Principles of Communications

Chapter IV: Digitization of Analog Signal

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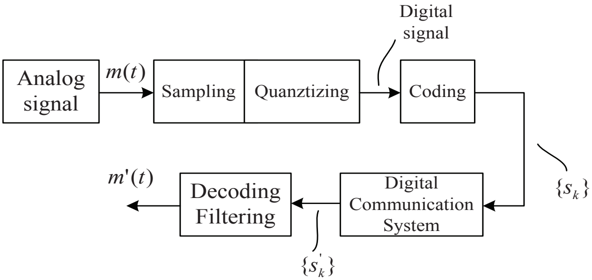
4.1 Introduction



In the nature, most signals, such as audio, image, are analog. One wants to transmit analog signal using digital techniques.

Analog to Digital

- Step 1. Sampling: from the continuous to the discrete.
- Step 2. Quantization: digital signal.
- Step 3. Code: data compression.

4.2 Sampling of Analog Signal Low-Pass Signal

4.2.1 Sampling of Low-Pass Signal

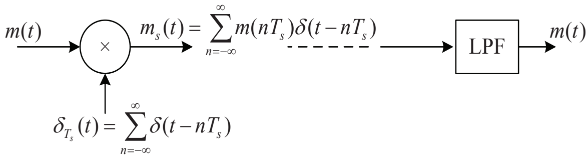
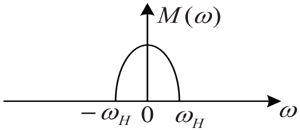
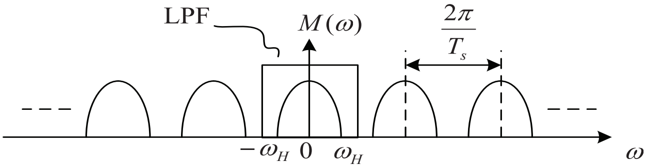


Figure: Scheme of sampling of analog signal

Low-pass sampling theorem:

If the highest frequency of a continuous analog signal $m(t)$ is less than f_H, and if its sampled by interval time $T_s \leq 1/2f_H$, then $m(t)$ can be completely decided by these samples.





Discussions

- $T_s \leq 1/2f_H \Leftrightarrow f_s \geq 2f_H$.
- $m(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\mathrm{sinc}\omega_H(t - nT_s)$. One can recover $m(t)$ utilizing $m(nT_s)$ through filtering.
- How about $f_s \leq 2f_H$? Aliasing? Can you draw a picture?

4.2.2 Sampling of Band-Pass Signal

A sufficient no-loss condition for sampling signals that do not have baseband components exists that involves the width of the non-zero frequency interval as opposed to its highest frequency component. The sufficient condition is that the frequency band of $m(t)$ is within the range

$$\left(\frac{N}{2}f_{\mathrm{s}}, \frac{N+1}{2}f_{\mathrm{s}}\right),$$

where N is some nonnegative integer. The reconstruction filter is an ideal bandpass filter with cutoffs at the upper and lower edges of the specified band, which is the difference between a pair of low-pass impulse responses

$$(N+1)f_s\ \mathrm{sinc}\left((N+1)\pi f_s t\right) - N f_s\ \mathrm{sinc}\left(N\pi f_s t\right).$$

See details in the webpage:
http://en.wikipedia.org/wiki/Sampling-theorem.

4.3 Quantization of Sampled Signal

1 Uniform Quantization

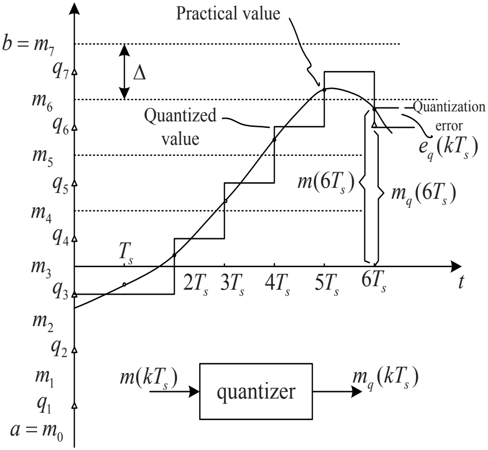


Figure: Quantization of sampled signal

Uniform quantization (quantizer parameters)

- the input max/min values are a and b, respectively.
- the number of quantization levels is M.
- the ith quantization interval is (m_{i-1}, m_i). Then quantization interval $\Delta_i = \Delta = (b - a)/M$ and the quantized output level $q_i = (m_{i-1} + m_i)/2$.

- Quantization error e_q:
  - If the sample $m(kT_s) \in [a, b]$, quantization error $e_q \leq \Delta/2$.
  - If the sample $m(kT_s) \notin [a, b]$ (overload area), quantization error e_q may be larger than $\Delta/2$.
  - When designing the quatizer, we always avoid the overload phenomenon.
- Quantization noise: Assume that $m(t)$ is stable random process with zero expect value and probability density function $f(x)$, and $m(t) \in [a, b]$. The quantization noise

$$N_q = E[|m(kT_s) - m_q(kT_s)|^2] = \sum_{i=1}^{M} \int_{m_{i-1}}^{m_i} (x - q_i)^2 f(x)dx.$$

- Quantization signal-to-noise ratio:

$$\frac{S}{N_q} = \frac{E[|m(kT_s)|^2}{E[|m(kT_s) - m_q(kT_s)|^2]}$$

.

- Example. The number of quantization levels is M, the pdf of m(t) $U[-a,a]$. Determine the $\frac{S}{N_q}$?
-

$$\begin{aligned}
N_q &= \sum_{i=1}^{M} \int_{m_{i-1}}^{m_i} (x - q_i)^2 \frac{1}{2a} dx \\
&= \sum_{i=1}^{M} \int_{-a+(i-1)\Delta}^{-a+i\Delta} (x + a - i\Delta + \frac{\Delta}{2})^2 \frac{1}{2a} dx \\
&= \sum_{i=1}^{M} \frac{1}{2a} \frac{\Delta^3}{12} \\
&= \frac{(\Delta)^2}{12} \quad (\text{since} \quad M\Delta = 2a).
\end{aligned}$$

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- $$S = E[m(kT_s)^2] = \int_{-a}^{a} x^2 \frac{1}{2a} dx = \frac{a^2}{3} = \frac{M^2}{12}\Delta^2$$
- Result: $(\frac{S}{N_q})_{\mathrm{dB}} = 20\log M = 6N$. Usually, we say, if the quantization bits are increased by 1, the $(\frac{S}{N_q})_{dB}$ increases 6dB.
- Drawbacks: uniform quantizer is not good for small signals. However, in the nature, such as speech signal, most cases are small.

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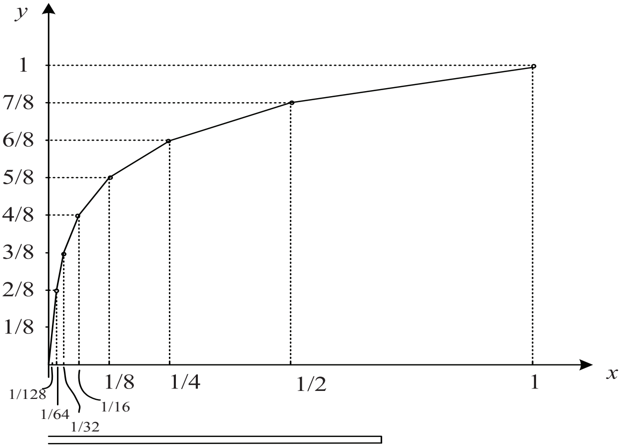
4.3.3 Non-uniform Quantization

In the non-uniform quantization, the interval of quantization varies with samples of a signal. It means that the response, $y = f(x)$, of the quantizer is nonlinear. In practice, ITU recommend two kinds of quantizer:

- A-law compression: $y = \begin{cases} \frac{Ax}{1+\ln A}, & 0 < x \le \frac{1}{A}, \\ \frac{1+\ln Ax}{1+\ln A}, & \frac{1}{A} \le x \le 1. \end{cases}$
- μ-law compression: $y = \frac{\ln(1+\mu x)}{\ln(1+\mu)}, \ 0 \le x \le 1.$

In America and Japan, μ-law 15 broke line compression $\mu = 255$ is exploited; In our country and Europe, A-law 13 broke line compression (A=87.6) is exploited.

A-law Broke Line Compression Characteristics



- The above characteristics is named A-law compression when A=87.6. See the pp.95, table 4.3.2.
- Why is 13 broke line?

4.4 Pulse Code Modulation - PCM

Table: Folded binary code and nature binary code

Polarity	Nature	Folded	Levels
+	1111	1111	15
	1110	1110	14
	1101	1101	13
	1100	1100	12
	1011	1011	11
	1010	1010	10
	1001	1001	9
	1000	1000	8
−	0111	0000	7
	0110	0001	6
	0101	0010	5
	0100	0011	4
	0011	0100	3
	0010	0101	2
	0001	0110	1
	0000	0111	0

- Folded binary code is good for small signals.

4.4.1 Principles of Pulse Code Modulation (PCM)

The PCM consists of three steps: sampling, quantization, and **coding**.

- A-law 13 broke line: the input range $[0, 1]$ is non-uniformly divided into 8 segments, [0, 1/128], ..., [1/4, 1/2], [1/2, 1].
- Each segment is uniformly divided into 16 smaller segments.

Coding rules: Folded binary code

- Polarity bit, named c_1, to denote the positive or negative of the pulse, e.g., positive – $c_1 = 1$ and negative – $c_1 = 0$.
- Segment bits, named $c_2c_3c_4$, to denote which segment the pulse locates.
- Inner-segment bits, named $c_5c_6c_7c_8$, to denote which quantization level the pulse is.

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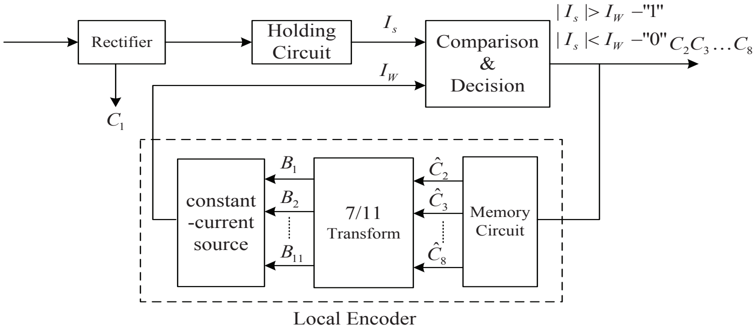
Define the minimum quantization interval $\Delta = \frac{1}{2048}$ as the quantization unit. Then, we have the table

No. of segment	beginning level	quantization interval
1 000	0	Δ
2 001	16Δ	Δ
3 010	32Δ	2Δ
4 011	64Δ	4Δ
5 100	128Δ	8Δ
6 101	256Δ	16Δ
7 110	512Δ	32Δ
8 111	1024Δ	64Δ

- Exe: pulse $I_s = 1260\Delta$, determine the A-law 13 broke line code.

PCM Encoder

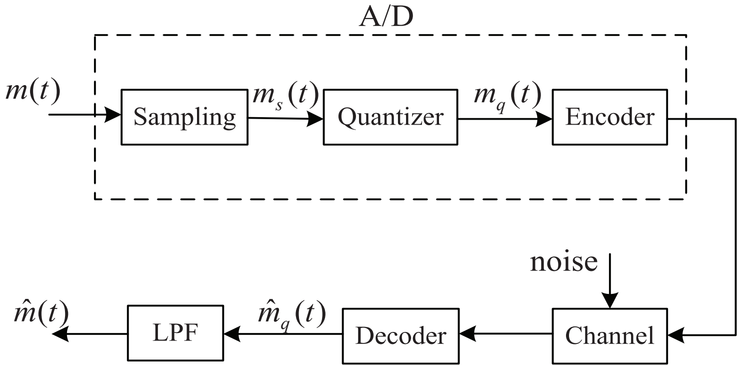


To determine c_i, set $\hat{c}_i$=1

Encoder Rate

- Assume the speech signal $[0, 4kHz]$.
- Sampling rate $f_s = 8k$ samples/s.
- PCM A-law 13 broke line rule: $R_b = 64\ kbit/s$.

Quantization Noise in PCM System



- The output of the LPF: $\hat{m}(t) = m(t) + n_q(t) + n_e(t)$.
- The output SNR: $\frac{S_o}{N_o} = \frac{E[m^2(t)]}{E[n_q^2(t)+E[n_e^2(t)]}$.
- Assume the input signal $m(t) \sim U[-a, a]$ and quantizer is with N bit (M levels) uniform quantization. Determine $\frac{S_o}{N_o}$?

- Signal to quantization noise ratio: $\frac{S_o}{N_q} = \frac{E[m^2(t)]}{n_q^2(t)} = M^2 = 2^{2N}$.
- Signal to noise ratio (error bits):
  - Noise power (error bits):

$$N_e = E[n_e^2(t)] = P_e \sum_{i=1}^{N} (2^{i-1}\Delta)^2 = \frac{2^{2N}-1}{3}\Delta^2 P_e \approx \frac{2^{2N}}{3}\Delta^2 P_e$$

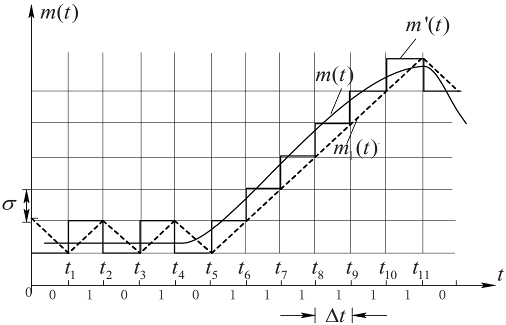
.

  - Signal power: $S_o = E[m^2(t)] = \int_{-a}^{a} x^2 \frac{1}{2a} dx = \frac{a^2}{3} = \frac{2^{2N}}{12}\Delta^2$. $(2a = 2^N \Delta)$
  - We have $\frac{S_o}{N_e} = \frac{1}{4P_e}$.
- We obtain the output SNR: $\frac{S_o}{N_o} = \frac{1}{\frac{N_q}{S_o} + \frac{N_e}{S_o}} = \frac{2^{2N}}{1+4P_e 2^{2N}}$.
- Discussions: (From the viewpoint of the practice)
  - High SNR: P_e is very small, $4P_e 2^{2N} << 1$, $\frac{S_o}{N_o} \approx \frac{S_o}{N_q}$.
  - Low SNR: P_e is is large, $4P_e 2^{2N} >> 1$, $\frac{S_o}{N_o} \approx \frac{S_o}{N_e}$.

4.6 Delta Modulation

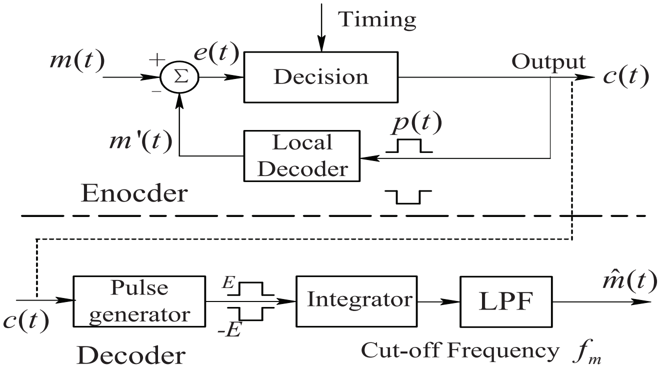
1. Principles of Delta Modulation

Definition: ΔM is to encode the difference between the neighboring samples.



- In the above figure, if Δt and σ are small enough, staircase waveform, $m'(t)$, can be arbitrarily close to $m(t)$.
- Encoding: "1" —- $m'(t)$ goes upstair δ and "0" —- $m'(t)$ goes downstair δ, $m'(t)$ is equivalent to binary sequence.

2. Encoder and Decoder



- Decision Rules:

$$m(t) - m'(t)|_{t=t_-} \begin{cases} > 0, & \text{“}1\text{''} \\ < 0, & \text{“}0\text{''}. \end{cases}$$

3. Overload Characteristic

- Quantization noise: $e_q(t) = m(t) - m'(t)$
  - Normal quantization noise: $|e_q(t)| \leq \sigma$.
  - Overload quantization noise: If the output $m'(t)$ of the local encoder cannot pace up with variation of $m(t)$, $|e_q(t)| > \sigma/2$.
- Maximum tracking slope:
  - Assume sampling interval Δt — sampling rate $f_s = \frac{1}{\Delta t}$;
  - Quantization step: σ.
  - Maximum tracking slope: $K = \sigma f_s$.
- To avoid overload quantization noise:

$$|\frac{dm(t)}{dt}| \leq \sigma f_s$$

.

- To elevate the tracking ability: increase quantization step σ and sampling rate f_s.
- Drawbacks: Increase quantization noise and transmission rate.

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4. Coding Range and Signal to Quantization SNR

Maximum coding level and Minimum coding level

- Maximum coding level: If input signal $m(t) = A\sin\omega_k t$, the maximum slope $K = A\omega_k$. To avoid overload noise, $A\omega_k \leq \sigma f_s$ is desired. So $A_{\max} = \frac{\sigma f_s}{2\pi f_k}$.
- Minimum coding level: If $m(t) \in \left[-\frac{\sigma}{2}, \frac{\sigma}{2}\right]$, the output coded sequence is $1010101010\cdots$ and cannot show the characteristic of the signal. So $A_{\min} = \frac{\sigma}{2}$.

Assume the input signal $m(t) = A\sin\omega_k t$. Quantization SNR

$$\frac{S_o}{N_q} = \frac{3}{8\pi^2}\left(\frac{f_s^3}{f_k^2 f_m}\right) \approx 0.04\frac{f_s^3}{f_k^2 f_m}$$
$$= 30\log f_s - 20\log f_k - 10\log f_m - 14(\text{dB}).$$

where f_m is the cut-off frequency of the low pass filter.

- $f_s \to 2f_s$, $\left(\frac{S_o}{N_q}\right)_{dB} \uparrow$ 9dB; $f_k \to 2f_k$, $\left(\frac{S_o}{N_q}\right)_{dB} \downarrow$ 6dB; $f_m \to 2f_m$, $\left(\frac{S_o}{N_q}\right)_{dB} \downarrow$ 3dB;

5. Comparisons of PCM and ΔM

	PCM	ΔM
Sampling Rate	$f \geq 2f_m$	related to the maximum tracking slope, usually larger than Nyquist rate.
Bit rate	$\geq 64kbit/s$	determined by f_s
$\frac{S_o}{N_q}$	$6N_{dB}$	$\frac{S_o}{N_q} = \frac{3}{8\pi^2}\left(\frac{f_s^2}{f_k^2 f_m}\right)$
Effects of error bits	sensitive	insensitive, applied to the system with high BER
Complexity	high, but widely used	low and only used in special applications. (**why?**)

Assume PCM and ΔM have the same transmission rate, R_b.

- ΔM: $f_s = R_b$.
- PCM: $f_s = 2f_m = \frac{R_b}{N} \longrightarrow R_b = 2Nf_m$

$$\frac{S_o}{N_q} \approx 10\log\left(0.04\frac{f_s^3}{f_k^2 f_m}\right)$$
$$= 10\log\left(0.32N^3\frac{f_m^2}{f_k^2}\right)$$

When $f_k = 1kHz$ and $f_m = 4kHz$, $\frac{S_o}{N_q} \approx 30\log 1.72N$ dB.