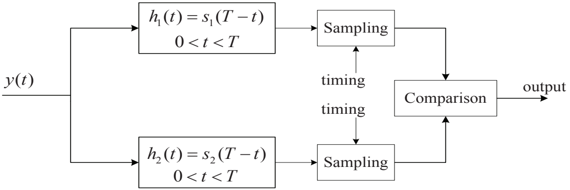
- When $H(\omega) = \left(KS(\omega)e^{j\omega t_0}\right)^*$, the inequality (*) "=" is reached. In this case,

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} KS^*(\omega)e^{j\omega(t-t_0)}d\omega = Ks(t_0 - t)$$

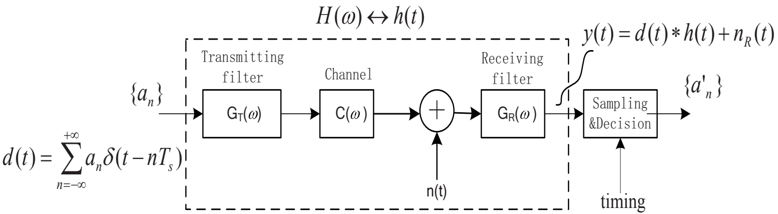
- The above fact shows that the maximum SNR at time t_0 can be obtained if we set $h(t) = s(t_0 - t)$. (Question: How to choose t_0?) $t_0 = T, 0$.
- Output of MF:

$$s_0(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(t-\tau)h(\tau)d\tau$$
$$= \int_{-\infty}^{\infty} s(t-\tau)Ks(T-\tau)d\tau = KR(t-T).$$

- Conclusion: MF is to perform correlation and its impulse response is $Ks(T-t)$. At the end of symbol, MF can obtain the maximum SNR $r_o = \frac{2E}{n_0}$.



8.9 Optimum Baseband Transmission System



- System characteristics: $H(\omega) = G_T(\omega)C(\omega)G_R(\omega)$.
- Demands for $H(\omega)$
  1. No inter-symbol interference: $\frac{1}{T}\sum_i H(\omega + \frac{2\pi i}{T})$, where $|\omega| \leq \frac{\pi}{T}$.
  2. Maximum SNR can be achieved: match filtering.

Assume $C(\omega) = 1$. We have $H(\omega) = G_T(\omega)G_R(\omega)$. To realize match filtering or realize maximum SNR, we desire

$$G_R(\omega) = G_T^*(\omega)e^{-j\omega t_0}$$

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Symbol Error Rate

Consider the baseband signal is an multi-level signal, e.g., the symbol has $L = 2M$ different levels: $\pm d, \pm 3d, \cdots, \pm(2M-1)d$ and assume that

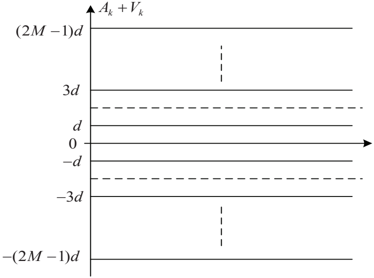
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|G_T(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G_R(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|d\omega = 1.$$

At the sampling instant:

$$r(kT_s) = A_k + n(kT_s) = A_k + V_k.$$

In the following three cases, there are error symbols if

1. $A_k = \pm d, \cdots, \pm(M-1)d$ and $|V_k| > d$.
2. $A_k = (2M-1)d$ and $V_k < -d$.
3. $A_k = -(2M-1)d$ and $V_k > d$.



So the symbol error rate is

$$P_e = \frac{1}{2M}[(2M-2)P(|V_k| > d) + P(V_k < -d) + P(V_k > d)]$$
$$= (1 - \frac{1}{2M})P(|V_k| > d).$$

$$P(|V_k| > d) = 2P(V_k > d) = \frac{2}{\sqrt{2\pi}\sigma_n}\int_d^{\infty} e^{-\frac{V_k^2}{2\sigma_n^2}} dV$$
$$= \frac{2}{\sqrt{\pi}}\int_{\frac{d}{\sqrt{2}\sigma_n}}^{\infty} e^{-z^2} dz = \mathrm{erfc}(\frac{d}{\sqrt{2}\sigma_n}).$$

So

$$P_e = (1 - \frac{1}{2M})\mathrm{erfc}(\frac{d}{\sqrt{2}\sigma_n}) = (1 - \frac{1}{L})\mathrm{erfc}(\frac{d}{\sqrt{2}\sigma_n}).$$

- Symbol energy:

$$E = E(A_k^2) = \frac{2d^2}{M}\sum_{i=1}^{M}(2i-1)^2 = \frac{d^2}{3}(4M^2-1).$$

Then we have $d^2 = \frac{3E}{L^2-1}$. $(L = 2M)$.

- Noise power:

$$\sigma_n^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{n_0}{2}|G_R(\omega)|^2 d\omega = \frac{n_0}{2}.$$

- The symbol error rate:

$$P_e = (1-\frac{1}{L})\mathrm{erfc}(\sqrt{\frac{3E}{(L^2-1)n_0}}).$$

$$A = \sum_{i=1}^{M}(2i)^2 = 4*\frac{1}{6}M(M+1)(2M+1)$$

$$B = \sum_{i=1}^{2M} i^2 = \frac{1}{6}2M(2M+1)(4M+1) \quad B - A = \tfrac{1}{3}M(4M^2-1).$$